Lyapunov spectrum properties

Reza Mohammadpour

ANR CODYS

November 20, 2020

Papers

1)R. Mohammadpour, Zero temperature limits of equilibrium states for subadditive potentials and approximation of the maximal Lyapunov exponent, *Topological Methods in Nonlinear Analysis*, **55(2)**,97—710(2020).

Papers

1)R. Mohammadpour, Zero temperature limits of equilibrium states for subadditive potentials and approximation of the maximal Lyapunov exponent, *Topological Methods in Nonlinear Analysis*, **55(2)**,97—710(2020).
2)R. Mohammadpour, Lyapunov spectrum properties and continuity of lower joint spectral radius ; `https://arxiv.org/abs/2001.03958`.

General setting for this talk

- X is a compact metric space that is endowed with metric d.

General setting for this talk

- X is a compact metric space that is endowed with metric d.
- (X, T) is a topological dynamical system (TDS).

General setting for this talk

- X is a compact metric space that is endowed with metric d.
- (X, T) is a topological dynamical system (TDS).
- $\mathcal{M}(X, T)$= the space of all T-invariant Borel probability measures on X.

General setting for this talk

- X is a compact metric space that is endowed with metric d.
- (X, T) is a topological dynamical system (TDS).
- $\mathcal{M}(X, T)$= the space of all T-invariant Borel probability measures on X.This space is a nonempty convex set and is compact with respect to the weak-$*$ topology.

General setting for this talk

- X is a compact metric space that is endowed with metric d.
- (X, T) is a topological dynamical system (TDS).
- $\mathcal{M}(X, T)$= the space of all T-invariant Borel probability measures on X.This space is a nonempty convex set and is compact with respect to the weak-$*$ topology.
- $\mathcal{E}(X, T) \subset \mathcal{M}(X, T) = \text{ext}(\mathcal{M}(X, T))$.

Birkhoff Theorem

Let $f : X \to \mathbb{R}$ be a continuous function.

Birkhoff Theorem

Let $f : X \to \mathbb{R}$ be a continuous function. We denote $S_n f(x) := \sum_{k=0}^{n-1} f(T^k(x))$ and call this a *Birkhoff sum*

Birkhoff Theorem

Let $f : X \to \mathbb{R}$ be a continuous function. We denote $S_n f(x) := \sum_{k=0}^{n-1} f(T^k(x))$ and call this a *Birkhoff sum* and we call

$$\lim_{n\to\infty} \frac{1}{n} S_n f(x)$$

a *Birkhoff average*.

Birkhoff Theorem

Let $f : X \to \mathbb{R}$ be a continuous function. We denote $S_n f(x) := \sum_{k=0}^{n-1} f(T^k(x))$ and call this a *Birkhoff sum* and we call

$$\lim_{n\to\infty} \frac{1}{n} S_n f(x)$$

a *Birkhoff average*.

- $\alpha(f) = \lim_{n\to\infty} \frac{1}{n} \inf_{x\in X} S_n f(x)$.

Birkhoff Theorem

Let $f : X \to \mathbb{R}$ be a continuous function. We denote $S_n f(x) := \sum_{k=0}^{n-1} f(T^k(x))$ and call this a *Birkhoff sum* and we call

$$\lim_{n\to\infty} \frac{1}{n} S_n f(x)$$

a *Birkhoff average*.

- $\alpha(f) = \lim_{n\to\infty} \frac{1}{n} \inf_{x\in X} S_n f(x)$.
- $\beta(f) = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} S_n f(x)$.

Birkhoff Theorem

Let $f : X \to \mathbb{R}$ be a continuous function. We denote $S_n f(x) := \sum_{k=0}^{n-1} f(T^k(x))$ and call this a *Birkhoff sum* and we call

$$\lim_{n\to\infty} \frac{1}{n} S_n f(x)$$

a *Birkhoff average*.

- $\alpha(f) = \lim_{n\to\infty} \frac{1}{n} \inf_{x\in X} S_n f(x)$.
- $\beta(f) = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} S_n f(x)$.

$\alpha(f)$ and $\beta(f)$ are called the *minimal and maximal ergodic averages of* f, respectively.

Ergodic optimization of Birkhoff averages

Since $\alpha(f) = -\beta(-f)$, let us focus the discussion on the quantity β.

Ergodic optimization of Birkhoff averages

Since $\alpha(f) = -\beta(-f)$, let us focus the discussion on the quantity β. It can also be characterized as

$$\beta(f) = \sup_{\mu \in \mathcal{M}(X,T)} \int f d\mu.$$

Ergodic optimization of Birkhoff averages

Since $\alpha(f) = -\beta(-f)$, let us focus the discussion on the quantity β. It can also be characterized as

$$\beta(f) = \sup_{\mu \in \mathcal{M}(X,T)} \int f d\mu.$$

Compactness of $\mathcal{M}(X, T)$ implies the following *attainability property*: there exists at least one measure $\mu \in \mathcal{M}(X, T)$ for which $\beta(f) = \int f d\mu$; such measures will be called *maximizing measures*,

Ergodic optimization of Birkhoff averages

Since $\alpha(f) = -\beta(-f)$, let us focus the discussion on the quantity β. It can also be characterized as

$$\beta(f) = \sup_{\mu \in \mathcal{M}(X,T)} \int f d\mu.$$

Compactness of $\mathcal{M}(X, T)$ implies the following *attainability property*: there exists at least one measure $\mu \in \mathcal{M}(X, T)$ for which $\beta(f) = \int f d\mu$; such measures will be called *maximizing measures*, denote $\mathcal{M}_{\max}(f)$.

Zero temperature

We study the behavior of the equilibrium measures (μ_t) for a potential tf when $t \to \infty$.

Zero temperature

We study the behavior of the equilibrium measures (μ_t) for a potential tf when $t \to \infty$.

- In the thermodynamic interpretation of the parameter t, it is the *inverse temperature*.

Zero temperature

We study the behavior of the equilibrium measures (μ_t) for a potential tf when $t \to \infty$.

- In the thermodynamic interpretation of the parameter t, it is the *inverse temperature*.
- The limits $t \to \infty$ are called *zero temperature limits*.

Zero temperature

We study the behavior of the equilibrium measures (μ_t) for a potential tf when $t \to \infty$.

- In the thermodynamic interpretation of the parameter t, it is the *inverse temperature*.
- The limits $t \to \infty$ are called *zero temperature limits*.
- The accumulation points of the measure (μ_t) as $t \to \infty$ are called *ground states*.

Zero temperature

We study the behavior of the equilibrium measures (μ_t) for a potential tf when $t \to \infty$.

- In the thermodynamic interpretation of the parameter t, it is the *inverse temperature*.
- The limits $t \to \infty$ are called *zero temperature limits*.
- The accumulation points of the measure (μ_t) as $t \to \infty$ are called *ground states*.
- Zero temperature limits laws are also related to ergodic optimization, because for $t \to \infty$ any accumulation point μ of the equilibrium measures (μ_t) will be a maximizing measure f.

Zero temperature

We study the behavior of the equilibrium measures (μ_t) for a potential tf when $t \to \infty$.

- In the thermodynamic interpretation of the parameter t, it is the *inverse temperature*.
- The limits $t \to \infty$ are called *zero temperature limits*.
- The accumulation points of the measure (μ_t) as $t \to \infty$ are called *ground states*.
- Zero temperature limits laws are also related to ergodic optimization, because for $t \to \infty$ any accumulation point μ of the equilibrium measures (μ_t) will be a maximizing measure f.
- $$\int f d\mu = \lim_{t\to\infty} \int f d\mu_t, \tag{1.1}$$

  and

  $$h_\mu(T) = \lim_{t\to\infty} h_{\mu_t}(T). \tag{1.2}$$

Multifractal formalism of Birkhoff averages

One may ask about the size of the set of points

$$E_f(\alpha) = \{x \in X : \frac{1}{n} S_n f(x) \to \alpha \;\text{ as }\; n \to \infty\},$$

which we call α*-level set of Birkhoff spectrum*,

Multifractal formalism of Birkhoff averages

One may ask about the size of the set of points

$$E_f(\alpha) = \{x \in X : \frac{1}{n} S_n f(x) \to \alpha \text{ as } n \to \infty\},$$

which we call α-*level set of Birkhoff spectrum*, for a given value α from the set

$$L = \{\alpha \in \mathbb{R} : \exists x \in X \text{ and } \lim_{n\to\infty} \frac{1}{n} S_n f(x) = \alpha\},$$

which we call *Birkhoff spectrum.*

Multifractal formalism of Birkhoff averages

One may ask about the size of the set of points

$$E_f(\alpha) = \{x \in X : \frac{1}{n} S_n f(x) \to \alpha \text{ as } n \to \infty\},$$

which we call *α-level set of Birkhoff spectrum*, for a given value α from the set

$$L = \{\alpha \in \mathbb{R} : \exists x \in X \text{ and } \lim_{n\to\infty} \frac{1}{n} S_n f(x) = \alpha\},$$

which we call *Birkhoff spectrum.* That size is usually calculated in terms of topological entropy. Let $Z \subset X$, we denote by $h_{top}(Z)$ topological entropy of T restricted to Z or, simply, the topological entropy of Z.

Restricted variational principle

We denote by $P_\Phi(t)$ the topological pressure for a potential $t\Phi$.

Restricted variational principle

We denote by $P_\Phi(t)$ the topological pressure for a potential $t\Phi$. It is well known that when (X, T) is a transitive subshift of finite type and f is a continuous function, then

$$E_f(\alpha) \neq \emptyset \Leftrightarrow \alpha \in \Omega := \{\int f d\mu : \mu \in \mathcal{M}(X, T)\},$$

Restricted variational principle

We denote by $P_\Phi(t)$ the topological pressure for a potential $t\Phi$. It is well known that when (X, T) is a transitive subshift of finite type and f is a continuous function, then

$$E_f(\alpha) \neq \emptyset \Leftrightarrow \alpha \in \Omega := \{\int f d\mu : \mu \in \mathcal{M}(X, T)\},$$

and

$$\begin{aligned} h_{top}(E_f(\alpha)) &= \inf_{t\in\mathbb{R}}\{P_f(t) - \alpha t : t \in \mathbb{R}\} \\ &= \sup\{h_\mu(T) : \mu \in \mathcal{M}(X, T) \text{ with} \int f d\mu = \alpha\} \ \forall\alpha \in \Omega. \end{aligned} \tag{1.3}$$

Restricted variational principle

We denote by $P_\Phi(t)$ the topological pressure for a potential $t\Phi$. It is well known that when (X, T) is a transitive subshift of finite type and f is a continuous function, then

$$E_f(\alpha) \neq \emptyset \Leftrightarrow \alpha \in \Omega := \{\int f d\mu : \mu \in \mathcal{M}(X, T)\},$$

and

$$\begin{aligned} h_{top}(E_f(\alpha)) &= \inf_{t\in\mathbb{R}}\{P_f(t) - \alpha t : t \in \mathbb{R}\} \\ &= \sup\{h_\mu(T) : \mu \in \mathcal{M}(X, T) \text{ with} \int f d\mu = \alpha\} \ \forall \alpha \in \Omega. \end{aligned} \tag{1.3}$$

(1.3) is called the *restricted variational principle*.

Potentials

Assume that ϕ_n is a continuous positive-valued function on X.

Potentials

Assume that ϕ_n is a continuous positive-valued function on X. We say that $\Phi := \{\log \phi_n\}_{n=1}^{\infty}$ is

Potentials

Assume that ϕ_n is a continuous positive-valued function on X. We say that $\Phi := \{\log \phi_n\}_{n=1}^{\infty}$ is

- a **subadditive potential** if

$$0 < \phi_{n+m}(x) \leq \phi_n(x)\phi_m(T^n(x)) \;\; \forall x \in X, m, n \in \mathbb{N}.$$

Potentials

Assume that ϕ_n is a continuous positive-valued function on X. We say that $\Phi := \{\log \phi_n\}_{n=1}^{\infty}$ is

- a **subadditive potential** if

$$0 < \phi_{n+m}(x) \leq \phi_n(x)\phi_m(T^n(x)) \;\; \forall x \in X, m, n \in \mathbb{N}.$$

- an **almost additive potential** if $\exists C \geq 1, \forall x \in X, m, n \in \mathbb{N}$, we have

$$C^{-1}\phi_n(x)\phi_m(T^n)(x) \leq \phi_{n+m}(x) \leq C\phi_n(x)\phi_m(T^n(x)) .$$

Potentials

Assume that ϕ_n is a continuous positive-valued function on X. We say that $\Phi := \{\log \phi_n\}_{n=1}^{\infty}$ is

- a **subadditive potential** if

$$0 < \phi_{n+m}(x) \leq \phi_n(x)\phi_m(T^n(x)) \;\; \forall x \in X, m, n \in \mathbb{N}.$$

- an **almost additive potential** if $\exists C \geq 1, \forall x \in X, m, n \in \mathbb{N}$, we have

$$C^{-1}\phi_n(x)\phi_m(T^n)(x) \leq \phi_{n+m}(x) \leq C\phi_n(x)\phi_m(T^n(x)) \, .$$

- an **additive potential** if

$$\phi_{n+m}(x) = \phi_n(x)\phi_m(T^n(x)) \;\; \forall x \in X, m, n \in \mathbb{N}.$$

Linear cocycles

The natural example of subadditive potentials is matrix cocycles.

Linear cocycles

The natural example of subadditive potentials is matrix cocycles. More precisely, given a measurable map $\mathcal{A} : X \to GL(d, \mathbb{R})$ taking values into the space $d \times d$ invertible matrices. We consider the products

$$\mathcal{A}^n(x) = \mathcal{A}(T^{n-1}(x)) \dots \mathcal{A}(T(x))\mathcal{A}(x).$$

The pair $(T, \mathcal{A})$ is called a *linear cocycle*.

Linear cocycles

The natural example of subadditive potentials is matrix cocycles. More precisely, given a measurable map $\mathcal{A}: X \to GL(d, \mathbb{R})$ taking values into the space $d \times d$ invertible matrices. We consider the products

$$\mathcal{A}^n(x) = \mathcal{A}(T^{n-1}(x)) \dots \mathcal{A}(T(x))\mathcal{A}(x).$$

The pair $(T, \mathcal{A})$ is called a *linear cocycle*. It induces a skew-product dynamics F on $X \times \mathbb{R}^d$ by $(x, v) \mapsto X \times \mathbb{R}^d$, whose n-th iterate is therefore

$$(x, v) \mapsto (T^n(x), \mathcal{A}^n(x)v).$$

If T is invertible then so is F. Moreover, $F^{-n}(x) = (T^{-n}(x), \mathcal{A}^{-n}(x)v)$ for each $n \geq 1$, where

$$\mathcal{A}^{-n}(x) := \mathcal{A}(T^{-n}(x))^{-1}\mathcal{A}(T^{-n+1}(x))^{-1} \dots \mathcal{A}(T^{-1}(x))^{-1}.$$

An example of linear cocycles: Random matrix cocycles

A simple class of linear cocycles is *one step cocycles* which is defined as follows.

- $X = \{1, ..., k\}^{\mathbb{Z}}$ is a symbolic space.

An example of linear cocycles: Random matrix cocycles

A simple class of linear cocycles is *one step cocycles* which is defined as follows.

- $X = \{1, ..., k\}^{\mathbb{Z}}$ is a symbolic space.
- $T : X \to X$ is a shift map, i.e. $T(x_l)_l = (x_{l+1})_l$

An example of linear cocycles: Random matrix cocycles

A simple class of linear cocycles is *one step cocycles* which is defined as follows.

- $X = \{1, ..., k\}^{\mathbb{Z}}$ is a symbolic space.
- $T : X \to X$ is a shift map, i.e. $T(x_l)_l = (x_{l+1})_l$
- Given a finite set of matrices $\mathcal{A} = \{A_1, \ldots, A_k\} \subset GL(d, \mathbb{R})$

An example of linear cocycles: Random matrix cocycles

A simple class of linear cocycles is *one step cocycles* which is defined as follows.

- $X = \{1, ..., k\}^{\mathbb{Z}}$ is a symbolic space.
- $T : X \to X$ is a shift map, i.e. $T(x_l)_l = (x_{l+1})_l$
- Given a finite set of matrices $\mathcal{A} = \{A_1, \ldots, A_k\} \subset GL(d, \mathbb{R})$

We define the function $\mathcal{A} : X \to GL(d, \mathbb{R})$ by

$$\mathcal{A}(x) = A_{x_0}.$$

An example of linear cocycles: Random matrix cocycles

A simple class of linear cocycles is *one step cocycles* which is defined as follows.

- $X = \{1, ..., k\}^{\mathbb{Z}}$ is a symbolic space.
- $T : X \to X$ is a shift map, i.e. $T(x_l)_l = (x_{l+1})_l$
- Given a finite set of matrices $\mathcal{A} = \{A_1, \ldots, A_k\} \subset GL(d, \mathbb{R})$

We define the function $\mathcal{A} : X \to GL(d, \mathbb{R})$ by

$$\mathcal{A}(x) = A_{x_0}.$$

In this case, we say that$(T, \mathcal{A})$ is a *one step cocycle*.

Lyapunov exponents

By Kingman's subadditive ergodic theorem, for any $\mu \in \mathcal{M}(X, T)$ and μ almost every $x \in X$ such that $\log^+ \|\mathcal{A}\| \in L^1(\mu)$, the following limit, called the *top Lyapunov exponent* at x, exists:

Lyapunov exponents

By Kingman's subadditive ergodic theorem, for any $\mu \in \mathcal{M}(X, T)$ and μ almost every $x \in X$ such that $\log^+ \|\mathcal{A}\| \in L^1(\mu)$, the following limit, called the *top Lyapunov exponent* at x, exists:

$$\chi(x, \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\|, \tag{1.4}$$

Lyapunov exponents

By Kingman's subadditive ergodic theorem, for any $\mu \in \mathcal{M}(X, T)$ and μ almost every $x \in X$ such that $\log^+ \|\mathcal{A}\| \in L^1(\mu)$, the following limit, called the *top Lyapunov exponent* at x, exists:

$$\chi(x, \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\|, \tag{1.4}$$

where $\|\mathcal{A}\|$ the Euclidean operator norm of a matrix $\mathcal{A}$ (i.e. the largest singular value of $\mathcal{A}$), that is subadditive i.e.,

$$0 < \|\mathcal{A}^{n+m}(x)\| \leq \|\mathcal{A}^n(x)\| \|\mathcal{A}^n(T^m(x))\| \ \ \forall x \in X, m, n \in \mathbb{N}.$$

Lyapunov exponents

By Kingman's subadditive ergodic theorem, for any $\mu \in \mathcal{M}(X, T)$ and μ almost every $x \in X$ such that $\log^+ \|\mathcal{A}\| \in L^1(\mu)$, the following limit, called the *top Lyapunov exponent* at x, exists:

$$\chi(x, \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\|, \tag{1.4}$$

where $\|\mathcal{A}\|$ the Euclidean operator norm of a matrix $\mathcal{A}$ (i.e. the largest singular value of $\mathcal{A}$), that is subadditive i.e.,

$$0 < \|\mathcal{A}^{n+m}(x)\| \leq \|\mathcal{A}^n(x)\|\|\mathcal{A}^n(T^m(x))\| \;\; \forall x \in X, m, n \in \mathbb{N}.$$

Let us denote $\chi(\mu, \mathcal{A}) = \int \chi(., \mathcal{A}) d\mu$. If the measure μ is ergodic then $\chi(x, \mathcal{A}) = \chi(\mu, \mathcal{A})$ for μ-almost every $x \in X$.

Ergodic optimization of Lyapunov exponents

- $\beta(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \|\mathcal{A}^n(x)\|$

Ergodic optimization of Lyapunov exponents

- $\beta(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \|\mathcal{A}^n(x)\| = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu, \mathcal{A})$.

Ergodic optimization of Lyapunov exponents

- $\beta(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \|\mathcal{A}^n(x)\| = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu,\mathcal{A})$.
- $\alpha(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \inf_{x\in X} \|\mathcal{A}^n(x)\|$

Ergodic optimization of Lyapunov exponents

- $\beta(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \|\mathcal{A}^n(x)\| = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu, \mathcal{A})$.
- $\alpha(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \inf_{x\in X} \|\mathcal{A}^n(x)\| = \inf_{\mu\in\mathcal{M}(X,T)} \chi(\mu, \mathcal{A})$.

Ergodic optimization of Lyapunov exponents

- $\beta(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \|\mathcal{A}^n(x)\| = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu,\mathcal{A})$.
- $\alpha(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \inf_{x\in X} \|\mathcal{A}^n(x)\| = \inf_{\mu\in\mathcal{M}(X,T)} \chi(\mu,\mathcal{A})$.

$\beta(\mathcal{A})$ is always attained by at least one measure (which will be called a *Lyapunov maximizing measure*, we denote by $\mathcal{M}_{\max}(\mathcal{A})$ the set of such measures),

Ergodic optimization of Lyapunov exponents

- $\beta(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \|\mathcal{A}^n(x)\| = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu,\mathcal{A})$.
- $\alpha(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \inf_{x\in X} \|\mathcal{A}^n(x)\| = \inf_{\mu\in\mathcal{M}(X,T)} \chi(\mu,\mathcal{A})$.

$\beta(\mathcal{A})$ is always attained by at least one measure (which will be called a *Lyapunov maximizing measure*, we denote by $\mathcal{M}_{\max}(\mathcal{A})$ the set of such measures), but that is not necessarily the case for $\alpha(\mathcal{A})$.

Continuity of the minimal Lyapunov exponent

In the one step cocycles case,

- Bochi and Morris proved $\alpha(\mathcal{A})$ is continuous under $1-$domination assumption.

Continuity of the minimal Lyapunov exponent

In the one step cocycles case,

- Bochi and Morris proved $\alpha(\mathcal{A})$ is continuous under $1-$domination assumption.
- Breuillard and Sert extended their result to the joint spectrum under domination assumption.

Continuity of the minimal Lyapunov exponent

In the one step cocycles case,

- Bochi and Morris proved $\alpha(\mathcal{A})$ is continuous under $1-$domination assumption.
- Breuillard and Sert extended their result to the joint spectrum under domination assumption. Moreover, they gave a counterexample that shows that we have discontinuity of the minimal Lyapunov exponent for generic cocycles.

Derivative cocycles

The linear cocycles generated by

Derivative cocycles

The linear cocycles generated by

- a diffemorphism map $T : M \to M$ on a closed Riemannian manifold M and

Derivative cocycles

The linear cocycles generated by

- a diffemorphism map $T : M \to M$ on a closed Riemannian manifold M and
- a family of maps $\mathcal{A}(x) := D_x T : T_x M \to T_{T(x)} M$

Derivative cocycles

The linear cocycles generated by

- a diffemorphism map $T : M \to M$ on a closed Riemannian manifold M and
- a family of maps $\mathcal{A}(x) := D_x T : T_x M \to T_{T(x)} M$

are called *derivative cocycles*.

Derivative cocycles

The linear cocycles generated by

- a diffemorphism map $T : M \to M$ on a closed Riemannian manifold M and
- a family of maps $\mathcal{A}(x) := D_x T : T_x M \to T_{T(x)} M$

are called *derivative cocycles*. Moreover, when $T : X \to X$ is an Anosov diffemophism (or expanding map), Bowen showed that there exists a symbolic coding of T by a subshift of finite type.

Derivative cocycles

The linear cocycles generated by

- a diffemorphism map $T : M \to M$ on a closed Riemannian manifold M and
- a family of maps $\mathcal{A}(x) := D_x T : T_x M \to T_{T(x)} M$

are called *derivative cocycles*. Moreover, when $T : X \to X$ is an Anosov diffemophism (or expanding map), Bowen showed that there exists a symbolic coding of T by a subshift of finite type. Therefore, we always assume that

Derivative cocycles

The linear cocycles generated by

- a diffemorphism map $T : M \to M$ on a closed Riemannian manifold M and
- a family of maps $\mathcal{A}(x) := D_x T : T_x M \to T_{T(x)} M$

are called *derivative cocycles*. Moreover, when $T : X \to X$ is an Anosov diffemophism (or expanding map), Bowen showed that there exists a symbolic coding of T by a subshift of finite type. Therefore, we always assume that

- $T : \Sigma \to \Sigma$ is a topologically mixing subshift of finite type

Derivative cocycles

The linear cocycles generated by

- a diffemorphism map $T : M \to M$ on a closed Riemannian manifold M and
- a family of maps $\mathcal{A}(x) := D_x T : T_x M \to T_{T(x)} M$

are called *derivative cocycles*. Moreover, when $T : X \to X$ is an Anosov diffemophism (or expanding map), Bowen showed that there exists a symbolic coding of T by a subshift of finite type. Therefore, we always assume that

- $T : \Sigma \to \Sigma$ is a topologically mixing subshift of finite type
- and $\mathcal{A} : X \to GL(d, \mathbb{R})$ is a Hölder continuous.

Derivative cocycles

Fix $\theta \in (0,1)$ and endow Σ with the metric d defined as follows: for $x = (x_i)_{i\in\mathbb{Z}}, y = (y_i)_{i\in\mathrm{Z}} \in \Sigma$, we have

$$d(x,y) = \theta^k$$

where k is the largest integer such that $x_i = y_i$ for all $|i| < k$. Equipped with such metric, the shift operator T becomes a hyperbolic homeomorphism of a compact metric space Σ.

Derivative cocycles

Fix $\theta \in (0,1)$ and endow Σ with the metric d defined as follows: for $x = (x_i)_{i\in\mathbb{Z}}, y = (y_i)_{i\in\mathrm{Z}} \in \Sigma$, we have

$$d(x,y) = \theta^k$$

where k is the largest integer such that $x_i = y_i$ for all $|i| < k$.
Equipped with such metric, the shift operator T becomes a hyperbolic homeomorphism of a compact metric space Σ.
In general, we know much more about one step cocycles that about the more general derivative cocycles, but here are some of the results known in the derivative cocycles situation.

Fiber bunched cocycles

A $r-$Hölder continuous function $\mathcal{A}$ is called *fiber bunched* if for any $x \in \Sigma$,

$$\|\mathcal{A}(x)\| \|\mathcal{A}(x)^{-1}\| \theta^r < 1,$$

where θ is the hyperbolicity constant defining the metric on the base Σ.

Fiber bunched cocycles

A $r-$Hölder continuous function $\mathcal{A}$ is called *fiber bunched* if for any $x \in \Sigma$,

$$\|\mathcal{A}(x)\|\|\mathcal{A}(x)^{-1}\|\theta^r < 1,$$

where θ is the hyperbolicity constant defining the metric on the base Σ. We say that the linear cocycle $(T, \mathcal{A})$ is fiber-bunched if its generator $\mathcal{A}$ is fiber-bunched. We denote by $H_b^r(\Sigma, GL(d, \mathbb{R})))$ the family of r-Hölder-continuous and fiber bunched functions.

Fiber bunched cocycles

Bonatti, Gómez-Mont, and Viana showed that the Hölder continuity and the fiber bunched assumption $\mathcal{A} \in H_b^r(\Sigma, GL(d, \mathbb{R}))$ imply the convergence of the *canonical holonomy* $H^{s/u}$.

Fiber bunched cocycles

Bonatti, Gómez-Mont, and Viana showed that the Hölder continuity and the fiber bunched assumption $\mathcal{A} \in H_b^r(\Sigma, GL(d, \mathbb{R}))$ imply the convergence of the *canonical holonomy* $H^{s/u}$. That means, for any $y \in W_{\text{loc}}^{s/u}(x)$,

$$H^s_{y \leftarrow x} := \lim_{n \to \infty} \mathcal{A}^n(y)^{-1} \mathcal{A}^n(x) \text{ and } H^u_{y \leftarrow x} := \lim_{n \to -\infty} \mathcal{A}^n(y)^{-1} \mathcal{A}^n(x).$$

Generic cocycles

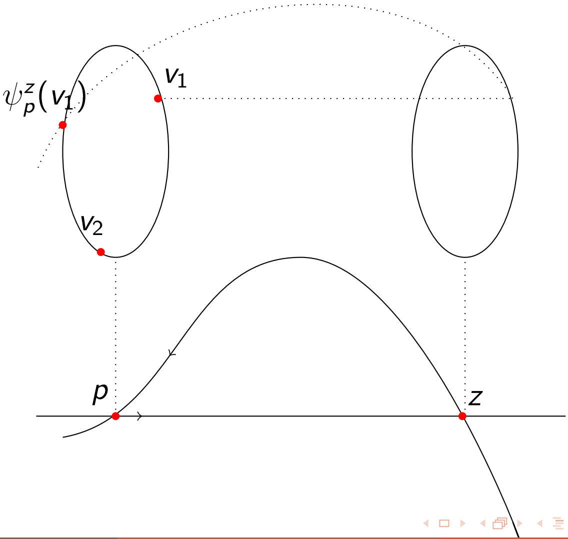
Definition (Bonatti and Viana)

The linear cocycle F satisfies the *pinching* and *twisting* condition if $\exists X \ni p = T^q(p)$ periodic point such that

- P all eigenvalues of $\mathcal{A}^q(p)$ have distinct values.
- T there exists a homoclinic point z of p such that ψ_z^p twists the eigendirections of $\mathcal{A}^q(p)$, where

$$\psi_z^p := H^s_{p \leftarrow z} \circ H^u_{z \leftarrow p}.$$

Generic cocycles



Results

$$\mathcal{W} := \{\mathcal{A} \in H_b^r(\Sigma, GL(d, \mathbb{R})) : \mathcal{A} \text{ is pinching and twisting}\}.$$

is open and dense.

Results

$$\mathcal{W} := \{\mathcal{A} \in H_b^r(\Sigma, GL(d,\mathbb{R})) : \mathcal{A} \text{ is pinching and twisting}\}.$$

is open and dense.

- Level set

$$E(\alpha) = \{x \in X : \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\| = \alpha\}.$$

Results

$$\mathcal{W} := \{\mathcal{A} \in H_b^r(\Sigma, GL(d, \mathbb{R})) : \mathcal{A} \text{ is pinching and twisting}\}.$$

is open and dense.

- Level set

$$E(\alpha) = \{x \in X : \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\| = \alpha\}.$$

- Lyapunov spectrum

$$L = \{\alpha : \exists x \in X \text{ such that } \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\| = \alpha\}.$$

Results

$$\mathcal{W} := \{\mathcal{A} \in H^r_b(\Sigma, GL(d,\mathbb{R})) : \mathcal{A} \text{ is pinching and twisting}\}.$$

is open and dense.

- Level set

$$E(\alpha) = \{x \in X : \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\| = \alpha\}.$$

- Lyapunov spectrum

$$L = \{\alpha : \exists x \in X \text{ such that } \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}^n(x)\| = \alpha\}.$$

- $\omega = \{\chi(\mu, \mathcal{A}); \mu \in \mathcal{M}(X, T)\}.$

Results

Lyapunov spectrum is closed and convex:

Results

Lyapunov spectrum is closed and convex:

- for one step cocycles under irreducibility assumption (Feng).

Results

Lyapunov spectrum is closed and convex:

- for one step cocycles under irreducibility assumption (Feng).
- for generic cocycles $\mathcal{W}$ (Park).

Results

Lyapunov spectrum is closed and convex:

- for one step cocycles under irreducibility assumption (Feng).
- for generic cocycles $\mathcal{W}$ (Park).
- for fiber bunched cocycles $H_b^r(\Sigma, GL(2,\mathbb{R}))$ (Mohammadpour).

Results

Theorem (Mohammadpour)

Let $\mathcal{A} \in \mathcal{W}$. *Then,*

$$L = \overline{\{\alpha, \;\; h_{top}(E(\alpha)) > 0\}}.$$

Furthermore, $\alpha \mapsto h_{top}(E(\alpha))$ *is concave for* $\alpha \in \mathring{L}$.

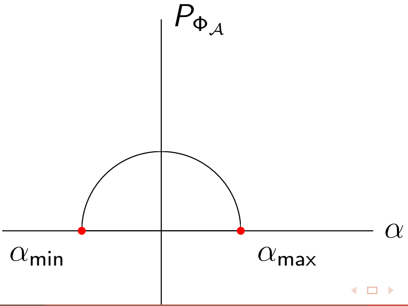
Results

Theorem (Mohammadpour)

Let $\mathcal{A} \in \mathcal{W}$. Then,

$$L = \overline{\{\alpha, \;\; h_{top}(E(\alpha)) > 0\}}.$$

Furthermore, $\alpha \mapsto h_{top}(E(\alpha))$ is concave for $\alpha \in \mathring{L}$.



Conjecture

Conjecture (Hunt and Ott)

For typical chaotic systems (X, T) and typical smooth functions f, the maximizing measure is unique and has low complexity.

Conjecture

Conjecture (Hunt and Ott)

For typical chaotic systems (X, T) and typical smooth functions f, the maximizing measure is unique and has low complexity.

In our context, low complexity means either zero topological entropy or supported on a periodic orbit.

Conjecture

Conjecture (Hunt and Ott)

For typical chaotic systems (X, T) and typical smooth functions f, the maximizing measure is unique and has low complexity.

In our context, low complexity means either zero topological entropy or supported on a periodic orbit.

Theorem (Contreras)

If (X, T) is an expanding map, then a generic Lipschitz function $f : X \to \mathbb{R}$ has a unique maximizing measure, which is supported on a periodic orbit. Moreover, the zero temperature limit exits.

Conjecture

Conjecture (Hunt and Ott)

For typical chaotic systems (X, T) and typical smooth functions f, the maximizing measure is unique and has low complexity.

In our context, low complexity means either zero topological entropy or supported on a periodic orbit.

Theorem (Contreras)

If (X, T) is an expanding map, then a generic Lipschitz function $f : X \to \mathbb{R}$ has a unique maximizing measure, which is supported on a periodic orbit. Moreover, the zero temperature limit exits.

Now, we have the following open question:

Is the conjecture true for the Lyapunov exponent? (That means, we replace Birkhoff sums by matrix products.)

Restricted varitional principle

Denote $\Phi_{\mathcal{A}} = \{\log \|\mathcal{A}^n\|\}$.

Restricted varitional principle

Denote $\Phi_{\mathcal{A}} = \{\log \|\mathcal{A}^n\|\}$.

Theorem (Mohammadpour)

Assume that $T : X \to X$ is a topologically mixing subshift of finite type on the compact metric space X. Suppose that $\mathcal{A} : X \to GL(d, \mathbb{R})$ belongs to generic cocycles $\mathcal{W}$. Then,

$$h_{top}(E(\alpha)) = \sup\{h_\mu(T) : \mu \in \mathcal{M}(X, T), \chi(\mu, \mathcal{A}) = \alpha\}$$
$$= \inf\{P_{\Phi_{\mathcal{A}}}(q) - \alpha.q : q \in \mathbb{R}\} \ \forall \alpha \in \omega.$$

Restricted varitional principle

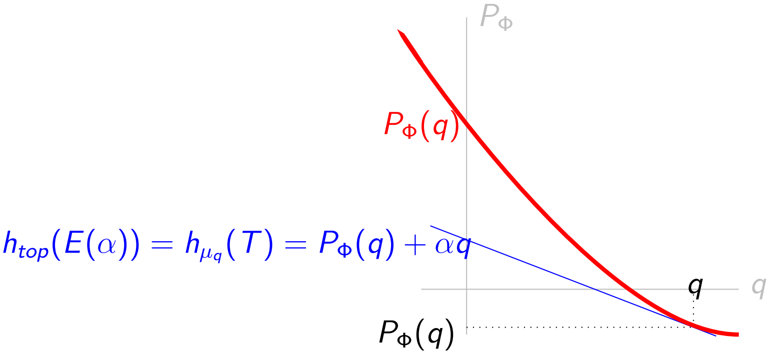


Figure: $P_\Phi(.)$ is a convex function for $q \in \mathbb{R}$. The blue line is tangent to $P_\Phi(.)$ at q with slope $-\alpha = P'_\Phi(q)$.

Zero temperature limit

Let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be a subadditive potential.

Zero temperature limit

Let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be a subadditive potential.

- $\chi(\mu, \Phi) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n(x) d\mu(x)$.

Zero temperature limit

Let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be a subadditive potential.

- $\chi(\mu, \Phi) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n(x) d\mu(x)$.
- $\beta(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \phi_n(x)$

Zero temperature limit

Let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be a subadditive potential.

- $\chi(\mu, \Phi) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n(x) d\mu(x)$.
- $\beta(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \phi_n(x) = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu, \Phi)$.

Zero temperature limit

Let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be a subadditive potential.

- $\chi(\mu, \Phi) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n(x) d\mu(x)$.
- $\beta(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \phi_n(x) = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu, \Phi)$.
- $\mathcal{M}_{\max}(\Phi) = \{\mu \in \mathcal{M}(X, T) : \beta(\Phi) = \chi(\mu, \Phi)\}$.

Zero temperature limit

Let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be a subadditive potential.

- $\chi(\mu, \Phi) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n(x) d\mu(x)$.
- $\beta(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \phi_n(x) = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu, \Phi)$.
- $\mathcal{M}_{\max}(\Phi) = \{\mu \in \mathcal{M}(X, T) : \beta(\Phi) = \chi(\mu, \Phi)\}$.

Theorem (Mohammadpour)

Let (X, T) be a TDS such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and topological entropy $h_{top}(T) < \infty$. Suppose that $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is a subadditive potential on the compact metric X. Then any weak* accumulation μ of a family of equilibrium measures (μ_t) for potentials $t\Phi$, where $t > 0$, has a Lyapunov maximizing measure for Φ.

Zero temperature limit

Let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be a subadditive potential.

- $\chi(\mu, \Phi) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n(x) d\mu(x)$.
- $\beta(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \phi_n(x) = \sup_{\mu\in\mathcal{M}(X,T)} \chi(\mu, \Phi)$.
- $\mathcal{M}_{\max}(\Phi) = \{\mu \in \mathcal{M}(X, T) : \beta(\Phi) = \chi(\mu, \Phi)\}$.

Theorem (Mohammadpour)

Let (X, T) be a TDS such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and topological entropy $h_{top}(T) < \infty$. Suppose that $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is a subadditive potential on the compact metric X. Then any weak accumulation μ of a family of equilibrium measures (μ_t) for potentials $t\Phi$, where $t > 0$, has a Lyapunov maximizing measure for Φ. Moreover,*

(i) $\chi(\mu, \Phi) = \lim_{t_i\to\infty} \chi(\mu_{t_i}, \Phi)$,

(ii) $h_\mu(T) = \lim_{t_i\to\infty} h_{\mu_{t_i}}(T) = \max\{h_\nu(T), \nu \in \mathcal{M}_{\max}(\Phi)\}$.

Zero temperature limits for Gibbs measures

We denote by $\mathcal{L}_n$ the set of admissible words Σ with length n.

Zero temperature limits for Gibbs measures

We denote by $\mathcal{L}_n$ the set of admissible words Σ with length n. We denote by $\Phi_{\mathcal{A}}$ the subadditive potential $\{\log \|\mathcal{A}^n\|\}$.

Zero temperature limits for Gibbs measures

We denote by $\mathcal{L}_n$ the set of admissible words Σ with length n. We denote by $\Phi_{\mathcal{A}}$ the subadditive potential $\{\log \|\mathcal{A}^n\|\}$.

Definition

We say that μ_t has the following Gibbs property for $t\Phi_{\mathcal{A}}$, where $t > 0$: There exists $C \geq 1$ such that for any $n \in \mathbb{N}$ and $[J] \in \mathcal{L}_n$, we have

$$C^{-1} \leq \frac{\mu_t([J])}{e^{-nP_{\mathcal{A}}(t)} \|\mathcal{A}^n(x)\|} \leq C,$$

for any $x \in [J]$.

Zero temperature limits for Gibbs measures

We denote by $\mathcal{L}_n$ the set of admissible words Σ with length n. We denote by $\Phi_{\mathcal{A}}$ the subadditive potential $\{\log \|\mathcal{A}^n\|\}$.

Definition

We say that μ_t has the following Gibbs property for $t\Phi_{\mathcal{A}}$, where $t > 0$: There exists $C \geq 1$ such that for any $n \in \mathbb{N}$ and $[J] \in \mathcal{L}_n$, we have

$$C^{-1} \leq \frac{\mu_t([J])}{e^{-nP_{\mathcal{A}}(t)}\|\mathcal{A}^n(x)\|} \leq C,$$

for any $x \in [J]$.

Theorem (Mohammadpour)

Let $\mathcal{A} \in \mathcal{W}$. Then the previous theorem holds for Gibbs measures.

Continuity of entropy spectrum

Theorem (Mohammadpour)

Suppose $\mathcal{A}_l, \mathcal{A} \in \mathcal{W}$ *with* $\mathcal{A}_l \to \mathcal{A}$*, and* $t_l, t \in \mathbb{R}_+$ *such that* $t_l \to t$*. Let* $\alpha_{t_l} = P'_{\Phi_{\mathcal{A}_l}}(t_l)$ *and* $\alpha_t = P'_{\Phi_{\mathcal{A}}}(t)$*.*

Continuity of entropy spectrum

Theorem (Mohammadpour)

Suppose $\mathcal{A}_l, \mathcal{A} \in \mathcal{W}$ *with* $\mathcal{A}_l \to \mathcal{A}$*, and* $t_l, t \in \mathbb{R}_+$ *such that* $t_l \to t$*. Let* $\alpha_{t_l} = P'_{\Phi_{\mathcal{A}_l}}(t_l)$ *and* $\alpha_t = P'_{\Phi_{\mathcal{A}}}(t)$*. Then*

$$\lim_{l\to\infty} h_{top}(E_{\mathcal{A}_l}(\alpha_l)) = h_{top}(E_{\mathcal{A}}(\alpha)).$$

Continuity of entropy spectrum

Theorem (Mohammadpour)

Suppose $\mathcal{A}_l, \mathcal{A} \in \mathcal{W}$ *with* $\mathcal{A}_l \to \mathcal{A}$, *and* $t_l, t \in \mathbb{R}_+$ *such that* $t_l \to t$. *Let* $\alpha_{t_l} = P'_{\Phi_{\mathcal{A}_l}}(t_l)$ *and* $\alpha_t = P'_{\Phi_{\mathcal{A}}}(t)$. *Then*

$$\lim_{l \to \infty} h_{top}(E_{\mathcal{A}_l}(\alpha_l)) = h_{top}(E_{\mathcal{A}}(\alpha)).$$

Moreover,

$$h_{top}(E(\alpha_t)) \to h_{top}(E(\beta(\mathcal{A})) \ \ when \ \ t \to \infty.$$

Continuity of the minimal Lyapunov exponent

Theorem (Mohammadpour)

Let (X, T) be a TDS such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and $h_{top}(T) < \infty$.

Continuity of the minimal Lyapunov exponent

Theorem (Mohammadpour)

Let (X, T) be a TDS such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and $h_{top}(T) < \infty$. Suppose that $\mathcal{A} : X \to GL(d, \mathbb{R})$ is a matrix cocycle over the TDS (X, T) and $(C_x)_{x \in X}$ is an invariant cone field on X.

Continuity of the minimal Lyapunov exponent

Theorem (Mohammadpour)

Let (X, T) be a TDS such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and $h_{top}(T) < \infty$. Suppose that $\mathcal{A} : X \to GL(d, \mathbb{R})$ is a matrix cocycle over the TDS (X, T) and $(C_x)_{x \in X}$ is an invariant cone field on X. Then $\alpha(\mathcal{A})$ can be approximated by the Lyapunov exponents of the equilibrium measures for the almost additive potential $t\Phi_{\mathcal{A}}$, where $t \in \mathbb{R}$. Moreover, a minimizing measure for $\mathcal{A}$ exists.

Continuity of the minimal Lyapunov exponent

Theorem (Mohammadpour)

Let (X, T) be a topologically mixing subshift of finite type. Suppose that $\mathcal{A}_n, \mathcal{A} : X \to GL(d, \mathbb{R})$ are matrix cocycles over (X, T), and $\Phi_{\mathcal{A}}$ has bounded distortion.

Continuity of the minimal Lyapunov exponent

Theorem (Mohammadpour)

Let (X, T) be a topologically mixing subshift of finite type. Suppose that $\mathcal{A}_n, \mathcal{A} : X \to GL(d, \mathbb{R})$ are matrix cocycles over (X, T), and $\Phi_{\mathcal{A}}$ has bounded distortion.Assume that $(C_x)_{x \in X}$ is an invariant cone field on X.

Continuity of the minimal Lyapunov exponent

Theorem (Mohammadpour)

Let (X, T) be a topologically mixing subshift of finite type. Suppose that $\mathcal{A}_n, \mathcal{A} : X \to GL(d, \mathbb{R})$ are matrix cocycles over (X, T), and $\Phi_{\mathcal{A}}$ has bounded distortion.Assume that $(C_x)_{x \in X}$ is an invariant cone field on X. Then $\alpha(\mathcal{A}_n) \to \alpha(\mathcal{A})$ when $\mathcal{A}_n \to \mathcal{A}$.

Continuity of the minimal Lyapunov exponent

Theorem (Mohammadpour)

Let (X, T) be a topologically mixing subshift of finite type. Suppose that $\mathcal{A}_n, \mathcal{A} : X \to GL(d, \mathbb{R})$ are matrix cocycles over (X, T), and $\Phi_{\mathcal{A}}$ has bounded distortion.Assume that $(C_x)_{x \in X}$ is an invariant cone field on X. Then $\alpha(\mathcal{A}_n) \to \alpha(\mathcal{A})$ when $\mathcal{A}_n \to \mathcal{A}$.

Definition (Bochi and Gourmelon)

We say that $\mathcal{A}$ is *i-dominated* if there exist constants $C > 1$, $0 < \tau < 1$ such that

$$\frac{\sigma_{i+1}(\mathcal{A}^n(x))}{\sigma_i(\mathcal{A}^n(x))} \leq C\tau^n, \ \ \forall n \in \mathbb{N}, x \in X.$$

Continuity of the minimal Lyapunov exponent

Theorem (Crovisier and Potrie)

Assume that $f \in \mathrm{Diff}^r(M)$. Let K be an invariant compact set and fix $d_+ \geq 1$. Then K is endowed with a Df -contracted cone-field C with dimension d_+ if and only if there exists a dominated splitting $T_K M = E \oplus_< F$ with $d_+ = \ dim(F)$.

Continuity of the minimal Lyapunov exponent

Theorem (Crovisier and Potrie)

Assume that $f \in \mathrm{Diff}^r(M)$. Let K be an invariant compact set and fix $d_+ \geq 1$. Then K is endowed with a Df -contracted cone-field C with dimension d_+ if and only if there exists a dominated splitting $T_K M = E \oplus_< F$ with $d_+ = \; dim(F)$.

Theorem (Mohammadpour)

Let $\mathcal{A}_k, \mathcal{A} \in H_b^r(\Sigma, GL(d, \mathbb{R}))$.

Continuity of the minimal Lyapunov exponent

Theorem (Crovisier and Potrie)

Assume that $f \in \mathrm{Diff}^r(M)$. Let K be an invariant compact set and fix $d_+ \geq 1$. Then K is endowed with a Df -contracted cone-field C with dimension d_+ if and only if there exists a dominated splitting $T_K M = E \oplus_< F$ with $d_+ = \ dim(F)$.

Theorem (Mohammadpour)

Let $\mathcal{A}_k, \mathcal{A} \in H_b^r(\Sigma, GL(d, \mathbb{R}))$. Assume that $\mathcal{A}_k$ and $\mathcal{A}$ satisfy $1-$domination. Then $\alpha(\mathcal{A}_n) \to \alpha(\mathcal{A})$, when $\mathcal{A}_n \to \mathcal{A}$.

Thank You!