Multidimensional Continued Fractions and Euclidean Dynamics Leiden March 2020

Minimal vectors in lattices over Gauss integers in $\mathbb{C}^2$

Nicolas Chevallier

- The space Ω of unimodular lattices in $\mathbb{C}^2$.
- Minimal vectors in a lattice.
- The diagonal flow $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.
- A subset T in Ω transverse to the flow.
- The first return map on T associated with the flow.

Ordinary continued fraction-minimal vector

- Let $M \in \mathrm{SL}(2,\mathbb{R})$ and let $\Lambda = M\mathbb{Z}^2$. A nonzero vector $u = (u_1, u_2) \in \Lambda$ is *minimal* if the only nonzero vectors $x = (x_1, x_2) \in \Lambda$ that are in the cylinder
$$C(u) = \{(x_1, x_2) : |x_1| \leq |u_1|,\ |x_2| \leq |u_2|\}.$$
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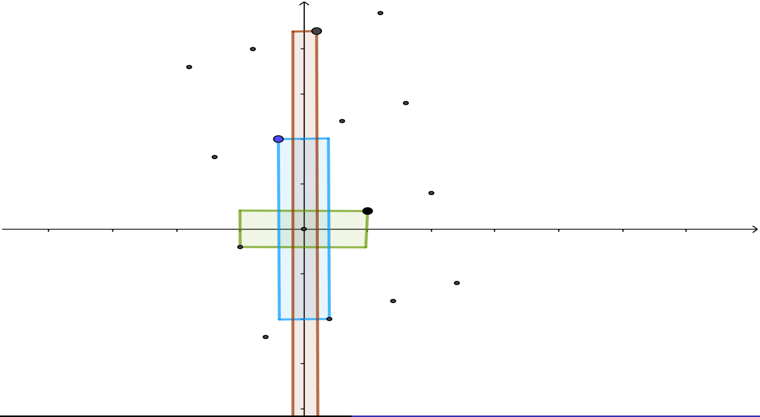

Ordinary continued fraction-minimal vector

- If $x \in \mathbb{R}$, $M = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ and $(p,q) \in \mathbb{Z}^2$, $q > 0$, then the vector $X = M \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p - qx \\ q \end{pmatrix}$ is a minimal in $\Lambda_x = M\mathbb{Z}^2$ iff (p,q) is a best approximation vector of x.

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- The sequence of minimal vectors $X_n = \begin{pmatrix} p_n - q_n x \\ q_n \end{pmatrix}$ is ordered according to the height q_n. The q_n are the denominators of the convergents of x.

Consecutive minimal vectors

- Two minimal vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in lattice Λ are *consecutive* if $|u_2| < |v_2|$ and if the only lattice points in the interior of the cylinder

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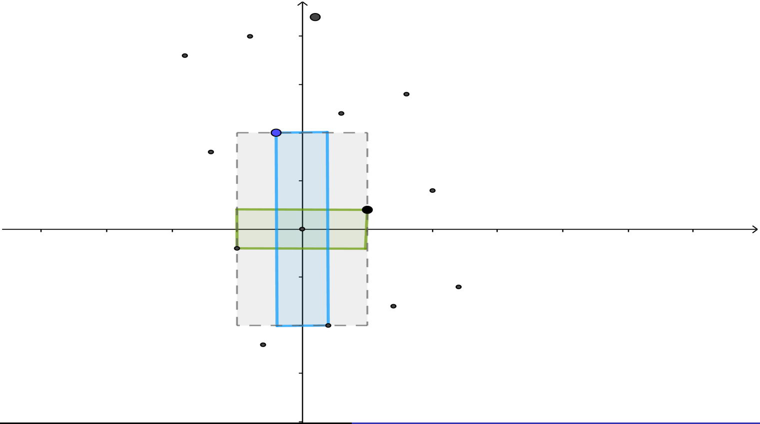

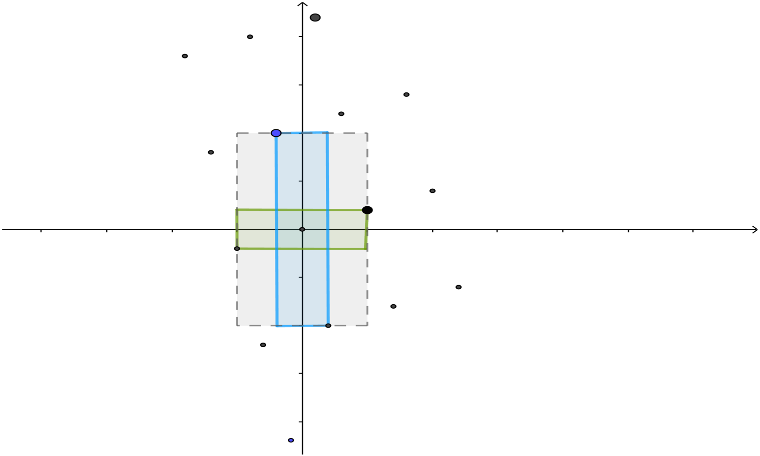

Lemma

If u and v are consective minimal vectors Λ *then* $\Lambda = \mathbb{Z}u + \mathbb{Z}v$.

Let T be the set of unimodular lattices Λ in $\mathbb{R}^2$ such that there exist two vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in Λ such that

Transversal in $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$

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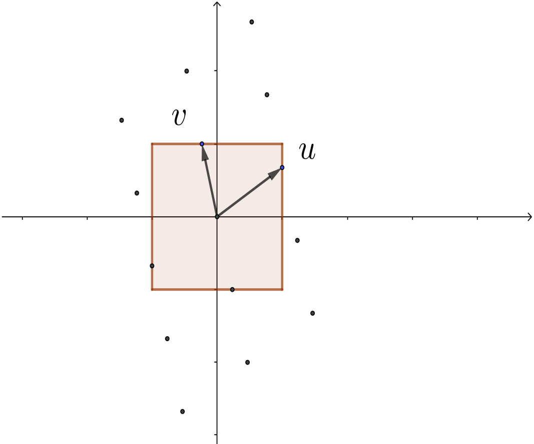
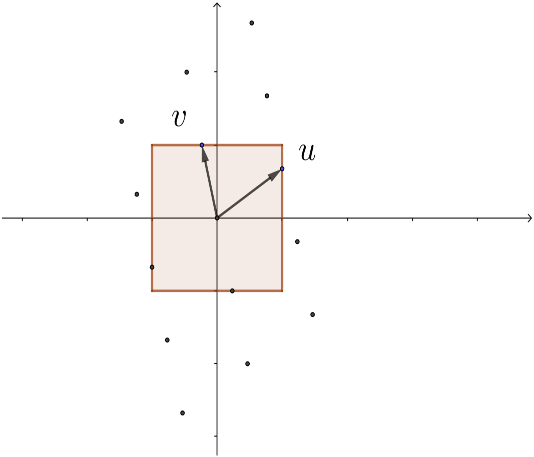
- $|u_2|,|v_1|<|u_1|=|v_2|=r$,
- The only nonzero vector of Λ in the ball $B_\infty(0,r)$ are $\pm u$ and $\pm v$.



The transversal T



Clearly u and v are two consecutive minimal vectors of Λ and $\lambda_1(\Lambda, \|.\|_\infty) = \lambda_2(\Lambda, \|.\|_\infty)$.

Let $u = (u_1, u_2)$ and
$v = (v_1, v_2)$ be in $\mathbb{R}^2$.
We want to know whether
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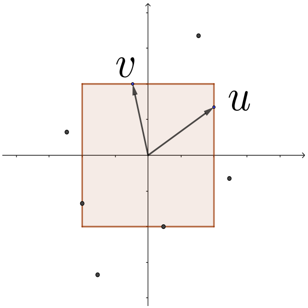

The conditions on u and v are

- $r = |u_1| = |v_2| > |u_2|, |v_1|$
- $u \pm v \notin C(u, v)$
- $u_1 v_2 - u_2 v_1 = \pm 1$

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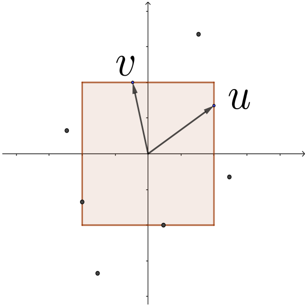


We obtain

$$\begin{cases} u = r(\varepsilon, y), \\ v = r(-\varepsilon x, 1) \end{cases}$$

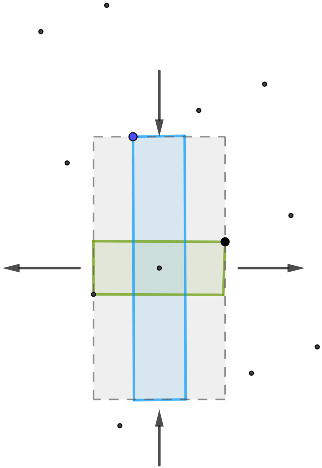
where $\varepsilon = \pm 1$, $x, y \in]0, 1[$ are free and $r = \frac{1}{1+xy}$

If $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two consecutive minimal of a lattice Λ, then

$$g_t\Lambda = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \Lambda \in T$$

where

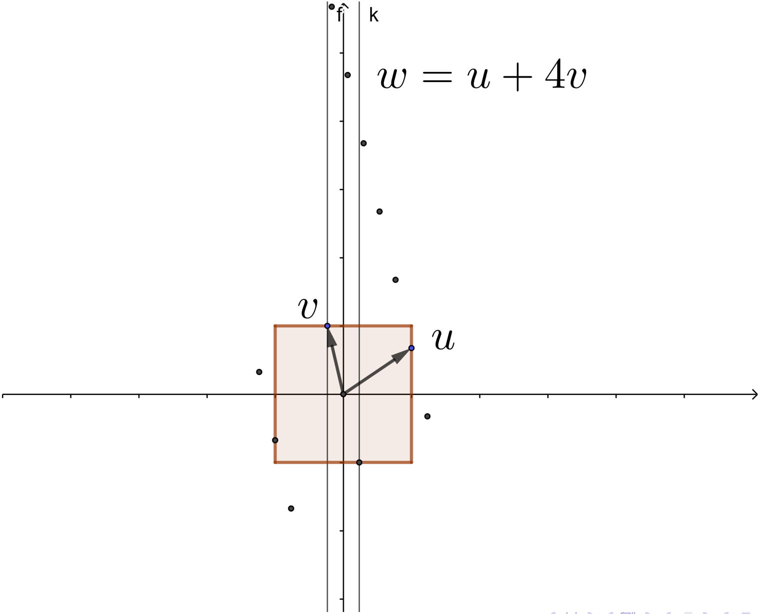
$$t = \frac{1}{2} \ln \frac{|v_2|}{|u_1|}.$$



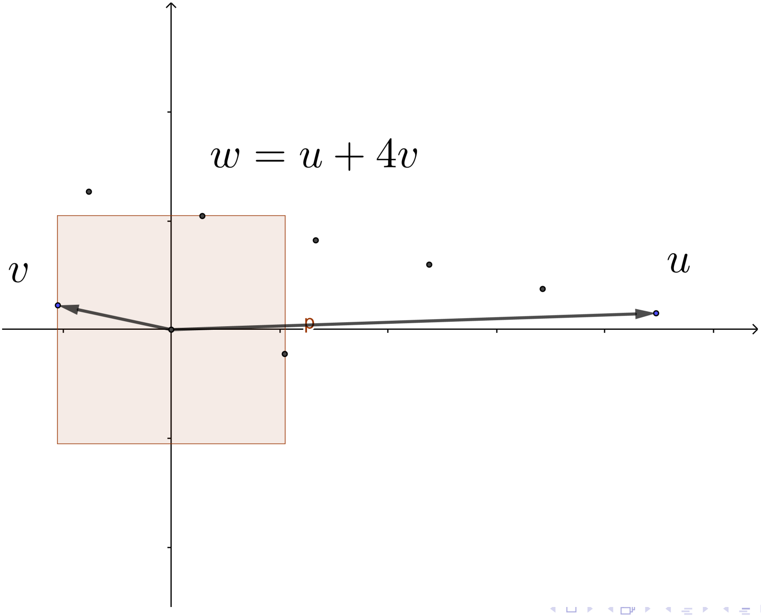
Actually, this holds when $\pm u$ and $\pm v$ are the only nonzero lattice vectors in the cylinder $C(u, v)$.

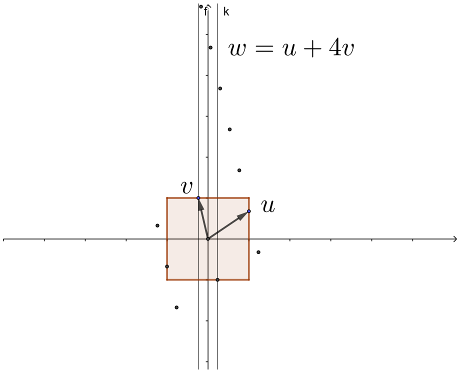
This is always true outside a set of zero measure in $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$:
The set of lattices with no nonzero point on the axes

$\Lambda \in T$, first return map



With $t = \frac{1}{2} \ln \frac{|w_2|}{|v_1|}$, again $g_t \Lambda \in T$





$$u = r(\varepsilon, y),\ v = r(-\varepsilon x, 1)$$
$$w = u + \lfloor 1/x \rfloor v = r(\varepsilon(1 - \lfloor 1/x \rfloor x), y + \lfloor 1/x \rfloor)$$
$$g_t v = r'(-\varepsilon, \frac{1}{y + \lfloor 1/x \rfloor}),\ g_t w = r'(\varepsilon\{1/x\}, 1)$$

Cylinders in $\mathbb{C}^2$

For $u = (u_1, u_2),\ v = (v_1, v_2) \in \mathbb{C}^2$

$$C(u) = \{(x_1, x_2) \in \mathbb{C}^2 : |x_1| \leq |u_1| \text{ and } |x_1| \leq |u_2|\}$$
$$C(u, v) = \{(x_1, x_2) \in \mathbb{C}^2 : |x_1| \leq |u_1| \text{ and } |x_1| \leq |v_2|\}$$

Definition

A Gauss lattice in a finite dimensional $\mathbb{C}$-vector space E is a subset that

- is submodule over the Gauss integers,
- is a discrete subset of E,
- generates the vector space E.

If z is a complex number

$$\Lambda_z = \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} \mathbb{Z}[i]^2$$

is a lattice in $\mathbb{C}^2$ with determinant 1.

The set of units in $\mathbb{Z}[i]$ is $\mathbb{U}_4 = \{\pm 1, \pm i\}$

$$\Omega_1 = \{\Lambda \text{ is a Gauss lattice in } \mathbb{C}^2 \text{ s.t. } \det_{\mathbb{C}}(\Lambda) \in \mathbb{U}_4\}$$

is the set of unimodular lattices in $\mathbb{C}^2$.

If $\Lambda = M\mathbb{Z}[i]^2 \in \Omega_1$ then the matrix M can be chosen in order that $\det_{\mathbb{C}}(M) = 1$, therefore

$$\Omega_1 \approx \mathrm{SL}(2, \mathbb{C}) / \mathrm{SL}(2, \mathbb{Z}[i])$$

Definition

Let Λ be a Gauss lattice in $\mathbb{C}^2$.

- A non zero vector $u = (u_1, u_2) \in \Lambda$ is a *minimal* vector in Λ if for every non zero $v \in \Lambda$, $v \in C(u) \Rightarrow |v_1| = |u_1|$ and $|v_2| = |u_2|$.
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Λ lattice in $\mathbb{C}^2$. Sequence of "all" minimal vectors,

$$(X_n(\Lambda))_{n\in D} = (z_{1n}, z_{2n})_{n\in D}$$

D interval $\subset \mathbb{Z}$.
$X_n(\Lambda)$ and $X_{n+1}(\Lambda)$ are consecutive and $(|z_{2n}|)_{n\in D}$ is increasing.

$$r_n(\Lambda) = |z_{1n}| \downarrow \text{ and } q_n(\Lambda) = |z_{2n}| \uparrow$$

Lemma

Let Λ be a lattice in $\mathbb{C}^2$ and let $(X_n(\Lambda))_{n \in D}$ be the sequence of minimal vectors of Λ.

- $\frac{1}{2}|\det_{\mathbb{C}}(\Lambda)| \leq q_{n+1}(\Lambda) r_n(\Lambda) \leq \frac{4}{\pi}|\det_{\mathbb{C}}(\Lambda)|$.
- $q_{n+14}(\Lambda) \geq C q_n(\Lambda)$ *where* $C = \frac{1}{2}(1 + \cos(\frac{2\pi}{7})) > 1.1234$
- $r_{n+56}(\Lambda) \leq \frac{1}{2} r_n(\Lambda)$.

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$$(\pi q_{n+1}(\Lambda)r_n(\Lambda))^2 \leq 16|\det_{\mathbb{R}}(\Lambda)| = 16|\det_{\mathbb{C}}(\Lambda)|^2.$$

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Next,

$$\begin{aligned}2q_{n+1}(\Lambda)r_n(\Lambda) &\geq q_{n+1}(\Lambda)r_n(\Lambda) + q_n(\Lambda)r_{n+1}(\Lambda)\\ &\geq |\det_{\mathbb{C}}(X_n(\Lambda), X_{n+1}(\Lambda))| \geq |\det_{\mathbb{C}}(\Lambda)|.\end{aligned}$$

Let $I = (1+i)\mathbb{Z}[i]$ and let $J = \frac{1}{1+i}(\mathbb{Z}[i] \setminus I)$.

Proposition

Let Λ be a Gauss lattice in $\mathbb{C}^2$. Suppose that $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two consecutive minimal vectors in Λ. Call L the lattice spanned by u and v. Then

1. *L has index 1 or 2: $[\Lambda : L] = \frac{|\det_{\mathbb{R}}(L)|}{|\det_{\mathbb{R}}(\Lambda)|} = 1$ or 2.*
2. *If L has index 2 then*

$$\Lambda = \{au + bv : (a, b) \in \mathbb{Z}[i]^2 \cup J^2\}$$

and $U = u, V = \frac{1}{1+i}(u+v)$ is a basis of Λ.

Proof.

Since u and v are consecutive minimal vectors,

- $|u_2| \leq |v_2|$ and $|v_1| \leq |u_1|$, hence $|\det_{\mathbb{C}}(\Lambda)| \leq |\det_{\mathbb{C}}(L)| \leq 2|u_1||v_2|$,
- $|\det_{\mathbb{R}}(L)| \leq 4|u_1|^2|v_2|^2 = 4\frac{\mathrm{Vol}(C(u,v))}{\pi^2} \leq 4\frac{16|\det_{\mathbb{R}}(\Lambda)|}{\pi^2}$.

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Therefore

$$\frac{|\det_{\mathbb{R}}(L)|}{|\det_{\mathbb{R}}(\Lambda)|} \leq \frac{64}{\pi^2} = 6,48\dots$$

This index is the square of the modulus of a Gauss integer, it is the sum of two squares.
Hence $[\Lambda : L] = 1, 2, 4$ or 5

Proof.

There exists a basis U, V of Λ and a, b and $c \in \mathbb{Z}[i]$ such that

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Changing U in $a^{-1}U$, we can suppose $a = 1$.

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Suppose that $c = 2$. There exists $g \in \mathbb{Z}[i]$ such that $|g - \frac{b}{c}| \leq \frac{1}{\sqrt{2}}$.
Since $|cg - b| \leq \sqrt{2}$, $|cg - b| = 0, 1$ or $\sqrt{2}$.

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$$w = V + gU = -\frac{b}{c}U + \frac{1}{c}v + gu = \frac{cg-b}{c}u + \frac{1}{c}v \in \Lambda$$

If $cg - b = 0$ then $w = \frac{1}{c}v \in \Lambda$, impossible for v is primitive.

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If $|cg - b| = 1$, by convexity $w \in$ the interior of $C(u, v)$ impossible.
If $|cg - b| = \sqrt{2}$ then $z = (\frac{cg-b}{c})^{-1} \in \mathbb{Z}[i]$ and the vector $w' = zw - u = \frac{z}{c}v$ is in Λ. Impossible because $|\frac{z}{c}| < 1$ and v is primitive. ......

QED

Reduced basis

Definition

Let E be a two-dimensional $\mathbb{C}$-vector space equipped with a $\mathbb{C}$-norm $\|.\|$. A basis (u, v) of a Gauss lattice $\Lambda = \mathbb{Z}[i]u + \mathbb{Z}[i]v$ is reduced with respect to the norm $\|.\|$ if $\|u\| = \lambda_1(\Lambda, \|.\|, \mathbb{Z}[i])$ and $\|v\| = \lambda_2(\Lambda, \|.\|, \mathbb{Z}[i])$.

E is a two dimensional $\mathbb{C}$-vector space equipped with an Hermitian norm $|.|_E$.

Gauss algorithm

E is a two dimensional $\mathbb{C}$-vector space equipped with an Hermitian norm $|.|_E$.

Input: A basis (u, v) of a Gauss lattice Λ in E.

1. If $|v|_E < |u|_E$, exchange $u \leftrightarrow v$.
2. $A :=$ False
3. Main loop: while $A =$ False
   1. Compute $w = (a + ib)u$ the orthogonal projection of v on the line $\mathbb{C}u$.
   2. Find the Gauss integer p closest to $a + ib$ and replace v by $v - pu$.
   3. If $|u|_E \leq |v|_E$, $A :=$ True, else exchange $u \leftrightarrow v$.

Output: (u, v) a reduced basis of Λ.

Proposition

The above algorithm find a reduced basis of $\Lambda = \mathbb{Z}[i]u + \mathbb{Z}[i]v$ *for the norm* $|.|_E$ *in finitely many steps.*

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Lyu, C. Porter, C. Ling, Lattice Reduction over Imaginary Quadratic Fields with an Application to Compute-and-Forward, *arXiv:* May 2019.

For $t > 0$ denote $|.|_t$ the Hermitian norm on $\mathbb{C}^2$ defined by

$$|(z_1, z_2)|_t^2 = |tz_1|^2 + |\tfrac{1}{t} z_2|^2.$$

Gauss algorithm and minimal vectors

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Proposition

Let v be a minimal vectors in a Gauss lattice $\Lambda \subset \mathbb{C}^2$. Set

$$s = \sqrt{\frac{4}{\pi}|\det_{\mathbb{C}}(\Lambda)|} \text{ and } t = \frac{s}{|v_1|}$$

Let (w, w') be a reduced basis of Λ with respect to the norm $|.|_t$. Then the next minimal vector after v is one of the vectors $zw + z'w'$ with $z, z' \in \mathbb{Z}[i]$ and $(|z|^2 + |z'|^2) < 23$.

Gauss algorithm and minimal vectors

Proposition

Let u and v be two consecutive minimal vector in a Gauss lattice $\Lambda \subset \mathbb{C}^2$ and let $L = \mathbb{Z}[i]u + \mathbb{Z}[i]v$. Set

$$s = \sqrt{\frac{4}{\pi}|\det_{\mathbb{C}}(\Lambda)|} \text{ and } t = \frac{s}{|v_1|},$$

$$\begin{cases} U = u \\ V = v \end{cases} \text{ or } \begin{cases} U = u \\ V = \frac{1}{1+i}(u+v) \end{cases} \quad \text{according } [L : \Lambda] = 1 \text{ or } 2$$

There is an absolute constant C such that Gauss algorithm associated with the Hermitian norm $|.|_t$, with input the basis U, V needs at most C steps.

Transversal

Let $\mathbb{U}_4$ be the group of units in $\mathbb{Z}[i]$. Let T be the set of Gauss unimodular lattices Λ in $\mathbb{C}^2$ such that $\det_{\mathbb{C}} \Lambda \in \mathbb{U}$ and such that there exists two vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in Λ

1. $|u_2|, |v_1| < |u_1| = |v_2| = r$,
2. the only nonzero vectors of Λ in the ball $B_\infty(0, r)$ are in $\mathbb{U}_4 u \cup \mathbb{U}_4 v$.

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$$\dim_{\mathbb{R}} T = 5$$

Parametrization of T_1

Let $\Psi_k : \mathbb{R} \times \mathbb{D}^2 \to \Omega_1,\ k = 1, 2$ be the maps defined by

$$\Psi_1(\theta, w_1, w_2) = \mathbb{Z}[i]u + \mathbb{Z}[i]v$$

where

$$\begin{aligned}
u &= u(\theta, w_1, w_2) = r(e^{i\theta}, e^{i\theta'} w_2),\\
v &= v(\theta, w_1, w_2) = r(e^{i\theta} w_1, e^{i\theta'}),\\
r &= \frac{1}{\sqrt{|1 - w_1 w_2|}},\\
\theta' &= -\theta - \arg(1 - w_1 w_2).
\end{aligned}$$

Then for all Λ in T_1 there exists exactly one element $(\theta, w_1, w_2) \in [0, \frac{\pi}{2}[\times \mathbb{D}^2$ such that $u(\theta, w_1, w_2)$, $v(\theta, w_1, w_2)$ are the minimal vectors associated with Λ.

$$
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v &= v(\theta, w_1, w_2) = r(e^{i\theta} w_1, e^{i\theta'}).
\end{aligned}
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$$\begin{aligned}|au - bv|_\infty &= r \max(|ae^{i\theta} - be^{i\theta} w_1|, |ae^{i\theta'} w_2 - be^{i\theta'}|) \\ &= r \max(|b||w_1 - \tfrac{a}{b}|, |a||w_2 - \tfrac{b}{a}|).\end{aligned}$$

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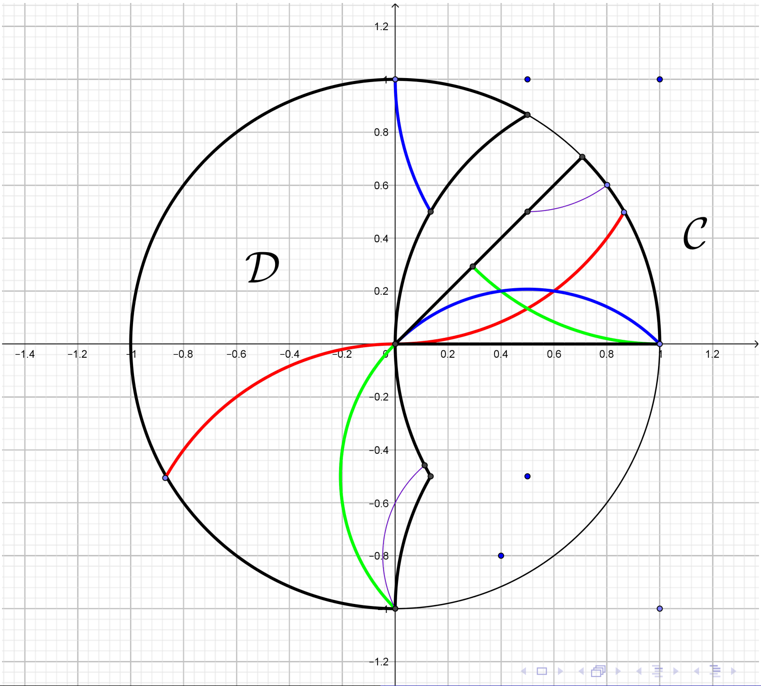
Hence $|au - bv|_\infty > r$ means that either

$$w_1 \notin D(\frac{a}{b}, \frac{1}{|b|})$$

or

$$w_2 \notin D(\frac{b}{a}, \frac{1}{|a|}).$$



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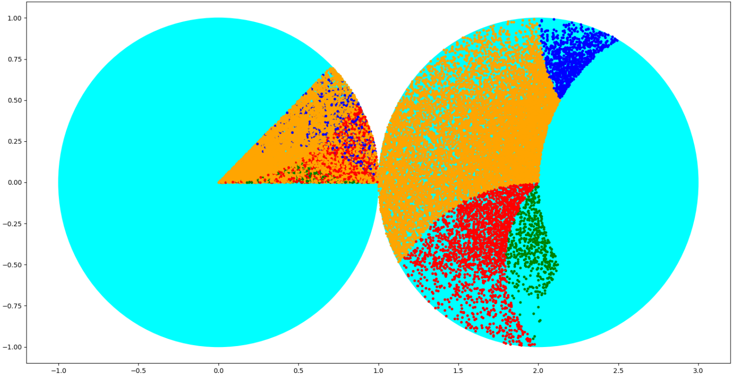
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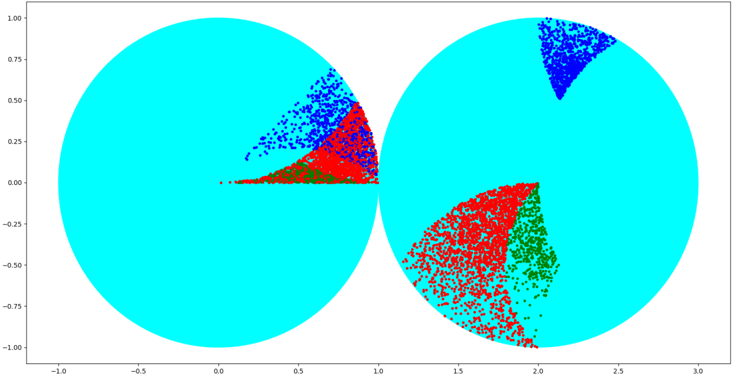
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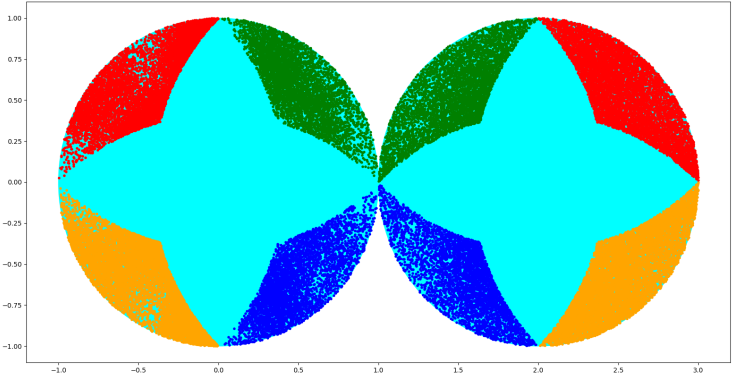
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Proposition

Choosing an appropriate normalization of the Haar, the flow g_t *induces on the transversal a measure* ν *which has the density*

$$f(\theta, w_1, w_2) = \frac{32}{|1 - w_1 w_2|^4}$$

with respect of the Lebesgue measure of $[0, \pi/2] \times \mathbb{D}^2$*, using the parametrization* Ψ_k*,* $k = 1, 2$*.*