Comparison of two approaches for the computation of spectral objects of transfer operators

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GREYC

Once upon a time…

My first article published in 2004 :

- *Computation of a class of continued fraction constants,* [HAL-00263891]
- Joint work with Brigitte Vallée (should have been co-author)
- Proof of convergence of the Daudé-Flajolet-Vallée method for the computation of continued fraction constants
- The constants are related to the spectrum of the transfer operator of the Gauss map

A (long standing) question from Valérie summarised in this message:

Bonjour,
est-ce que vous seriez libres le jeudi 17 décembre après-midi pour faire une session spécialisée Estimations numériques pour les opérateurs de transfer ?
On pourrait avoir un exposé de Loick (si tu es disponible et d'accord Loick) et comparer avec l'approche de Julia Slipantschuk et Oscar Bandtlow au Queen Mary college.
Je joins leur article. Charles a entendu un exposé de Julia sur le sujet. On pourrait même lui demande d'en faire un?
Si vous êtes partants et disponibles, je ferai alors l'annonce au niveau de l'ANR.

La réunion ANR générale aura lieu le matin.

Amicalement,
Valérie

Lagrange approximation of transfer operators associated with holomorphic data

Oscar F. Bandtlow and Julia Slipantschuk

Quick overview of both approaches

Bandtlow-Slipantschuk	Daudé-Flajolet-Vallée

computation of spectral objects of transfer operators $\mathbf{G}$

interval and circle maps	interval maps
general transfer operators	"truncated" and "classical" transfer operators

Finite section method or Galerkin method : $\pi_n \circ \mathbf{G} \circ \pi_n$

$\pi_n =$ Lagrange interpolation on n points	$\pi_n =$ Taylor expansion of order n

very similar complex contraction properties

Exponential convergence (in n) of the spectral objects

all the constants are not explicit $\longrightarrow$ non proven values	all the constants are explicit (Gauss map) $\longrightarrow$ proven values

Finite section method

infinite dimensional Banach spaces

finite dimensional

$$\begin{tikzcd}
\mathbf{G} \arrow[rr, dashed, "\text{Hard/non explicit}"] \arrow[d, "\begin{array}{c}\pi_n = \text{finite rank}\\ \text{projection}\end{array}"'] & & \lambda,\ f_\lambda \\
\begin{array}{c}\pi_n \circ \mathbf{G} \circ \pi_n \\ \pi_n \circ \mathbf{G}\end{array} \arrow[rr, "\text{matricial computations}"'] & & \lambda_n,\ f_{\lambda_n} \arrow[u, "\text{convergence?}"']
\end{tikzcd}$$

explicit

contraction property $\Rightarrow$ $||\mathbf{G} - \pi_n \circ \mathbf{G}|| = O(\theta^n), \quad \theta < 1$

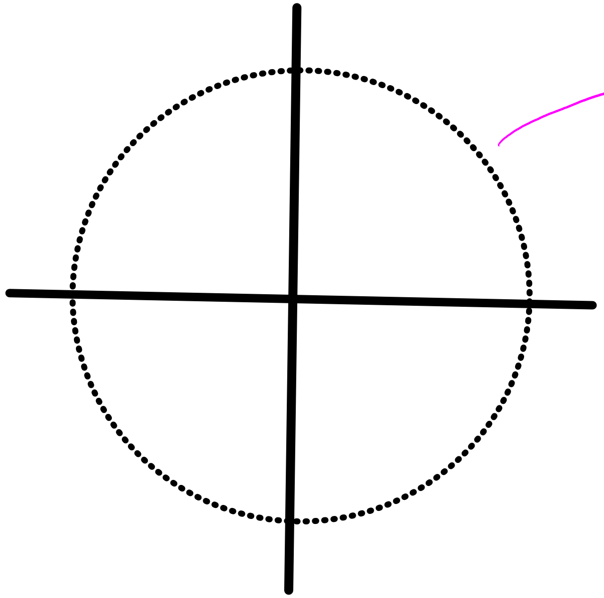
$$\Rightarrow \begin{cases} ||f_\lambda - f_{\lambda_n}|| = O(\theta^n) \\ |\lambda - \lambda_n| = O(\theta^n) \end{cases}$$

- non explicit constants
- involve $\|(I - zG)^{-1}\|$

Real and complex domains

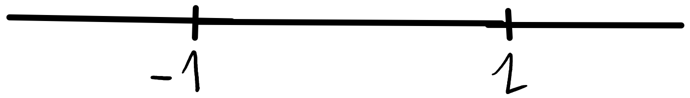
Bandtlow-Slipantschuk

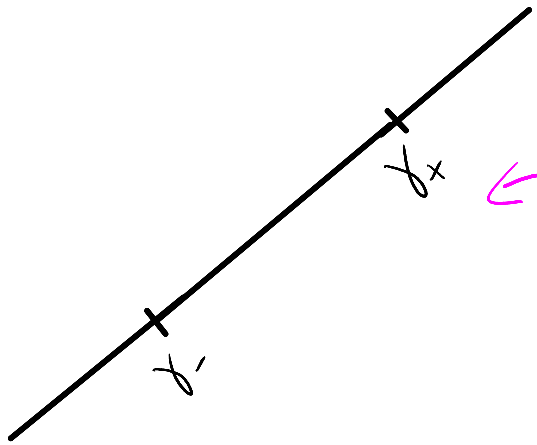
Torus



$$\sigma(z) = \frac{1}{2}(z + z^{-1})$$

Interval





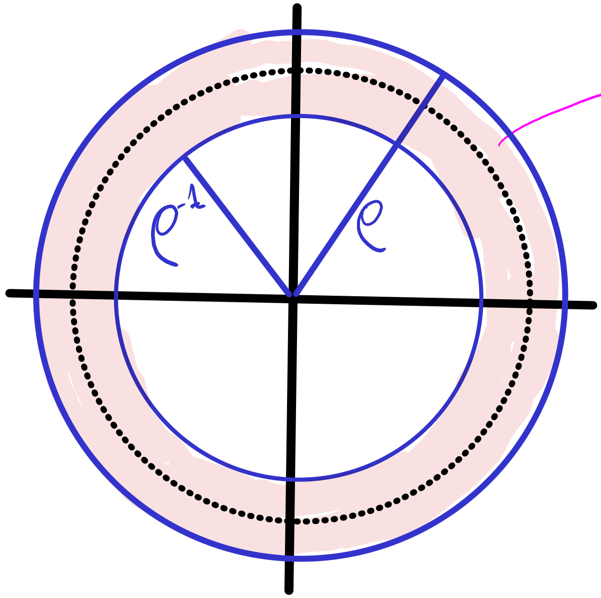
$$\alpha_\gamma(z) = \frac{\gamma_+ - \gamma_-}{2} z + \frac{\gamma_+ + \gamma_-}{2}$$

Daudé-Flajolet-Vallée

Extension to complex domains

Bandtlow-Slipantschuk

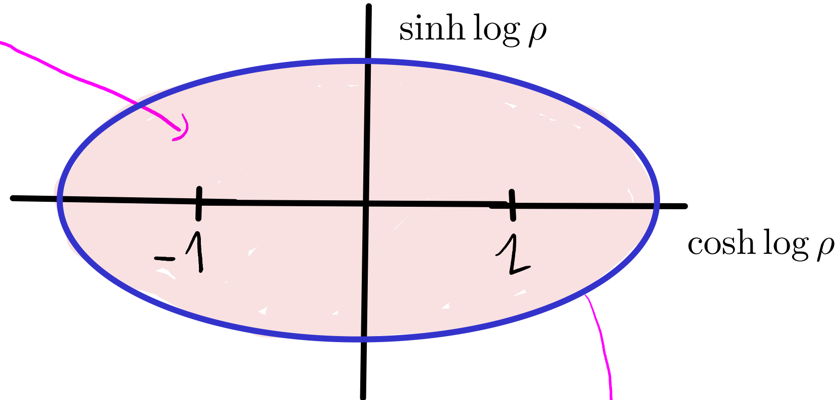
Torus ⇝ Annulus



$$A_\rho = \{z \in \mathbb{C}; \rho^{-1} < |z| < \rho\}$$

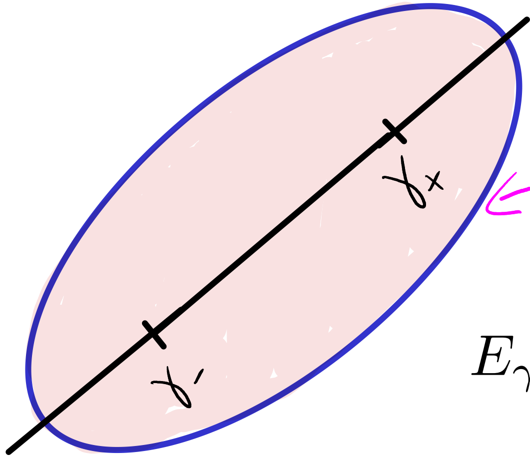
Interval ⇝ elliptic domain

$$\sigma(z) = \frac{1}{2}(z + z^{-1})$$



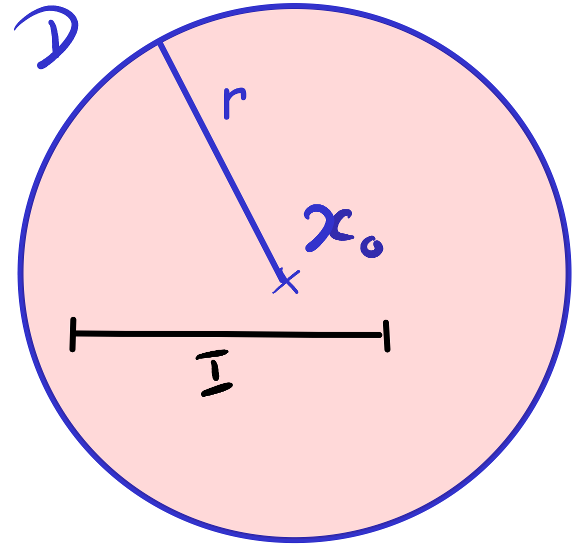
$$E_\rho = \sigma(A_\rho)$$

$$\alpha_\gamma(z) = \frac{\gamma_+ - \gamma_-}{2} z + \frac{\gamma_+ + \gamma_-}{2}$$



$$E_{\gamma,\rho} = \alpha_\gamma(E_\rho)$$

Daudé-Flajolet-Vallée

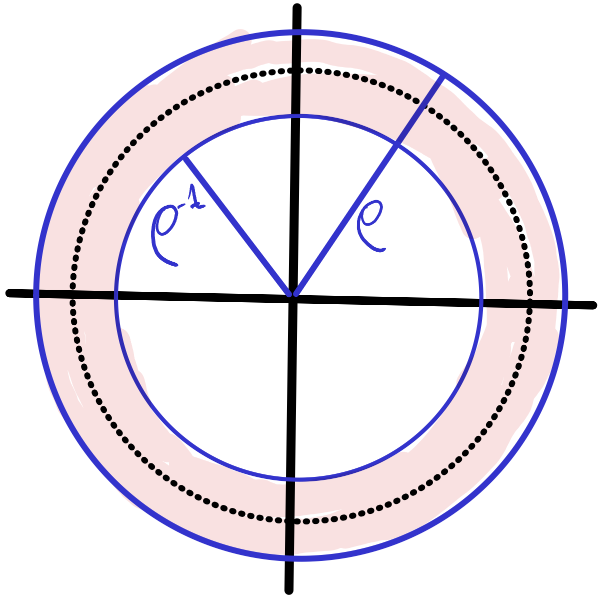


$$I \subset D(x_0, r)$$

Hardy functional spaces

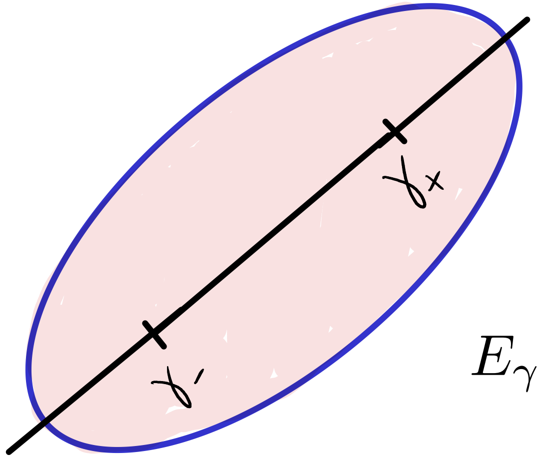
Bandtlow-Slipantschuk

Torus $\rightsquigarrow$ Annulus



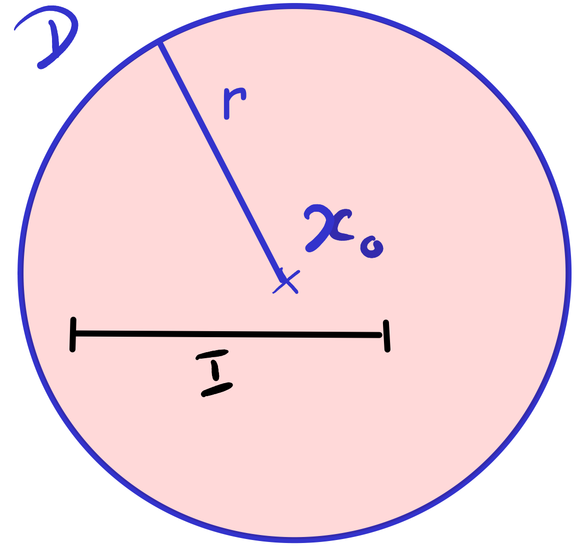
$A_\rho = \{z \in \mathbb{C}; \rho^{-1} < |z| < \rho\}$

Interval $\rightsquigarrow$ elliptic domain



$E_{\gamma,\rho} = \alpha_\gamma(E_\rho)$

Daudé-Flajolet-Vallée



$I \subset D(x_0, r)$

Hardy space : $H_\infty(D) = \{f : D \mapsto \mathbb{C};\ f \text{ holomorphic}\};\qquad ||f||_D = \sup_{z\in D} |f(z)|$

Family of transfer operators

Bandtlow-Slipantschuk

- Holomorphic map weight system $\rightsquigarrow$ D = a complex domain

Interval / Elliptic domain

$$\mathbf{G}[f](z) = \sum_{i\in\mathcal{I}} W_i(z) f\left(\Phi_i(z)\right)$$

$\mathcal{I}$ = indices $\quad W_i \in H_\infty(D) \quad \Phi_i : D \mapsto D \in H_\infty(D)$

$$\sum_{i\in\mathcal{I}} ||W_i||_D < \infty;$$

Torus / Annulus : $\quad \tau : \mathbb{T} \mapsto \mathbb{T}$

$$\mathbf{G}[f](z) = \sum_{x\in\mathbb{T};\tau(x)=z} \frac{w(x)}{|\tau'(x)|} f(x) \quad \in H_\infty(D)$$

Daudé-Flajolet-Vallée

Interval / Disc

$$\tau : I \mapsto I$$

$$\mathbf{G}_s[f](z) = \sum_{x\in\mathbb{T};\tau(x)=z} \frac{1}{|\tau'(x)|^s} f(x)$$

some

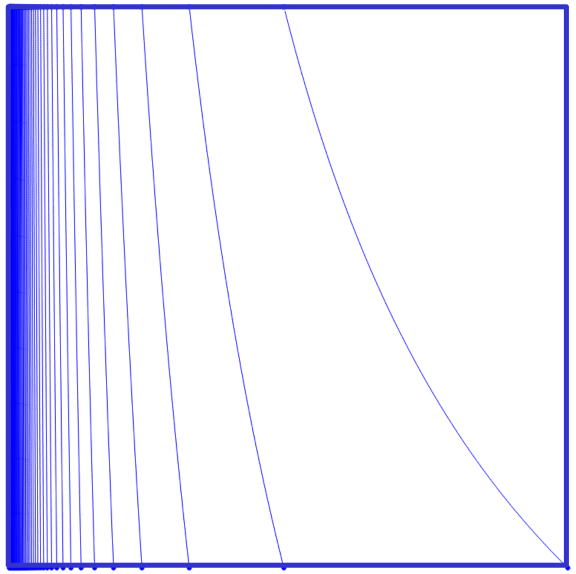
$\in H_\infty(D)$

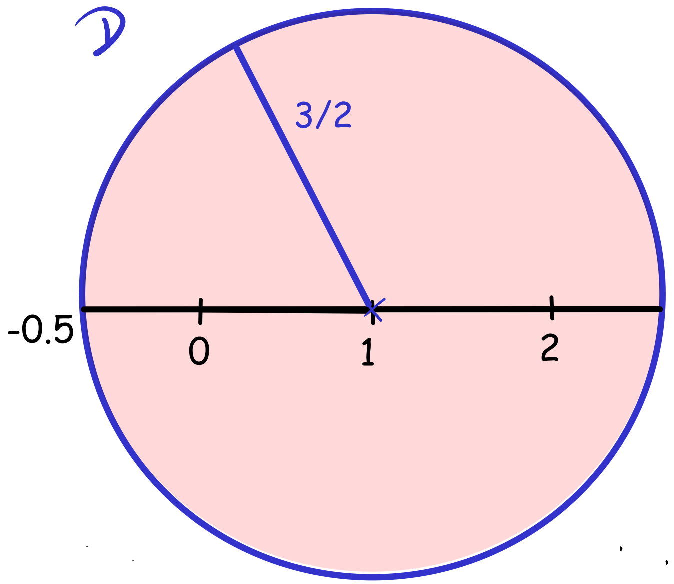
some = transfer operator with constraint

$\boxed{\mathbf{G} \text{ or } \mathbf{G}_s \text{ are a continuous operator on } H_\infty(D)}$

Example: Gauss map

$I=[0,1]$, $T(x)=\{1/x\}$





Transfer operators:

$$\mathbf{G}_s[f](x) = \sum_{m\geq 1} \frac{1}{(m+x)^s} f\left(\frac{1}{m+x}\right)$$

Transfer operators with constraints: $\mathcal{A} \subset \mathbb{N}^\star$

$$\mathbf{G}_{s,\mathcal{A}}[f](x) = \sum_{m\in\mathcal{A}} \frac{1}{(m+x)^s} f\left(\frac{1}{m+x}\right)$$

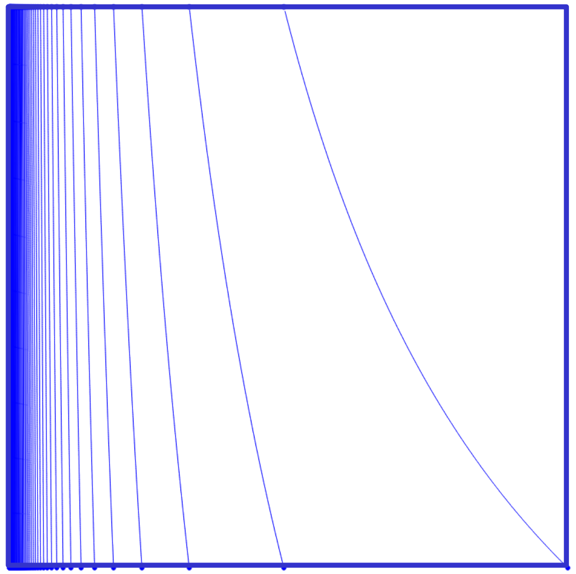
Notation: $h_m(x) = \dfrac{1}{m+x}$

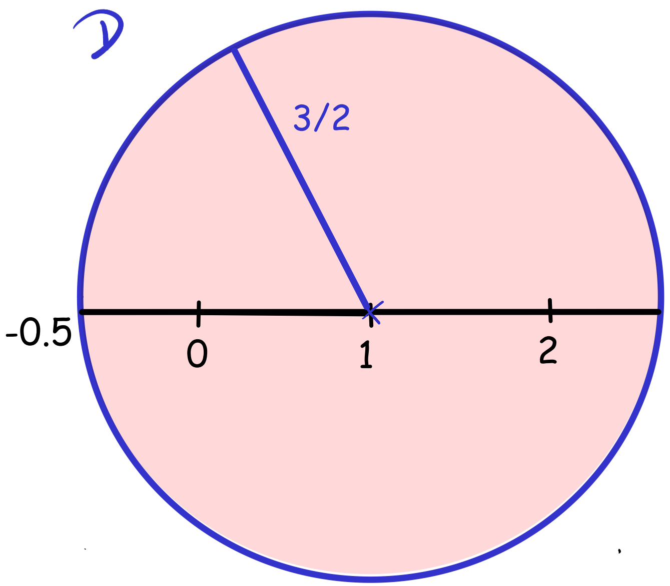
$I = [0,1]$ $\quad \mathcal{D} = D(1, \frac{3}{2})$ $\quad x_0 = 1$ $\quad r = \frac{3}{2}$

$h_m(\mathcal{D}) \subset D(1,1) \subset \mathcal{D}$

Example : Gauss map

$I=[0,1]$, $T(x)=\{1/x\}$





On $\mathcal{A}_s(D)$ G_s and $G_{s,\mathcal{A}}$ are compact and admit a unique dominant eigenvalue $\lambda(s)$ and $\lambda_{\mathcal{A}}(s)$

- $-\lambda'(1)$ = entropy
- $\lambda(2) \rightsquigarrow$ Lattice reduction algorithm (Gauss)
- $\lambda_{\mathcal{A}}(s) \rightsquigarrow$ Hausdorff dimension of $E_{\mathcal{A}} = \{ x \in [0,1] \mid$ the digits of the CFE of x are in $\mathcal{A} \}$
- ......

Other examples in the article

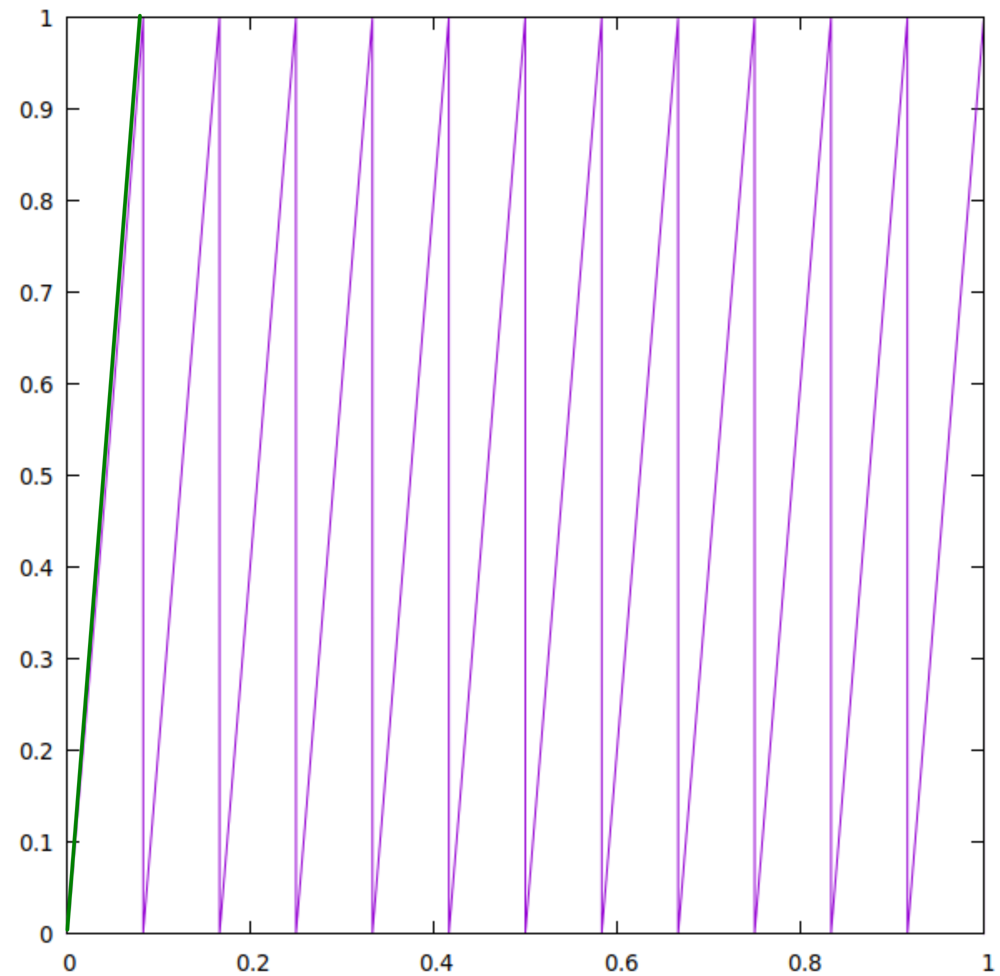
$T\colon [0,1] \to [0,1]$ given by

$$T(x) = \begin{cases} \frac{11x}{1-x} & 0 \leq x < \frac{1}{12}, \\ 12x - i & \frac{i}{12} < x \leq \frac{i+1}{12}, \end{cases}$$

$$\{\Phi_i\}_{i=0}^{11} \text{ with } \Phi_0(x) = \frac{x}{11+x}$$

$$\Phi_i(x) = \frac{x+i}{12} \text{ for } i = 1, \ldots, 11,$$

$$W_i(x) = \Phi_i'(x)$$



$E_{\gamma,\rho}$ avec $\gamma = (0,1)$ et $\rho \in]10, 20[$

Other examples in the article

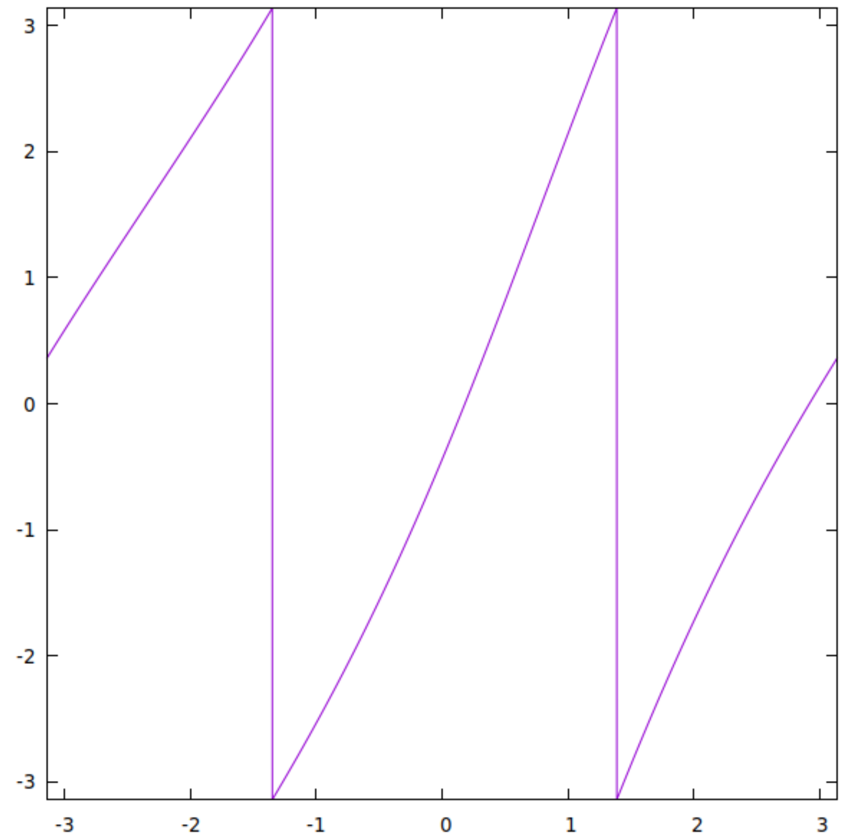
on the torus, consider the Blaschke product

$$\tau(z) = \left(\frac{z-\mu}{1-\bar{\mu}z}\right)^2, \quad |\mu| < 1/3$$

$$w(z) = 1/\tau'(z)$$

The associated transfert operator acts on $H_\infty(A_\rho)$
for a suitable annulus A_ρ

ρ ?



$\mu = \frac{1+i}{10}$

Other examples in the article

Lyapunov exponent and transfer operators

- $\mathcal{A} = \{A_1, \dots, A_k\}$ 2×2 positive invertible matrices
- $p = \{p_1, \dots, p_k\}$ a probability vector $\rightsquigarrow$ $\mathbb{P}_p =$ associated Bernoulli measure on $\{1, \dots, K\}^{\mathbb{N}}$

$$\Lambda = \Lambda(\mathcal{A}, p) = \lim_{n \to \infty} \frac{1}{n} \int \log \| A_{\omega_1} \cdots A_{\omega_n} \| \, d\mathbb{P}_p(\omega)$$

- for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{A}$ let $w_A(z) = (a+c-b-d)z + b + d$ and $\phi_A(z) = \dfrac{(a-b)z+b}{w_A(z)}$

$$\mathbf{G}[f] = \sum_{i=1}^{k} p_i \; f \circ \phi_{A_i}$$

$\rightsquigarrow$ $\begin{cases} 1 \text{ eigenvalue/eigenvector.} \\ h^\star \text{ eigenfunctional} \end{cases}$

$$\boxed{\phi_A(D(\tfrac{1}{2}, \tfrac{1}{2})) \subsetneq D(\tfrac{1}{2}, \rho) \qquad \rho < 1/2}$$

$$\mathbf{M}[f] = \sum_{i=1}^{k} p_i \log(w_{A_i}) \; f \circ \phi_{A_i}$$

$$\boxed{\Lambda = h^\star(\mathbf{M}[1])}$$

Finite section method

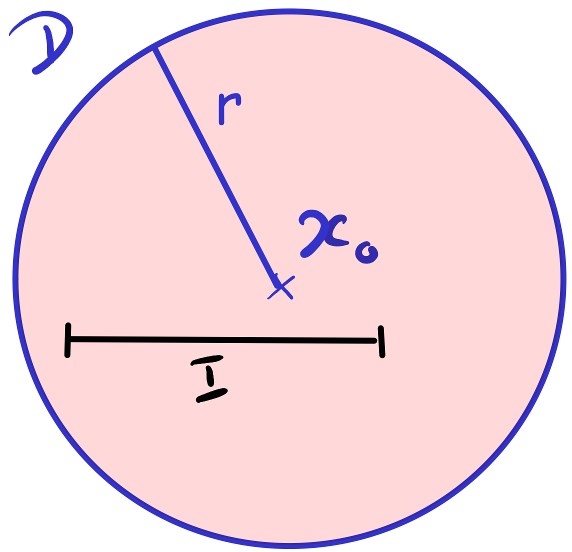
$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{G} \arrow[r, dashed, "\text{Hard/non explicit}"] \arrow[d, "\substack{\pi_n = \text{finite rank}\\ \text{projection}}"'] & \lambda,\ f_\lambda,\ f^\star_\lambda \\
\substack{\pi_n \circ \mathbf{G} \circ \pi_n \\ \pi_n \circ \mathbf{G}} \arrow[r, "\text{matricial computations}"'] & \lambda_n,\ f_{\lambda_n},\ f^\star_{\lambda_n} \arrow[u, "\text{convergence ?}"']
\end{tikzcd}$$

contraction property $\Rightarrow$ $||\mathbf{G} - \pi_n \circ \mathbf{G}|| = O(\theta^n), \quad \theta < 1$

$$\Rightarrow \begin{cases} ||f_\lambda - f_{\lambda_n}|| = O(\theta^n) \\ |\lambda - \lambda_n| = O(\theta^n) \end{cases}$$

Finite rank projection

Daudé-Flajolet-Vallée



$I \subset D(x_0, r)$

- $D = D(x_0, r), \qquad f \in H_\infty(D)$

$$f(z) = \sum_{n \geq 0} f_n (z - x_0)^n$$

- $\pi_n =$ truncated taylor expansion

$$\pi_n[f](z) = \sum_{k=0}^{n} f_n (z - x_0)^n$$

- The matrix is computable

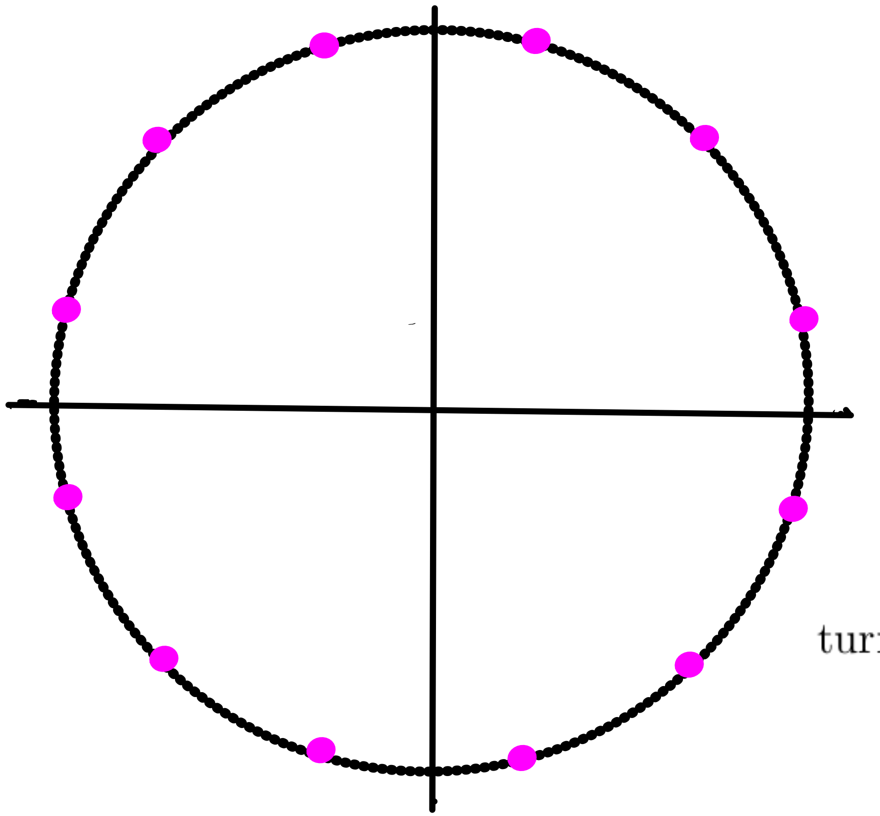
$$[(z-x_0)^j]\mathbf{G}_{s,\mathcal{A}}[(z-x_0)^i] = (-1)^i \sum_{u=0}^{j} \binom{j}{u} (-x_0)^{j-u} \binom{s+u+i-1}{i} \zeta_{\mathcal{A}}(s+i+u, x_0)$$

$$\zeta_A(s, x) = \sum_{m \in A} \frac{1}{(m+x)^s}$$

Finite rank projection

Bandtlow-Slipantschuk

Torus (unit circle)



π_n = Lagrange interpolation polynomial on 2n equidistant points (roots of the unity)

$$z_k = \exp\left(\frac{2k+1}{2n} i\pi\right), \qquad k = 0, \ldots, 2n-1$$

$$\pi_n[f](z) = \frac{1}{2n} \sum_{l=-n}^{n-1} c_{l,2n}(f) z^l$$

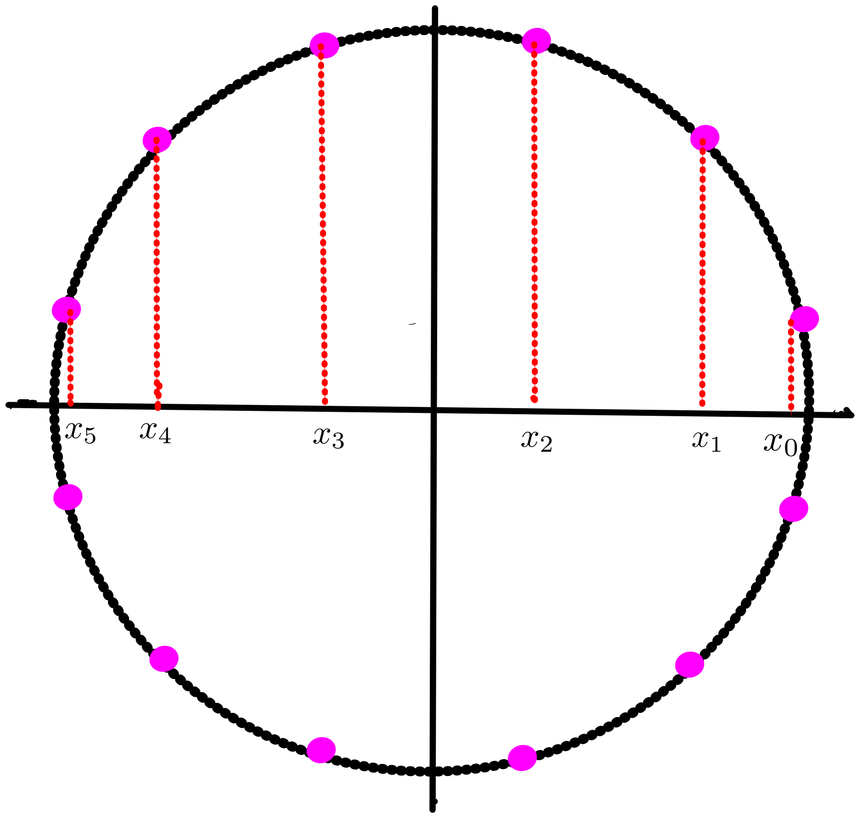
$$c_{l,2n}(f) = \sum_{k=0}^{2n-1} f(z_k) z_k^{-l} \qquad (f \in C(\mathbb{T}))$$

turn out to be the discrete Fourier transform of the sequence $f(z_0), f(z_1), \ldots, f(z_{2n-1})$

Finite rank projection

Bandtlow-Slipantschuk

Interval [-1,1]



π_n = Lagrange-Chebyshev interpolation polynomial on the Chebyshev nodes of order n

$$x_k = \cos(\frac{2k+1}{2n}\pi), \qquad k = 0, \ldots, n-1$$

$$\pi_n[f](x) = \frac{1}{2n} \sum_{k=0}^{n-1} f(x_k) \frac{T_n(x)}{T_n'(x)(x - x_k)}$$

$T_n =$ Chebyshev polynomial of degree n

$$T_n(\cos(\theta)) = \cos(n\theta)$$

Finite rank projection : convergence

Major drawback : $\boxed{\pi_n \not\to Id}$

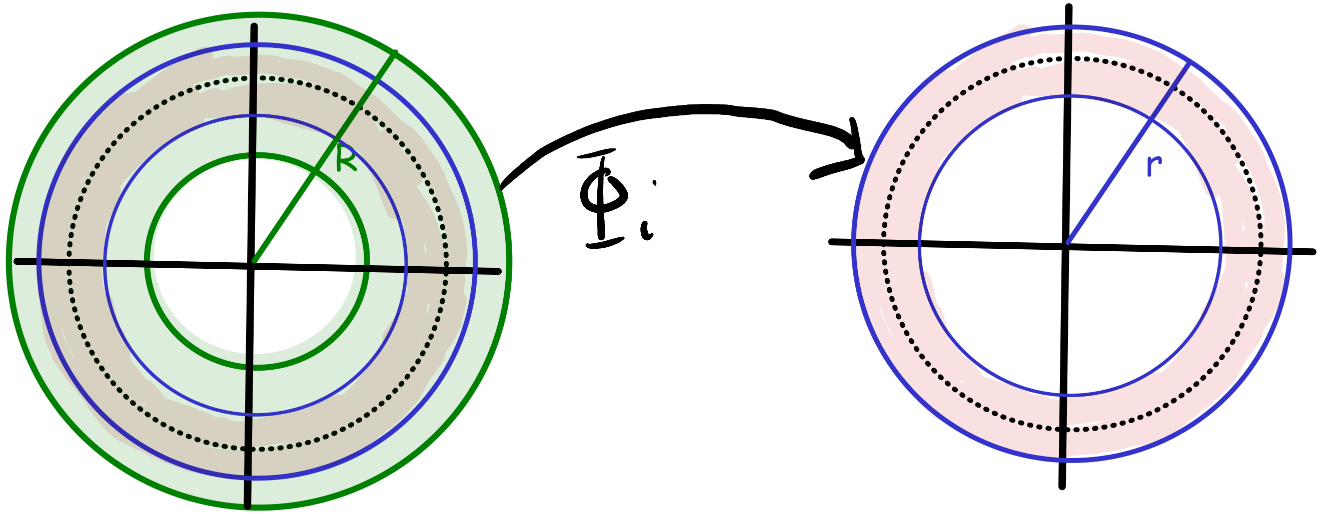
Conclusion : witout additional hypotheses, no efficient convergence speed can be obtained

Contraction property : $\mathbf{G}[f](z) = \sum_{i \in \mathcal{I}} W_i(z) f\left(\Phi_i(z)\right)$

Consider R>r>0

- D_R = large domain
- D_r = small domain
- $W_i \in H_\infty(D_R)$
- $\sum_{i \in \mathcal{I}} \|W_i\|_{D_R} < \infty$

$$\boxed{\Phi_i : D_R \to D_r \in H_\infty(D_R)}$$



Finite rank projection : convergence

Contraction property : $\mathbf{G}[f](z) = \sum_{i\in\mathcal{I}} W_i(z) f(\Phi_i(z))$

Consider R>r>0

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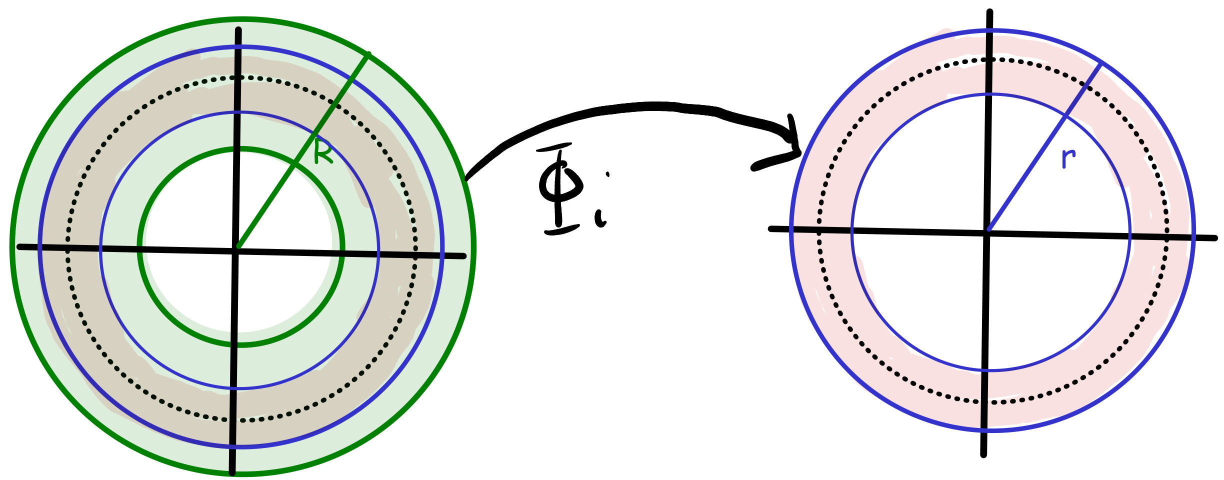


Torus

$\Phi_i : A_R \to A_r$

Interval

$\Phi_i : E_{\sigma,R} \to E_{\sigma,r}$

DFV

$\Phi_i : D(x_0,R) \to D(x_0,r)$

Convergence of operators

D_R = large domain

D_r = small domain

$W_i \in H_\infty(D_R)$

$$\sum_{i \in \mathcal{I}} ||W_i||_{D_R} < \infty$$

$$\mathbf{G}[f](z) = \sum_{i \in \mathcal{I}} W_i(z) f\left(\Phi_i(z)\right)$$

Norm convergence of operators

$$||\mathbf{G} - \pi_n \circ \mathbf{G}||_{D_r} = O\left(\left(\frac{r}{R}\right)^n\right)$$

$$||\mathbf{G} - \mathbf{G} \circ \pi_n||_{D_R} = O\left(\left(\frac{r}{R}\right)^n\right)$$

Constants are explicit !

Convergence of spectral objects

D_R = large domain $\quad D_r$ = small domain $\quad \mathbf{G}[f](z) = \sum_{i \in \mathcal{I}} W_i(z) f\left(\Phi_i(z)\right)$

Classical result: norm convergence of operators entails convergence of spectral objects

Main result

- If $\mu_n \in \operatorname{spec}(\pi_n \circ \mathbf{G})$ converges to μ then $\mu \in \operatorname{spec}(\mathbf{G})$.
- If $\mu \in \operatorname{spec}(\mathbf{G})$ then there exists a sequence $\mu_n \in \operatorname{spec}(\pi_n \circ \mathbf{G})$ such that

$$|\mu - \mu_n| = O\left(\left(\frac{r}{R}\right)^n\right) \quad (\mu \text{ simple})$$

- Consider $\mu \in \operatorname{spec}(\mathbf{G})$, $\mu_n \in \operatorname{spec}(\pi_n \circ \mathbf{G})$, $\mu_n \to \mu$, h_n the normalised eigenvector associated to μ_n, $\mathcal{P}$ the spectral projection associated with μ

$$||\mathcal{P}[h_n] - h_n||_{D_r} = O\left(\left(\frac{r}{R}\right)^n\right)$$

- Remains true with $\mathbf{G}^\star, \pi_n^\star, h_n^\star, \mathcal{P}^\star$ and D_R

Constants are not explicit !

About the constants

- They involve the norm of the resolvant operator $||(\mathbf{I} - z\mathbf{G})^{-1}||_{D_R}$
- If $\mathbf{G}$ is normal $(\mathbf{G}\mathbf{G}^\star = \mathbf{G}^\star\mathbf{G})$ then

$$||(\mathbf{I} - z\mathbf{G})^{-1}||_{D_R} = \frac{1}{\mathrm{dist}(z, \mathrm{spec}(\mathbf{G}))}$$

- For the Gauss map, the transfer operator is known to be normal on (other) Hardy spaces

$\Rightarrow$ polynomial time algorithm

A first comparison

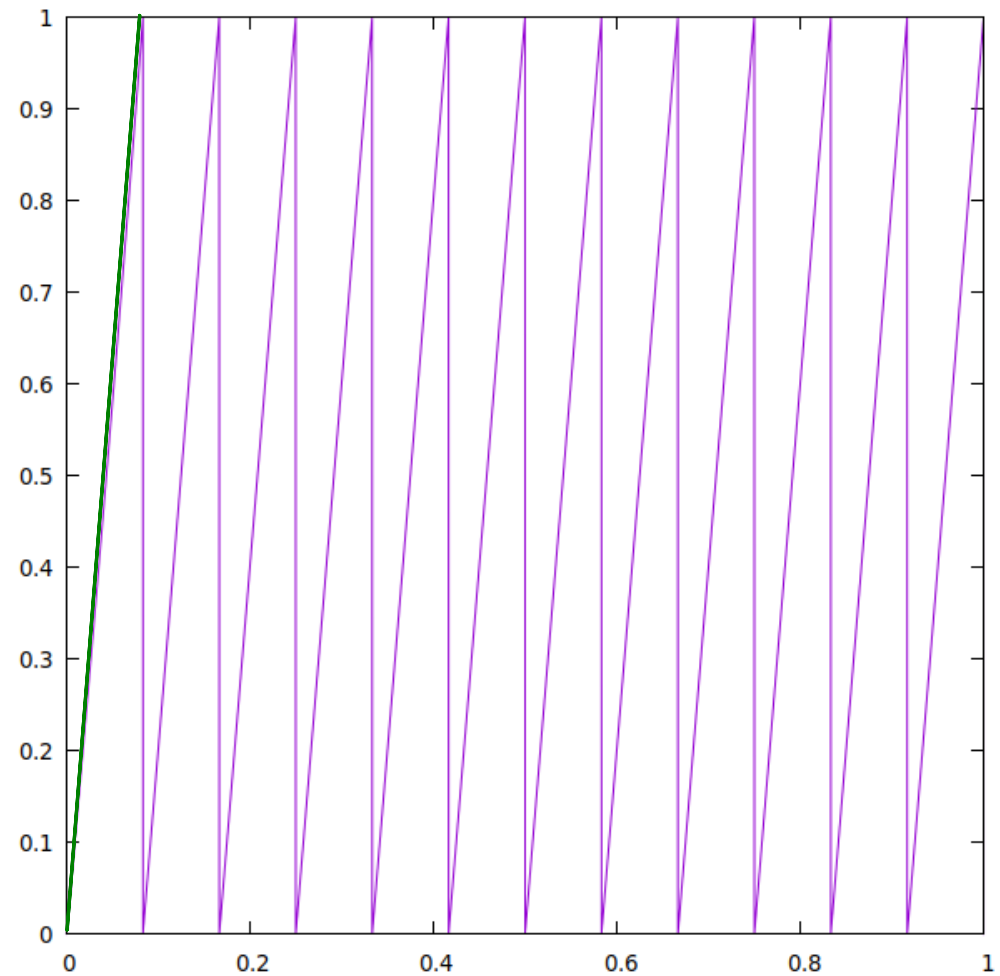
$T\colon [0,1] \to [0,1]$ given by

$$T(x) = \begin{cases} \frac{11x}{1-x} & 0 \le x < \frac{1}{12}, \\ 12x - i & \frac{i}{12} < x \le \frac{i+1}{12}, \end{cases}$$

$$W_i(x) = \Phi_i'(x)$$

$\{\Phi_i\}_{i=0}^{11}$ with $\Phi_0(x) = \frac{x}{11+x}$

$$\Phi_i(x) = \frac{x+i}{12} \text{ for } i = 1, \ldots, 11,$$



Aim: compute the second eigenvalue (decay of correlation)

A first comparison

2	0.0899609091606775605271181343464894043253988825186233560706
12	0.0900761270052955777611934889786172935065120132357444867624
22	0.0900761270052955778472464929999485481037667345626173002032
32	0.0900761270052955778472464929999485626943805990355232965894
42	0.0900761270052955778472464929999485626943805990355246068579

32	0.0900761270052955778472464929999485626943798480989565925680
42	0.0900761270052955778472464929999485626943805990355246068006