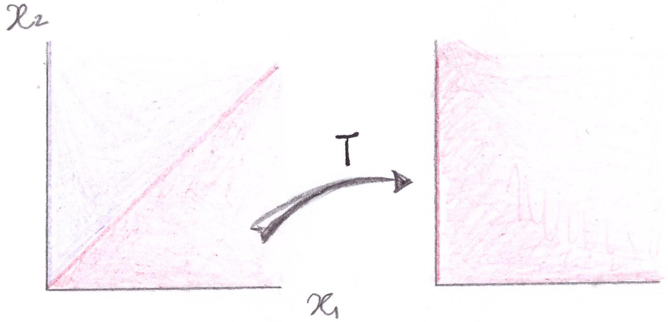
Euclide algorithm

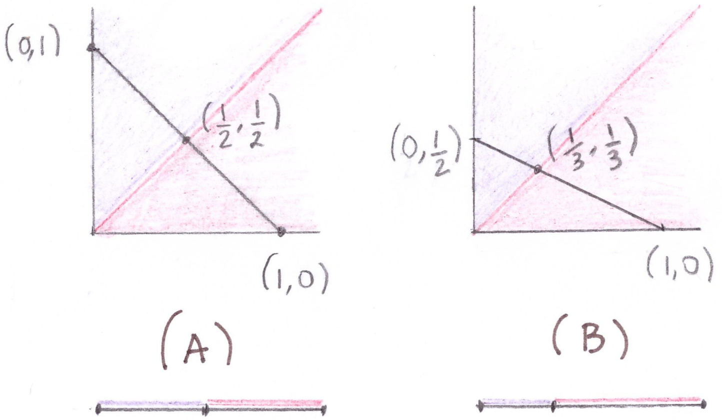
$$T : \mathbb{R}_+^2 \xrightarrow{\sim} \mathbb{R}_+^2$$

$$(x_1, x_2) \longrightarrow \begin{cases} (x_1 - x_2, x_2) & \text{if } x_1 > x_2 \\ (x_1, x_2 - x_1) & \text{otherwise.} \end{cases}$$



$x_1 > x_2$

$x_2 > x_1$



$$T^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

projectively,

$$(1 - x_2/x_1,\ x_2/x_1)$$

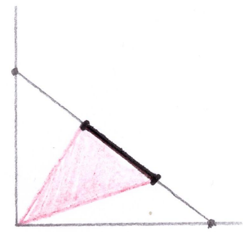
$$(x_1/x_2,\ 1 - x_1/x_2)$$

$\forall q \in \mathbb{R}_+^2$,

$$P\mathbb{R}_+^2 \cong \{ \lambda \in \mathbb{R}_+^2 \mid \langle \lambda, q \rangle = 1 \}$$

(A) $q = (1,1)$ (B) $q = (1,2)$

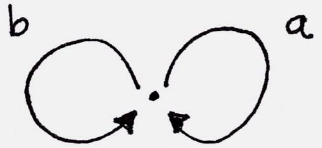
Associated family of measures ν_q



$$\nu_q(\longleftrightarrow) := \mathrm{Leb}(\triangle)$$

for $M \geqslant 0$, $\nu_q(M.\Delta) = \nu_{q.M}(\Delta)$

Simplicial Systems



$\mathcal{A} = \{a, b\}$

coding:

$$P\mathbb{R}^2_+ \xrightarrow{\sim} \mathcal{A}^{\mathbb{N}}$$
$$x \longrightarrow w(x)$$

every measure ν_q induces a stochastic process $(w_n)_{n\in\mathbb{N}}$ which laws depend only on the past $\rightsquigarrow$ Random walk

$$\mathbb{P}_q(a) = \frac{q_a}{q_a + q_b}$$

$$\mathbb{P}_q(aa \mid a) = \mathbb{P}_{q\cdot\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)}(a) = \frac{q_a + q_b}{q_a + 2q_b}$$

$$M_a := \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$M_b := \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

$$M_{\gamma_1\gamma_2\ldots\gamma_n} := M_{\gamma_1}\cdot M_{\gamma_2}\cdots M_{\gamma_n}$$

$$\mathbb{P}_q(\gamma.a \mid \gamma) = \mathbb{P}_{qM_\gamma}(a)$$

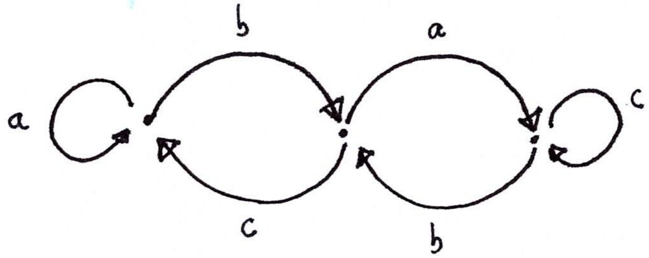
q is a finite dimensional memory vector called distortion vector.

The simplex $P\mathbb{R}^2_+$ can be thought as the event space.
The associated σ-algebra is generated by set of points whose coding starts with a given finite path.

$\mathcal{A} = \{a, b, c\}$

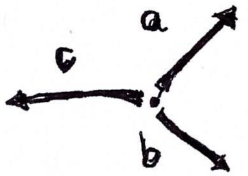
$M_a = E_{b,a}$

$M_c = E_{a,c}$



Rauzy diagram for 3-Interval Exchange Transfo.

$$(x_a, x_b, x_c) \longrightarrow \begin{cases} (x_a, x_b, x_c - x_a) & \text{si } a \\ (x_a - x_c, x_b, x_c) & \text{si } c \end{cases}$$

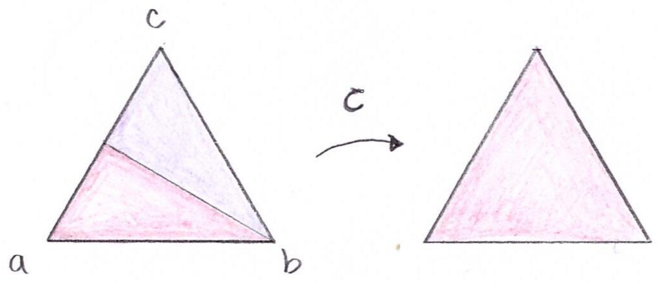


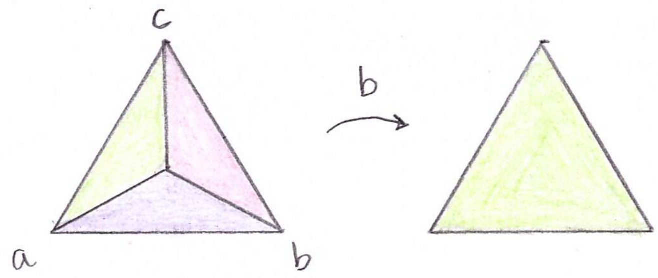
$$M_a = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

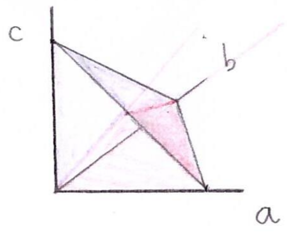
$$M_b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$M_c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(x_a, x_b, x_c) \longrightarrow \begin{cases} (x_a, x_b - x_a, x_c - x_a) & \text{si } a \\ (x_a - x_b, x_b, x_c - x_b) & \text{si } b \\ (x_a - x_c, x_b - x_c, x_c) & \text{si } c \end{cases}$$



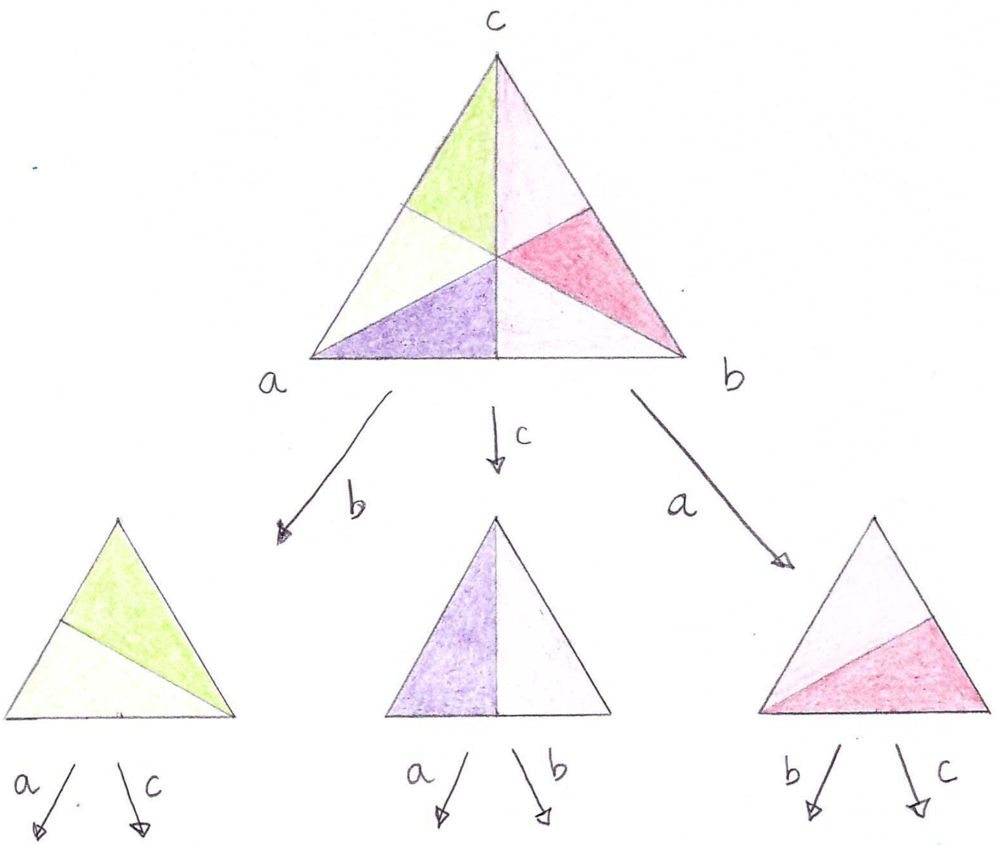




Simplex splitting CF

- Poincaré algorithm

$$T: (x_a, x_b, x_c) \longrightarrow \begin{cases} (x_a - x_b,\ x_b - x_c,\ x_c) & \text{if } x_a > x_b > x_c \\ (x_a,\ x_b - x_c,\ x_c - x_a) & \text{if } x_b > x_c > x_a \\ \vdots & \end{cases}$$



$$(x_a, x_b, x_c)$$

$$\Big\downarrow c$$

$$(\underbrace{x_a - x_c}_{y_a},\ \underbrace{x_b - x_c}_{y_b},\ \underset{y_c}{x_c})$$

$$\Big\downarrow b$$

$$(y_a - y_b,\ y_b,\ y_c)$$

$$= (x_a - x_b,\ x_b - x_c,\ x_c)$$

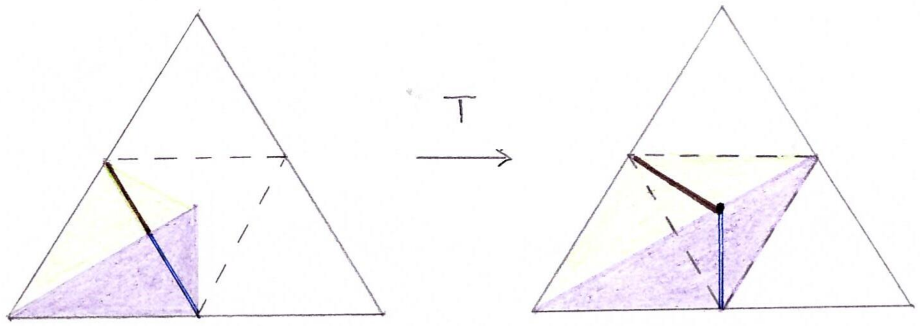
- Selmer algorithm

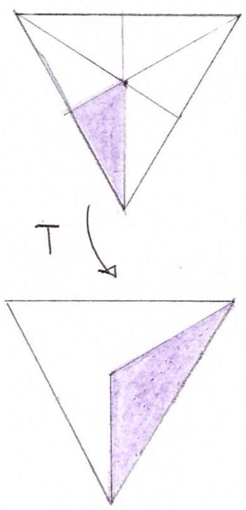
$$T: (x_a, x_b, x_c) \longrightarrow (x_a - x_c, x_b, x_c) \text{ if } x_a > x_b > x_c$$

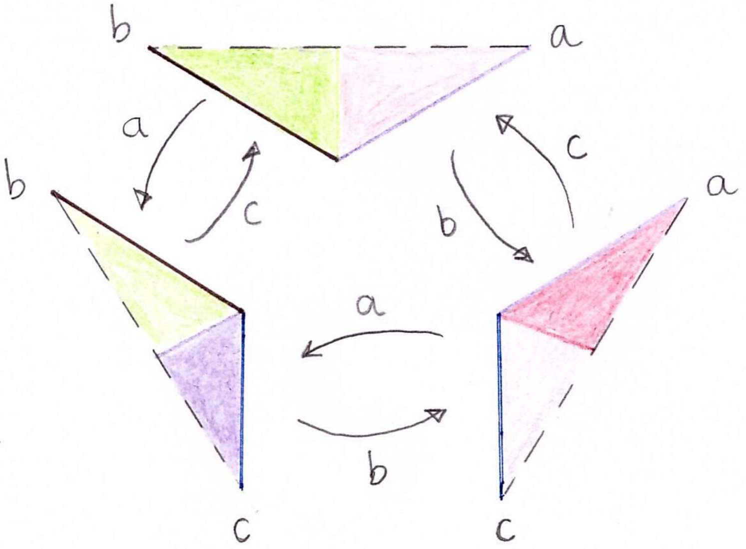
$\vdots$

$$\left\{ (x_a, x_b, x_c) \;\middle|\; \begin{array}{l} x_i < x_j + x_k \\ \forall \{i,j,k\} = \{a,b,c\} \end{array} \right\}$$

is an invariant subset for T.



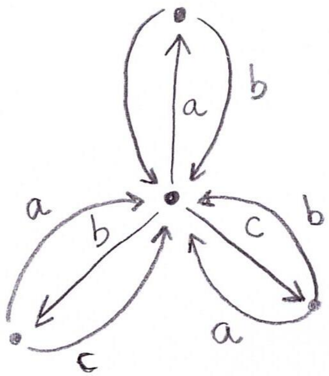




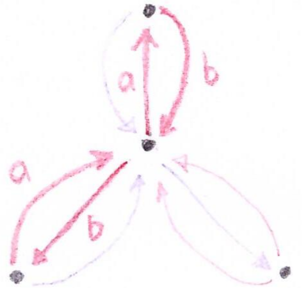
Cassaigne algorithm

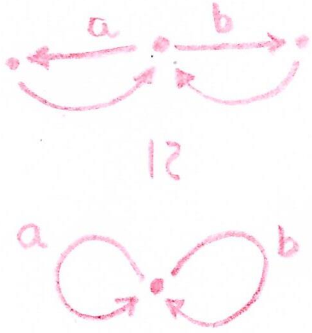
Ergodicity

Poincaré:



$q_a, q_b \gg q_c$

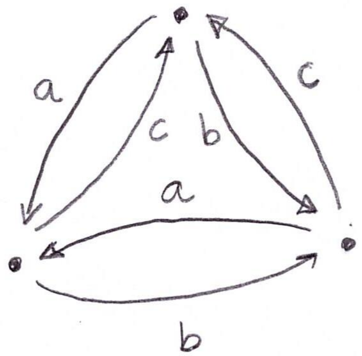




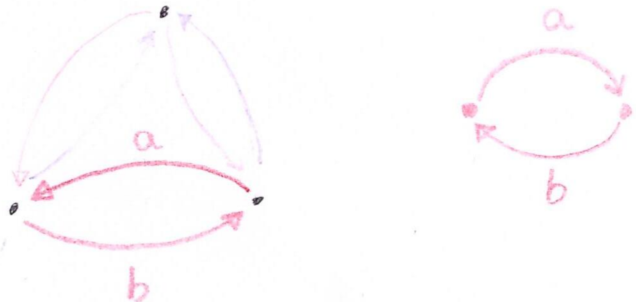
Theorem (Nogueira 1995):

For almost every $x \in \mathbb{R}_+^{A}$,
$T^n x$ converges projectively to a vertex of the simplex.

Selmer:



$q_a, q_b \gg q_c$



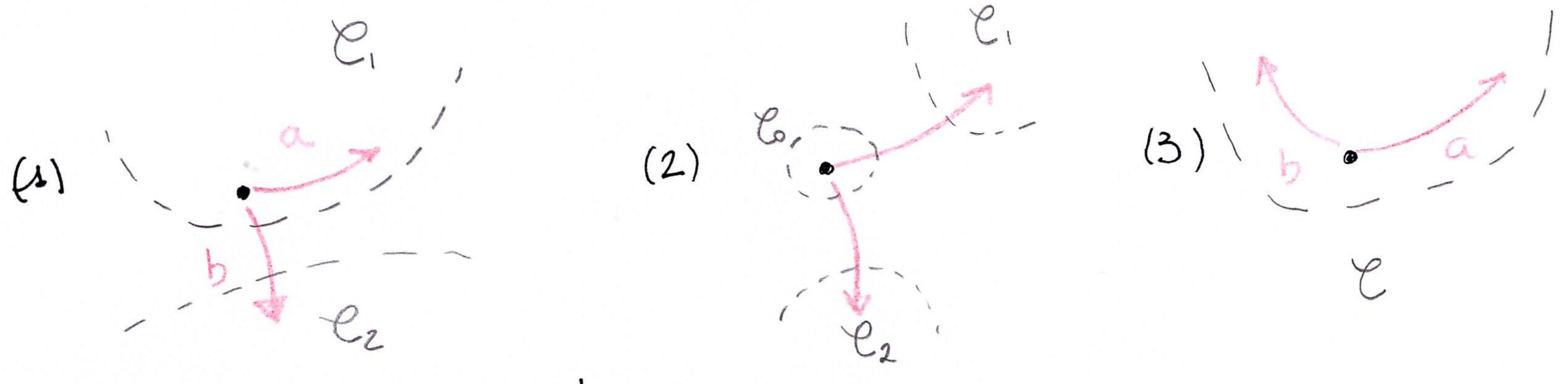
Theorem (Schweiger 2000):

The projectivized map on the simplex is ergodic with respect to a unique measure equivalent to Lebesgue.

Theorem (F.):

Let $\mathcal{L}$ a non-trivial subset of $\mathcal{A}$, consider the corresponding degenerate subgraph, assume there is a vertex with two outgoing edges labelled in $\mathcal{L}$, then there are

Three cases



where $\mathcal{C}_i$ are strongly connected components.

If case (3) does not occur, then the projectivized map has a unique ergodic measure equivalent to Lebesgue measure.

Moreover it induces the unique measure of maximal entropy for the suspension flow.

Other applications to estimate fractal dimensions.

For instance asymtotic of the dimension of Rauzy gaskets.