On Cross Sections to the Horocycle and Geodesic Flows on Quotients by Hecke Triangle Groups G_q

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The BCZ Map

The Stern-Brocot tree

- The rationals can be enumerated using the **Stern-Brocot** process :
  - Given any two **unimodular** rationals $\frac{a_0}{q_0}, \frac{a_1}{q_1} \in \mathbb{Q}$, insert their **mediant** $\frac{a_0}{q_0} \oplus \frac{a_1}{q_1} := \frac{a_0+a_1}{q_0+q_1}$ between them.
  - The pairs $\frac{a_0}{q_0}, \frac{a_0+a_1}{q_0+q_1}$ and $\frac{a_0+a_1}{q_0+q_1}, \frac{a_1}{q_1}$ are unimodular.
  - The repetitive application of the Stern-Brocot process exhausts the rationals between $\frac{a_0}{q_0}$ and $\frac{a_1}{q_1}$.

The Farey fractions

- The **Farey fractions** $\mathscr{F}(Q)$ at level Q

$$\mathscr{F}(Q) := \left\{ \frac{a}{q} \middle| 0 \le a \le q \le Q,\ \gcd(a,q) = 1 \right\}.$$

  - The first fraction to appear between two successive fractions $\frac{a_0}{q_0}, \frac{a_1}{q_1} \in \mathscr{F}(Q)$ is $\frac{a_0+a_1}{q_0+q_1} \in \mathscr{F}(q_0+q_1)$.

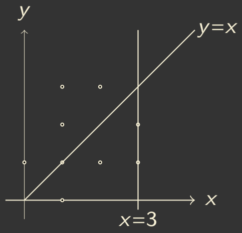

Figure: The Farey fractions at level $Q=3$ are given by the slopes of visible integer points inside the triangle bounded by the lines $y=0$, $y=x$, and $x=3$.

Some properties of the Farey fractions

- If $\frac{a_0}{q_0}, \frac{a_1}{q_1} \in \mathscr{F}(Q)$ are successive, then $0 < q_0, q_1 \leq Q$, and $q_1 + q_2 > Q$.
- If $\frac{a_0}{q_0}, \frac{a_1}{q_1} \in \mathscr{F}(Q)$ are successive, then they are unimodular. I.e. $\frac{a_1}{q_1} - \frac{a_0}{q_0} = \frac{1}{q_0 q_1}$.
- If $\frac{a_0}{q_0}, \frac{a_1}{q_1}, \frac{a_2}{q_2} \in \mathscr{F}(Q)$ are successive, then

$$
\begin{aligned}
a_2 &= \left\lfloor \frac{Q + q_0}{q_1} \right\rfloor a_1 - a_0 \\
q_2 &= \left\lfloor \frac{Q + q_0}{q_1} \right\rfloor q_1 - q_0
\end{aligned}
$$

The BCZ dynamical system

- Boca-Cobeli-Zaharescu (2001) introduced a dynamical system that tracks the successive normalized denominators of the Farey fractions at any level.
- If $\frac{a_0}{q_0}, \frac{a_1}{q_1}, \frac{a_2}{q_2} \in \mathscr{F}(Q)$ are successive, then
  - $\left(\frac{q_0}{Q}, \frac{q_1}{Q}\right)$ and $\left(\frac{q_1}{Q}, \frac{q_2}{Q}\right)$ belong to the **Farey triangle**

$$\mathcal{T} := \left\{(a,b) \in \mathbb{R}^2 \middle| 0 < a, b \leq 1,\ a+b > 1\right\},$$

  - $\left(\frac{q_1}{Q}, \frac{q_2}{Q}\right) = T\left(\frac{q_0}{Q}, \frac{q_1}{Q}\right)$, where $T : \mathcal{T} \to \mathcal{T}$ is the **BCZ map**

$$T(a,b) := \left(b, \left\lfloor \frac{1+a}{b} \right\rfloor b - a\right).$$

The Farey triangle

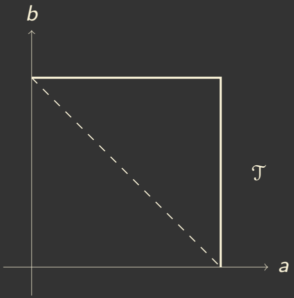

Figure: The Farey triangle $\mathcal{T}$.

Applications of the BCZ map to Farey statistics

- For any Q, consider the measure

$$\begin{aligned}\rho_Q &= \frac{1}{\#\mathscr{F}(Q)-1}\sum_{\frac{a_0}{q_0},\frac{a_1}{q_1}\in\mathscr{F}(Q):\text{succ}}\delta_{\left(\frac{q_0}{Q},\frac{q_1}{Q}\right)}\\ &= \frac{1}{\#\mathscr{F}(Q)-1}\sum_{i=0}^{\#\mathscr{F}(Q)-2}\delta_{T^i\left(\frac{0}{Q},\frac{1}{Q}\right)}\end{aligned}$$

on $\mathcal{T}$.

Theorem (Boca-Cobeli-Zaharescu (2001))

The measures ρ_Q weak-$*$ converge to the probability measure $m := 2\,da\,db$ on $\mathcal{T}$.

Example: distribution of Farey gaps

Corollary (Boca-Cobeli-Zaharescu (2001))

Let

$$f_Q(t) := \frac{1}{\#\mathscr{F}(Q)} \# \left\{ succ.\ \frac{a_0}{q_0}, \frac{a_1}{q_1} \in \mathscr{F}(Q) \,\middle|\, Q^2 \left(\frac{a_1}{q_1} - \frac{a_0}{q_0} \right) \geq t \right\}.$$

Then $\lim_{Q\to\infty} f_Q(t) = f(t)$*, where*

$$f(t) := m\left(\left\{ (a,b) \in \mathscr{T} \,\middle|\, \frac{1}{ab} \geq t \right\} \right).$$

- Note: $\#\mathscr{F}(Q) \sim \frac{3}{\pi^2} Q^2$.

Cross section to the horocycle Flow

- Write $X_2 := \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$.

Theorem (Athreya-Cheung (2014))

Consider the function $P : \mathscr{T} \to X_2$ given for any $(a,b) \in \mathscr{T}$ by $P(a,b) := \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \mathrm{SL}(2,\mathbb{Z})$. The following are true:

1. *The set $P(\mathscr{T})$ is in bijection with the collection of lattices in X_2 that have a horizontal vector of length not exceeding 1.*
2. *The set $P(\mathscr{T})$ is a Poincaré cross section to the horocycle flow $h_s := \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} \curvearrowright X_2$ with roof function $R(a,b) := \frac{1}{ab}$, and first return map T.*

Goal

- Extend the previous ideas to the family of Hecke triangle groups.

The λ_q-CF Algorithm

The Hecke triangle groups

- For $q = 3, 4, 5, \cdots$, the **Hecke triangle group** G_q is the subgroup of $\mathrm{SL}(2, \mathbb{R})$ generated by
  - $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
  - $T_q := \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}$

  where $\lambda_q := 2\cos\frac{\pi}{q}$.
- We have $G_3 = \mathrm{SL}(2, \mathbb{Z})$.

The λ_q-continued fraction algorithm

- As is customary, we write

$$U_q := T_q S = \begin{pmatrix} \lambda_q & -1 \\ 1 & 0 \end{pmatrix}.$$

  - conjugate to a rotation by $\frac{\pi}{q}$
  - rotates vectors on the ellipses given by $Q_q(x,y) := x^2 - \lambda_q xy + y^2$

- For $i = 0, 1, \cdots, 2q-1$, we write

$$\mathfrak{w}_i^q := U_q^i (1,0)^T.$$

  - Note that $\mathfrak{w}_0^q = (1,0)^T$ and $\mathfrak{w}_{q-1}^q = (0,1)^T$.

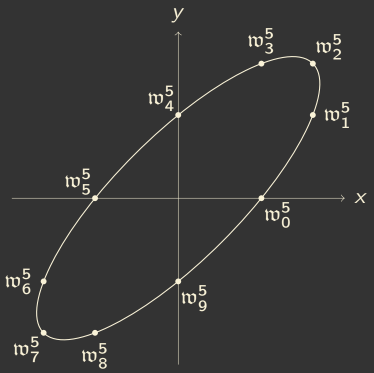

Figure: The vectors $\{\mathfrak{w}_i^5\}_{i=0}^9$ along with the ellipse $Q_5((x,y)^T) = x^2 - \lambda_5 xy + y^2 = 1$. Note that λ_5 is the golden ratio $\varphi = (1+\sqrt{5})/2$.

The λ_q-continued fraction algorithm

- For $i = 0, 1, \cdots, 2q-2$, we write

$$\Sigma_i^q := (0, \infty)\mathfrak{w}_i^q + [0, \infty)\mathfrak{w}_{i+1}^q$$

for the sector in $\mathbb{R}^2$ bound by $\mathfrak{w}_i^q$ (inclusive), and $\mathfrak{w}_{i+1}^q$ (exclusive).

  - The matrix T_q^{-1} sends Σ_0^q to the first quadrant $\mathrm{FQ} := \cup_{i=0}^{q-2}\Sigma_i^q$, reduces Q_q-values of vectors except for those that are parallel to $(1,0)^T$.
  - The matrix U_q^{-1} sends Σ_i^q to $\Sigma_{i-1 \mod 2q}$, maintains the Q_q-values of vectors.

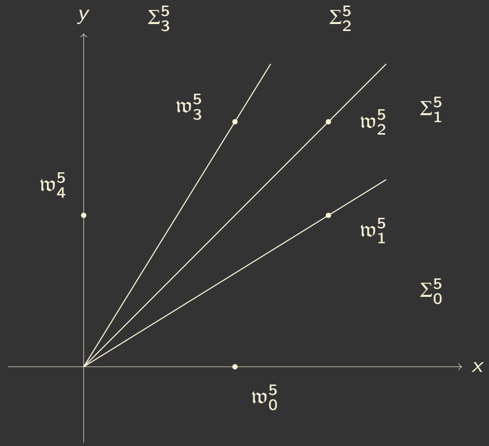

Figure: The vectors $\{\mathfrak{w}_i^5\}_{i=0}^4$, and the sectors $\{\Sigma_i^5\}_{i=0}^3$.

The λ_q-continued fraction algorithm

Definition

For any $\mathbf{u} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, if $\mathbf{u} \in \Sigma_{i_0}^q$ with $i_0 \in \{0, 1, \cdots, 2q-2\}$, then the λ_q-**continued fraction algorithm** sends $\mathbf{u}$ to the $T_q^{-1} U_q^{-i_0} \mathbf{u}$ in the first quadrant $\mathrm{FQ} := \cup_{i=0}^{q-2} \Sigma_i^q$.

Consequences of the λ_q-CF Algorithm

Some consequences of the λ_q-CF algorithm:

- All non-trivial discrete linear orbits of G_q on the plane $\mathbb{R}^2$ are homothetic dilations of

$$\Lambda_q := G_q(1,0)^T.$$

- There exists a G_q-**Stern-Brocot process** that exhausts all the elements of Λ_q in the sector spanned by any unimodular pair $\mathbf{u}, \mathbf{v} \in \Lambda_q$ (i.e. $\mathbf{u} \wedge \mathbf{v} = 1$).
- The elements of the $\mathrm{SL}(2,\mathbb{R})$ orbit $\mathrm{SL}(2,\mathbb{R}) \cdot \Lambda_q$ can be bijectively identified with the elements of $X_q := \mathrm{SL}(2,\mathbb{R})/G_q$.

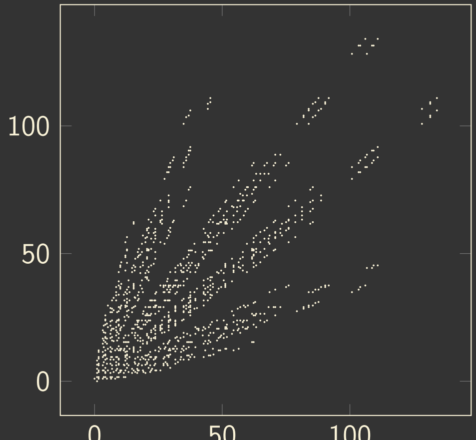

Figure: The elements of Λ_5 in the first quadrant generated using $N = 5$ applications of the G_5-Stern-Brocot process.

The BCZ Map Analogue for G_q

Theorem (Taha (2019))

*Consider the G_q-**Farey triangle***

$$\mathcal{T}^q = \left\{(a,b) \in \mathbb{R}^2 \middle| 0 < a \leq 1,\ 1 - \lambda_q a < b \leq 1\right\},$$

and the maps $P_q : \mathcal{T}^q \to X_q$ given for any $(a,b) \in \mathcal{T}^q$ by $P_q(a,b) := \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} G_q$. The following are true:

1. *The set $P_q(\mathcal{T}^q)$ is in bijection with the elements in the $\mathrm{SL}(2,\mathbb{R})$-orbit of Λ_q that have a horizontal vector of length not exceeding 1.*
2. *The set $P_q(\mathcal{T}^q)$ is a Poincaré cross section to the horocycle flow $h_s \curvearrowright X_q$ with explicit roof function R_q and first return map BCZ_q.*

The G_5-Farey triangle

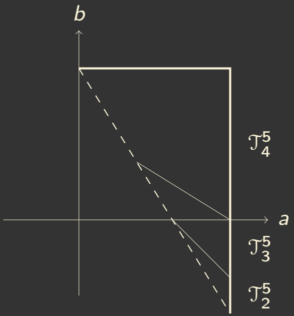

Figure: The G_5-Farey triangle $\mathcal{T}^5$.

The Farey fractions analogue for G_q

- For any $A \in \mathrm{SL}(2,\mathbb{R})$, $\tau > 0$, and interval $I \subseteq \mathbb{R}$, we denote by

$$\mathscr{F}_I(A\Lambda_q,\tau) := \{\mathbf{u} \in A\Lambda_q \cap ((0,\tau] \times \mathbb{R}) \,|\, \mathrm{slope}(\mathbf{u}) \in I\}$$

the Farey fractions analogue for G_q.

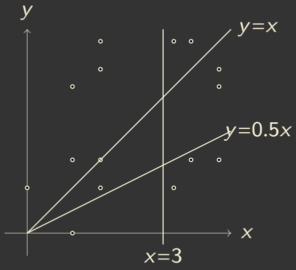

Figure: The set $\mathscr{F}_{[0.5,1]}(\Lambda_5,3)$ is the collection of points from Λ_5 inside the triangle bounded by the lines $y = 0.5x$, $y = x$, and $x = 3$.

Applications of the G_q-BCZ map to G_q-Farey statistics

1. The G_q-**next-vector algorithm**: The *best* algorithm for generating elements of Λ_q in a box.
2. G_q-**Farey statistics**: equidistribution of slopes and slope gap distribution of the elements of Λ_q in homothetic dilations of triangles
3. Quadratic growth of the number of elements of Λ_q in homothetic dilations of *measureable* shapes.

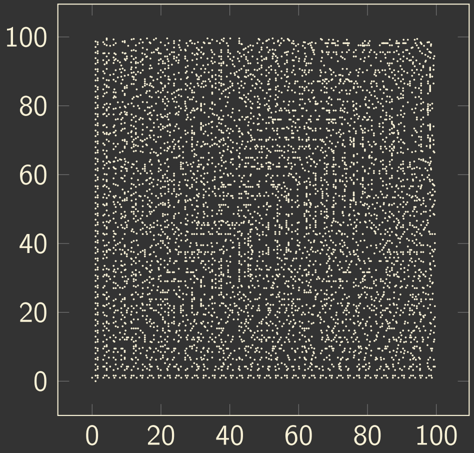

Figure: The elements of Λ_5 in the box $[0, 100]^2$ generated using the G_5-next-vector algorithm.

The AN Natural Extension of λ_q-CFs

- Arnoux-Nogueira (1993) introduced a heuristic for building models of natural extensions of vectorial continued fraction algorithms.
- The λ_q-CF algorithm can be expressed as a function $\mathbf{u} \mapsto A_q(\mathbf{u})^{-1}\mathbf{u}$ on the first quadrant FQ, where $A_q : \mathrm{FQ} \to \mathrm{SL}(2,\mathbb{R})$ is constant on the sectors $\{\Sigma_i^q\}_{i=0}^{q-2}$.
- Writing $\widehat{\mathrm{FQ} \times \mathrm{FQ}} := \left\{(\mathbf{u},\mathbf{v}) \in \mathrm{FQ} \times \mathrm{FQ} \middle| \mathbf{u} \cdot \mathbf{v} = 1\right\}$, the map A_q can be extended to $\widehat{A}_q : \widehat{\mathrm{FQ} \times \mathrm{FQ}} \to \widehat{\mathrm{FQ} \times \mathrm{FQ}}$ via $(\mathbf{u},\mathbf{v}) \mapsto (A_q(\mathbf{u})^{-1}\mathbf{u}, A(\mathbf{u})^T\mathbf{v})$.
- The space $\widehat{\mathrm{FQ} \times \mathrm{FQ}}$ comes with a natural "geodesic flow" $\varphi^t(\mathbf{u},\mathbf{v}) := (e^t\mathbf{u}, e^{-t}\mathbf{v})$.

- It can be seen that the elements of the suspension of the horocycle flow over the G_q-Farey triangles

$$S_{R_q}\mathcal{T}^q := \left\{((a,b),s)\middle|(a,b)\in\mathcal{T}^q,\ s\in[0,R_q(a,b))\right\}$$

can be, as G_q-cosets, be bijectively identified with the element of the portion of $\widehat{\mathrm{FQ}\times\mathrm{FQ}}$ lying above particular polygons (the G_q-**Stern-Brocot polygon** $\mathcal{P}^q$)

$$S\mathcal{P}^q := \{(\mathbf{u},\mathbf{v})\in\widehat{\mathrm{FQ}\times\mathrm{FQ}}\mid\mathbf{u}\in\mathcal{P}^q\}.$$

- This provides a coding of the geodesic flow on X_q alternative to the usual Bowen-Series coding.

The G_5-Stern-Brocot polygon

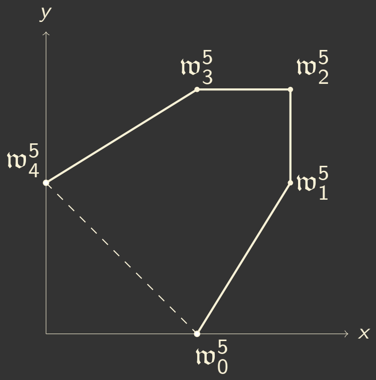

Figure: The G_5-Stern-Brocot polygon.

Double regular n-gons

- Particular groups of symmetries (i.e. the Veech groups) of double regular n-gons with n odd are conjugate to the Hecke triangle groups G_n.

Figure: Double-Pentagon Shaped Tile, 15th Century Iran, the Metropolitan Museum of Art.

- The previous results can be used to study the linear flow and its closed trajectories on translation surfaces arising from those polygons. E.g. Davis-Lelievre (2018) with the λ_5-continued fraction algorithm.

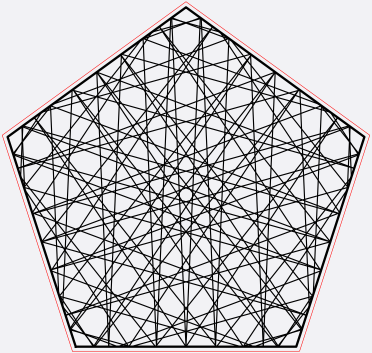

Figure: A periodic path on the pentagon. Source: https://www.swarthmore.edu/NatSci/ddavis3/pents/index.html

Thank you!