Simplicity of Lyapunov exponents for integral cocycles over countable shifts

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$\Omega = \bigcup_{n\geq 0} \Lambda^n =$ set of finite words

$\Sigma(\underline{\ell}) = \{x \in \Sigma \text{ starting by } \underline{\ell}\} =$ cylinder associated to $\underline{\ell} \in \Omega$.

Bounded distortion

Definition

$\mu = f$-inv. prob. measure has *bounded distortion* if $\exists\, C \geq 1$ s.t.

$$C^{-1}\mu(\Sigma(\underline{\ell}_1))\cdot\mu(\Sigma(\underline{\ell}_2)) \leq \mu(\Sigma(\underline{\ell}_1\underline{\ell}_2)) \leq C\mu(\Sigma(\underline{\ell}_1))\cdot\mu(\Sigma(\underline{\ell}_2))$$

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Bounded distortion $\implies$ ergodicity, mixing, ...

Example (Bernoulli shifts)

$\Lambda = \{1, \dots, k\}$, $\nu(\{i\}) = p_i \in (0,1)$, $\sum\limits_{i=1}^{k} p_i = 1 \implies \mu = \nu^{\mathbb{N}}$ has bounded distortion (with $C = 1$).

"Bounded distortion = almost Bernoulli"

Locally constant integrable cocycles

Let G be a matrix group acting on $\mathbb{K}^d$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbf{H}$.

Example (some classical groups)

$G = GL(d, \mathbb{R})$, $SL(d, \mathbb{R})$, $Sp(d, \mathbb{R})$, $O(p, q)$, $U_{\mathbb{C}}(p, q)$, $U_{\mathbf{H}}(p, q)$, $p + q = d$, $O(d, \mathbf{H})$, etc.

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Definition (*Locally constant integrable cocycle*)

It is $A:\Sigma\to G$, $A((x_i)_{i\in\mathbb{N}}) = A_{x_0}\in G$ s.t. $\int_\Sigma \log\|A^{\pm1}(\underline{x})\|\,d\mu(\underline{x})$ $= \sum\limits_{\ell\in\Lambda}\mu(\Sigma(\ell))\log\|A_\ell^{\pm1}\| < \infty$.

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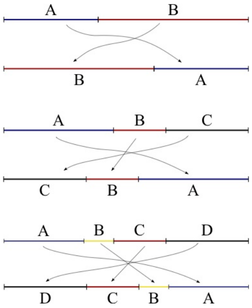
By Oseledets theorem, we have Lyapunov exponents $\theta_1 \geq \cdots \geq \theta_d$ describing the fiber dynamics of

$$(f, A) : \Sigma \times \mathbb{K}^d \to \Sigma \times \mathbb{K}^d, \quad (f, A)(\underline{x}, v) = (f(\underline{x}), A(\underline{x})v)$$

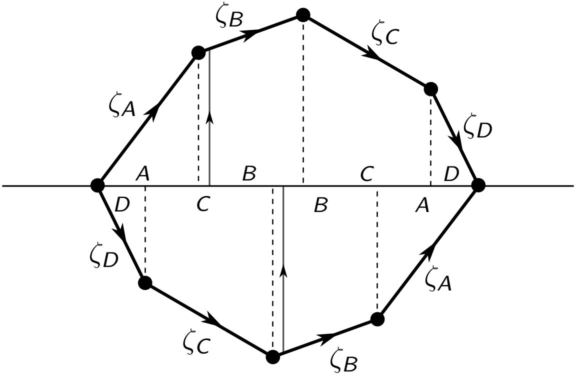
Rauzy–Veech algorithm (I): interval exchange maps

An *interval exchange map* of $d \geq 2$ subintervals of a bounded open interval I is an injective map $T : D_T \to D_{T^{-1}}$ where $D_T, D_{T^{-1}} \subset I$ with $\#(I - D_T) = \#(I - D_{T^{-1}}) = d - 1$ and $T|_{\text{c.c. of } D_T}$ is a translation onto a c.c. of $D_{T^{-1}}$.



Rauzy–Veech algorithm (II): translation surfaces

The first return map of a translation flow on a *translation surface* is an interval exchange map.



Rauzy–Veech algorithm (III): renormalization

An *elementary step* of the Rauzy–Veech algorithm takes an i.e.m. T to the i.e.m. $\widehat{T}$ = first return of T-orbits to $\widehat{I} = (a, b)$ where $a = \min\{l - D_T, l - D_{T^{-1}}\}$ and $b = \max\{l - D_T, l - D_{T^{-1}}\}$.

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Example (Elementary step on i.e.m. with $d = 2$)

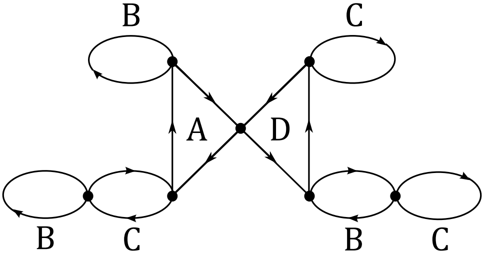
The i.e.m. $T(x) = \begin{cases} x + \alpha, & \text{if } 0 < x < 1 - \alpha \\ x - (1 - \alpha), & \text{if } 1 - \alpha < x < 1 \end{cases}$ is taken to $\widehat{T}(x) = \begin{cases} x + \alpha, & \text{if } 0 < x < 1 - 2\alpha \\ x - (1 - 2\alpha), & \text{if } 1 - 2\alpha < x < 1 - \alpha \end{cases}$ for $0 < \alpha < \frac{1}{2}$.

Remark

The previous example is an elementary step of the Euclidean division algorithm on the lengths $1 - \alpha$ and α of the c.c. of D_T.

Rauzy–Veech algorithm (IV): finite shifts

The Rauzy–Veech algorithm is coded by a finite graph (*Rauzy diagram*) whose arrows are decorated with $d \times d$ matrices $B_\gamma = \mathrm{Id} + E_{\alpha\beta} \in SL(d, \mathbb{Z})$ (where $E_{\alpha\beta}$ = elementary matrix).



Roughly speaking, the vertices of this graph account for the permutations of subintervals and the matrices B_γ encode the changes of lengths of subintervals.

Rauzy–Veech algorithm (IV): countable shifts

After *accelerating* (Zorich, Yoccoz, Avila-Viana, etc.), one gets a loc. constant integ. (Kontsevich-Zorich) cocycle over a countable shift with a (Masur-Veech) prob. meas. having *bounded distortion*.

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The Lyapunov exponents of this cocycle have the form

$$\theta_1 > \theta_2 \geq \cdots \geq \theta_g \geq \underbrace{0 = \cdots = 0}_{s-1 \text{ times}} \geq -\theta_g \geq \cdots \geq -\theta_2 > -\theta_1$$

where $d = 2g + s - 1$ and g is the genus of the underlying translation surfaces.

Remark

This structure of the Lyapunov spectrum is explained by the fact that $B_\gamma \in SL(d, \mathbb{Z})$ corresponds to an action in the first relative homology of a translation surface, i.e., B_γ is *symplectic in disguise*.

Rauzy gasket (I): special systems of isometries

A *special system of isometries* is a collection $\phi_j : [0, x_j] \to [y_j, 1]$, $1 \leq j \leq 3$, of translations between intervals with $x_1 + x_2 + x_3 = 1$.

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Some eclectic motivations for the study of systems of isometries come from:

- pseudo-groups of rotations and free actions on $\mathbb{R}$-trees (Levitt, Gaboriau, Paulin),
- Novikov's problem (Dynnikov, de Leo),
- letter freq. in ternary episturmian words (Arnoux, Starosta).

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The parameter space is $\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}$. Let $\Delta_j = \{(x_1, x_2, x_3) \in \Delta : x_j \geq \sum_{k \neq j} x_k\}$, $1 \leq j \leq 3$. The projectivizations of

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

induce weakly contracting maps $f_j : \Delta \to \Delta_j$, $j = 1, 2, 3$.

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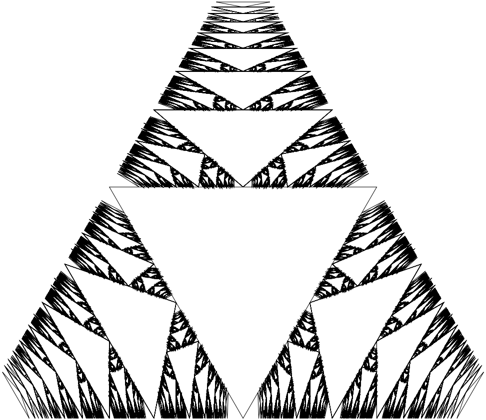
induce weakly contracting maps $f_j \colon \Delta \to \Delta_j$, $j = 1, 2, 3$.

We renormalize a parameter outside the hole $\Delta \setminus \bigcup_{j=1}^3 \Delta_j$ by applying the inverse of the appropriate f_j.

Rauzy gasket (III): Rauzy gasket

The *Rauzy gasket* is the non-empty compact subset R of Δ s.t.

$$R = f_1(R) \cup f_2(R) \cup f_3(R).$$



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The Rauzy-Dynnikov algorithm is also coded by a finite graph whose arrows are decorated by matrices in $SL(3,\mathbb{Z})$.

By *accelerating* (and using thermodynamical methods), one gets a loc. constant integ. cocycle over a countable shift with a (Avila-Hubert-Skripchenko) prob. meas. with bounded distortion.

Remark

An analogous discussion holds for the *fully subtractive algorithm* because it is "dual" to the Rauzy-Dynnikov algorithm (as it was first noticed by Arnoux-Starosta).

Selmer and Cassaigne algorithms

The (homogenous) Selmer algorithm sends $(x_1, x_2, x_3) \in \Delta$ with $0 < x_1 < x_2 < x_3$ to $\frac{1}{1-x_1}(x_1, x_2, x_3 - x_1) \in \Delta$, etc.

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It was noticed by Cassaigne the dynamics can be also encoded by the algorithm

$$(x_1, x_2, x_3) \in \Delta \mapsto \begin{cases} (x_1 - x_3, x_3, x_2) & \text{if } x_1 > x_3 \\ (x_2, x_1, x_3 - x_1) & \text{if } x_1 < x_3 \end{cases}$$

Countable shifts

In the same spirit of the Rauzy-Veech algorithm, the Cassaigne algorithm was accelerated by Fougeron-Skripchenko to produce a loc. constant integ. cocycle over a countable shift with a prob. meas. with bounded distortion.

Furstenberg simplicity criterion for $SL(2,\mathbb{R})$

Theorem (Furstenberg)

Consider a loc. constant $SL(2,\mathbb{R})$-valued cocycle $A:\Sigma\to SL(2,\mathbb{R})$ over a Bernoulli shift (Σ,μ) (i.e., a finitely supported random walk on $SL(2,\mathbb{R})$). If

- *the group generated by $A(\Sigma)$ is not contained in a conjugated of $SO(2,\mathbb{R})$ nor preserves a line in $\mathbb{R}^2$, and*
- *the group generated by $A(\Sigma)$ does not preserve a pair of lines,*

then $\theta_1 > 0$.

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Remark

We shall see that these assumptions are particular instances of *Zariski-density* (cf. Guivarc'h-Raugi, Goldsheid-Margulis) and *pinching and twisting* (cf. Avila-Viana).

Guivarc'h-Raugi and Goldsheid-Margulis simplicity criteria

Let G be a semisimple Lie group.

Theorem (Guivarc'h-Raugi, Goldsheid-Margulis)

Consider a loc. constant G-valued cocycle $A : \Sigma \to G$ over a Bernoulli shift (Σ, μ) (i.e., a finitely supported random walk on G). If the monoid generated by $A(\Sigma)$ is Zariski dense in G, then the Lyapunov exponents are "as simple as possible": the Lyapunov vector belongs to the interior of the Weyl chamber.

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Example

For $G = SO(p,q)$, $q \leq p$, a typical diagonal matrix has the form $\mathrm{diag}(e^{\theta_1}, \dots, e^{\theta_q}, \underbrace{1, \dots, 1}_{p-q \text{ times}}, e^{-\theta_q}, \dots, e^{-\theta_1})$ and, hence, a "simple" Lyapunov spectrum is $\theta_1 > \dots > \theta_q > 0^{p-q} > -\theta_q > \dots > -\theta_1$.

Avila-Viana simplicity criteria (I)

Theorem (I)

Consider a loc. constant integ. cocycle $A : \Sigma \to GL(d, \mathbb{C})$ over a countable shift with a prob. meas. having bounded distortion. If

- *A is pinching:* $\exists$ *a finite word* $\underline{\ell} = (\ell_0, \dots, \ell_{n-1}) \in \Omega$ *s.t. all eigenvalues of* $A^{\underline{\ell}} := A_{\ell_{n-1}} \dots A_{\ell_0}$ *have distinct moduli;*
- *A is twisting:* $\exists$ *a finite word* $\underline{k} \in \Omega$ *s.t.* $A^{\underline{k}}(F) \cap F' = \{0\}$ *for all* $A^{\underline{\ell}}$*-invariant n-plane F and* $(d-n)$*-plane* F', $1 \leq n \leq d$,

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Remark

It is *crucial* to get a *single* element $A^{\underline{k}}$ twisting *all* $A^{\underline{\ell}}$–invariant subspaces at the *same* time: e.g., $A_0 = \mathrm{diag}(2, 1/2)$ and A_1=rotation by $\pi/2$ lead to zero exponents.

Avila-Viana simplicity criteria (II)

Theorem (II)

Consider a loc. constant integ. cocycle $A : \Sigma \to Sp(2d, \mathbb{R})$ over a countable shift with a prob. meas. having bounded distortion. If

- *A is pinching: $\exists$ a finite word $\underline{\ell} \in \Omega$ s.t. all eigenvalues of $A^{\underline{\ell}}$ are real and simple;*
- *A is twisting: $\exists$ $\underline{k} \in \Omega$ s.t. $A^{\underline{k}}(F) \cap F' = \{0\}$ for all $A^{\underline{\ell}}$-inv. isotropic F and coisotropic F' with $\dim F + \dim F' = 2d$,*

then the Lyapunov spectrum of A is simple.

Zariski-density versus pinching and twisting

Zariski density implies pinching and twisting, but the converse is *not* true in general.

On the other hand, Zariski density is *easier* to check in practice and it allows for an *uniform* statement independently of the semisimple Lie group G.

For this reason, Möller, Yoccoz and I looked for simplicity criteria *closer* to the Zariski density assumption of Guivarc'h-Raugi and Goldsheid-Margulis.

Galois-theoretical simplicity criterion

Theorem (M.-Möller-Yoccoz)

Let $G = SL(d)$ or $Sp(2d)$. Consider a loc. const. integ. cocycle $A : \Sigma \to G(\mathbb{Z})$ over a countable shift with a prob. meas. with bounded distortion. If the monoid generated by $A(\Sigma)$ contains two Galois-pinching elements (i.e., their characteristic polynomials are irreducible with all roots $\in \mathbb{R}$ and largest possible Galois group) with disjoint splitting fields, then A has simple Lyapunov spectrum.

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Remark

This result holds for other semisimple Lie groups (after adjusting the definition of Galois-pinching).

Galois-pinching and Zariski density

The Galois-theoretical simplicity criterion is closer to Zariski density thanks to the following results of Prasad-Rapinchuk:

- any Zariski-dense subgroup of $G(\mathbb{Z})$ contains a Galois-pinching element;
- if a monoid $\Gamma \subset G(\mathbb{Z})$ contains a Galois-pinching element γ_1 and an element γ_2 of infinite order not commuting with γ_1, then the Zariski closure of Γ is G or the subgroup H associated to the longest roots.

How to check the Galois-theoretical criterion in practice?

Let $P(x) = x^4 + ax^3 + bx^2 + ax + 1 \in \mathbb{Z}[x]$ be the characteristic polynomial of an element of $Sp(4, \mathbb{Z})$.

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By Galois theory, P has the largest possible Galois group if and only if $\Delta_1 := a^2 - 4b + 8$, $\Delta_2 := (b + 2 + 2a)(b + 2 - 2a)$ and $\Delta_1\Delta_2$ are not squares. In this case, the quadratic subfields of the splitting field are $\mathbb{Q}(\sqrt{\Delta_1})$, $\mathbb{Q}(\sqrt{\Delta_2})$, $\mathbb{Q}(\sqrt{\Delta_1\Delta_2})$.

Moreover, P has real, positive, simple roots if and only if $\Delta_1 > 0$, $t := -a - 4 > 0$ and $d := b + 2 + 2a > 0$.

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Moreover, P has real, positive, simple roots if and only if $\Delta_1 > 0$, $t := -a - 4 > 0$ and $d := b + 2 + 2a > 0$.

Remark

Rivin found some interesting polynomial time (mostly probabilistic) algorithms to compute Galois groups of characteristic polynomials, to determine the Zariski denseness assumption in Guivarc'h-Raugi and Goldsheid-Margulis theorems when $G = SL(d)$ or $Sp(2d)$, etc.

Zorich phenomenon, square-tiled surfaces, ...

Avila-Viana famously used their simplicity criterion to establish that the Lyapunov spectrum of the accelerations of the Rauzy-Veech algorithm have the form

$$\theta_1 > \theta_2 > \cdots > \theta_g > 0^{s-1} > -\theta_g > \cdots > -\theta_2 > -\theta_1,$$

so that the deviations of ergodic averages of almost all i.e.m. T are

$$\sum_{n=0}^{N-1} f(T^n(x)) = \left(\int f\right) N + c_2(f,x) N^{\theta_2/\theta_1} + \dots$$

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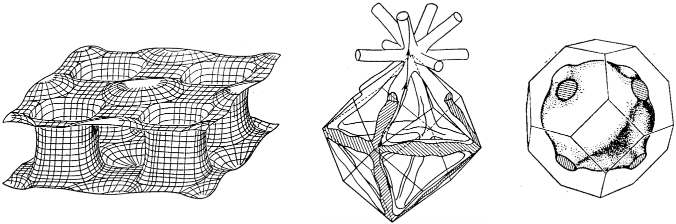
M.-Möller-Yoccoz used the Galois-theoretical criterion together with Faltings' theorem (on rational points on genus > 1 curves) to show the simplicity of the Lyapunov spectra of the cocycles over finite extensions of the Gauss map associated to the "majority" of square-tiled surfaces in $\mathcal{H}(4)$.

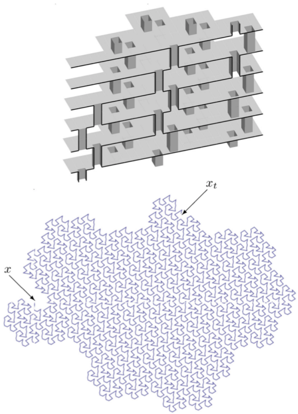
Novikov's problem (I)

Novikov wanted to understand the trajectories of electrons in a metal subjected to an uniform magnetic field.

Mathematically, some of these electrons travel along the intersection of Fermi surfaces and the family of planes orthogonal to the magnetic field.



Novikov's problem (II)



The diffusion rate of typical trajectories is driven by $-\theta_3/\theta_1$ where $\theta_1 \geq \theta_2 \geq \theta_3$ are the Lyapunov exponents of the Rauzy gasket.

Novikov's problem (III)

By the duality with the fully subtractive algorithm, one has $\theta_1 > 0 > \theta_2 \geq \theta_3$ (after Avila-Delecroix).

Since $\theta_1 + \theta_2 + \theta_3 = 0$ (because the cocycle is $SL(3, \mathbb{Z})$-valued), Avila-Hubert-Skripchenko could apply the Galois-theoretical simplicity criterion to get $\theta_1 > 0 > \theta_2 > \theta_3$ and, *a fortiori*,

$$-\theta_3/\theta_1 > 1/2$$

Uniform approximation exponent of Cassaigne's algorithm

In a recent work, Fougeron-Skripchenko used Lagarias' work and the Galois-theoretical simplicity criterion to check that the positivity of the uniform approximation exponent

$$1 - \theta_2/\theta_1$$

of Cassaigne algorithm: in a nutshell, the Lyapunov spectrum $\theta_1 > \theta_2 > \theta_3$ of this algorithm is simple because the matrices

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

are Galois-pinching with disjoint splitting fields (as their characteristic polynomials $x^3 - 4x^2 + 1$ and $x^3 - 6x^2 + 1$ have discriminants 229 and $3^3 \cdot 31$, etc.).

Concluding remarks

It is clear that this list of applications of these simplicity criteria based on Galois theory and Zariski density is incomplete, and I hope that these techniques (together with suitable numerical support) will fit the context of many others MCF algorithms.

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Thank you! Merci! Spasíbo!