Recurrence of substitutive Sturmian words

A probabilistic study

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Joint work with

Brigitte Vallée

Réunion du projet CODYS,

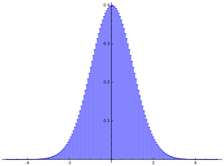
Paris, 18 December, 2019.

Context

- Probabilistic analysis

Object/experiment/execution?

$\Rightarrow$ Models, averages, distribution?



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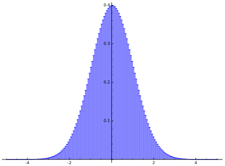

- Word Combinatorics

Study of *words*
$\Rightarrow$ subwords (factors), frequencies

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$\sigma\colon 0 \mapsto 01,\ 1 \mapsto 10$
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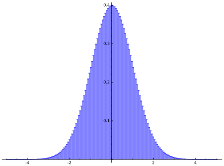

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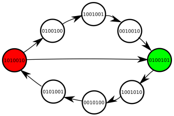
- Sturmian Words

simplest not eventually periodic.
$\Rightarrow$ recurrence: worst case, average?



Plan of the talk

1. Sturmian words
   - General Sturmian words
   - Substitutive words
2. Recurrence function
   - Definition and classical results
   - Our models and results for generic words
3. Substitutive Sturmian words
   - Model for quadratic irrationals
   - Main result
4. Toolbox for the proofs
5. Conclusion

Complexity and Sturmian words

Definition

Complexity function of an infinite word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$

$$p_{\mathbf{u}} \colon \mathbb{N} \to \mathbb{N}, \qquad p_{\mathbf{u}}(n) = \#\{\text{factors of length } n \text{ in } \mathbf{u}\}.$$

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Important property

$$\mathbf{u} \in \mathcal{A}^{\mathbb{N}} \text{ is not eventually periodic}$$
$$\iff p_{\mathbf{u}}(n+1) > p_{\mathbf{u}}(n) \text{ for all } n \in \mathbb{N}$$
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Definition

$\mathbf{u} \in \{0,1\}^{\mathbb{N}}$ is Sturmian $\iff p_{\mathbf{u}}(n) = n+1$ for each $n \geq 0$.

Explicit construction (discrete coding)

Given $\alpha, \beta \in [0,1)$ we define

$$\underline{\mathfrak{S}}_{\alpha,\beta}(n) = \lfloor (n+1)\,\alpha + \beta \rfloor - \lfloor n\,\alpha + \beta \rfloor \ ,$$
$$\overline{\mathfrak{S}}_{\alpha,\beta}(n) = \lceil (n+1)\,\alpha + \beta \rceil - \lceil n\,\alpha + \beta \rceil \ ,$$

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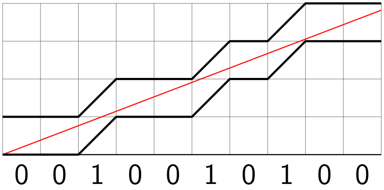

Figure: Sequences $\underline{\mathfrak{S}}_{\alpha,\beta}$ and $\overline{\mathfrak{S}}_{\alpha,\beta}$ are discrete codings of $y = \alpha x + \beta$.

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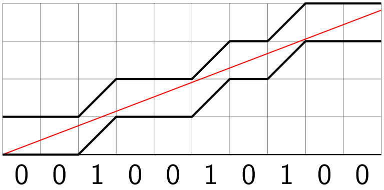

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Theorem [Morse & Hedlund '40]

- $\mathbf{u}$ is Sturmian $\iff$ there are $\alpha, \beta \in [0,1)$, α irrational, such that

$$u_i = \underline{\mathfrak{S}}_{\alpha,\beta}(i)\,, \quad \text{for all} \quad i \geq 0\,, \text{ or } u_i = \overline{\mathfrak{S}}_{\alpha,\beta}(i)\,, \quad \text{for all} \quad i \geq 0\,.$$

Substitutive words

Definition (Substitutive word)

A word $\boldsymbol{u}$ is substitutive iff $\sigma(\boldsymbol{u}) = \boldsymbol{u}$ for a primitive morphism σ.

Primitivity: σ is primitive iff the associated *matrix* M_σ is primitive

$$M_\sigma = \begin{array}{c c} & \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \left[\begin{array}{cc} |\sigma(0)|_0 & |\sigma(0)|_1 \\ |\sigma(1)|_0 & |\sigma(1)|_1 \end{array} \right] \end{array}$$

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its *fixed point* $\boldsymbol{f}_\infty$ can be constructed by iteration

$$\boldsymbol{f}_0 = \mathbf{0},\ \boldsymbol{f}_1 = \mathbf{01},\ \boldsymbol{f}_2 = \mathbf{010},\ \boldsymbol{f}_3 = \mathbf{01001}, \ldots\ \boldsymbol{f}_\infty = \mathbf{0100101001001}\ldots.$$

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Reminder for CFEs

$$\alpha = [a_1, a_2, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

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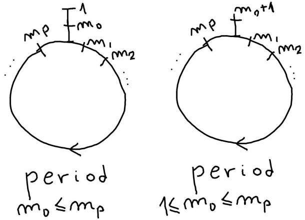
Theorem (Characterization by continued fractions)

The Sturmian word $\underline{\mathfrak{S}}(\alpha, \alpha)$ *is substitutive*

$$\Longleftrightarrow$$

α *is qi and preperiod is of form given here.*



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$$R_{\mathbf{u}}(n) \geq n + p_{\mathbf{u}}(n) - 1.$$

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- Inequality relating the functions,

$$R_{\mathbf{u}}(n) \geq \underbrace{n}_{\text{first factor}} + \underbrace{p_{\mathbf{u}}(n) - 1}_{\substack{\text{count } +1 \\ \text{for every other factor}}}\ .$$

Recurrence of Sturmian words: a link to arithmetic

Theorem (Morse, Hedlund, 1940)

The recurrence function is piecewise affine and satisfies

$$R_\alpha(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha)\,, \qquad \text{for } q_{k-1}(\alpha) \le n < q_k(\alpha).$$

Truncating the expansion at depth k we get a convergent

$$\frac{p_k(\alpha)}{q_k(\alpha)} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \frac{1}{a_k}}}\,.$$

The denominators $q_k(\alpha)$ are called the continuants of α and

$$q_{k+1}(\alpha) = a_{k+1} q_k(\alpha) + q_{k-1}(\alpha)\,.$$

Recurrence quotient

$$S(\alpha, n) := \frac{R_\alpha(n) + 1}{n} = 1 + \frac{q_{k-1}(\alpha) + q_k(\alpha)}{n}\,, \quad q_{k-1}(\alpha) \leq n < q_k(\alpha)\,.$$

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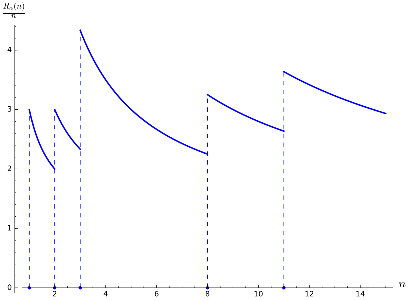
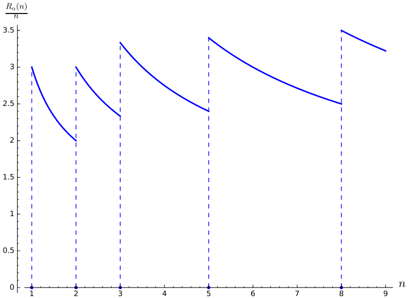


Recurrence quotient $\alpha = e^{-1}$.



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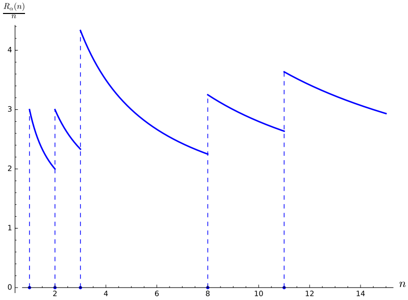

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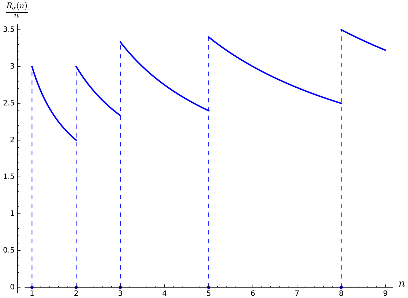

Recurrence quotient $\alpha = \phi^{-2}$.

Shape depends strongly on α and position of n within $[q_{k-1}, q_k)$:

- **Worst case.** On left $S(\alpha, q_{k-1}) = 2 + a_k + O(1/a_k)$.
- **Best case.** On right $S(\alpha, q_k - 1) = 2 + O(1/a_k)$.

Studies of the recurrence function

- Previous studies of $R_\alpha(n)$ give
  - information about extreme cases.
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$$\limsup_{n\to\infty} \frac{S(\alpha, n)}{\log n} = \infty\,, \quad \lim_{n\to\infty} \frac{S(\alpha, n)}{(\log n)^{1+\epsilon}} = 0\,.$$

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- In our probabilistic setting we
  - fix an integer n (we want $n \to \infty$ ...)
  - pick an irrational α uniformly at random from
    (1) the "generic" reals from $[0,1]$
    (2) quadratic irrationals of *"size"* $\leq D$ and let $D \to \infty$.
  - study expectations $\mathbb{E}_\alpha[S(\alpha,n)]$, distributions $\mathbb{P}_\alpha\left(S(\alpha,n) \leq \lambda\right)$

Theorem (uniform $\alpha \in (0,1)$, [R.,Vallée,17])

The random variable $\alpha \mapsto S(\alpha, n)$ admits a limiting distribution

$$\lim_{n\to\infty} \mathbb{P}(\alpha : S(\alpha,n) \leq \lambda) = \int_{[2,\lambda]} g(y)dy\,,$$

for $\lambda \geq 2$ (and 0 otherwise), where the density g equals

$$g(\lambda) = \begin{cases} \frac{12}{\pi^2}\frac{1}{\lambda-1}\log(1+\frac{\lambda-2}{1}) & \textit{if } \lambda \in [2,3] \\ \frac{12}{\pi^2}\frac{1}{\lambda-1}\log(1+\frac{1}{\lambda-2}) & \textit{if } \lambda \in [3,\infty) \end{cases}.$$

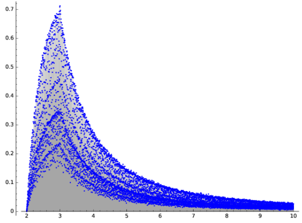

Figure: Histogram with $\epsilon = 1/n$.

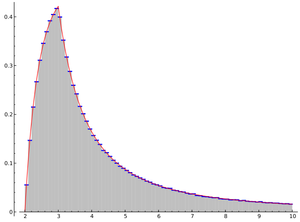

Figure: Limit density.

For $q_{k-1}(\alpha) \leq n < q_k(\alpha)$,

$$S(\alpha, n) = 1 + \tfrac{q_{k-1}(\alpha)+q_k(\alpha)}{n} = 1 + \tfrac{q_k(\alpha)}{n}\left(\tfrac{q_{k-1}(\alpha)}{q_k(\alpha)} + 1\right)$$
$$= f\left(\tfrac{q_{k-1}(\alpha)}{q_k(\alpha)}, \tfrac{q_k(\alpha)}{n}\right),$$

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$$f(x, y) = 1 + y(1 + x), \quad (x, y) \in \mathcal{D} := \{(x, y) \in \mathbb{R}_{\geq 0} \colon xy \leq 1 < y\}.$$

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Limit distribution for $\alpha \mapsto S(\alpha, n)$ (+ more general class) given by

$$\lim_{n\to\infty} \mathbb{P}\left(\alpha : S(\alpha, n) \leq \lambda\right) = \frac{6}{\pi^2} \iint_{\mathcal{D}_\lambda} \omega(x,y)dxdy\,,$$

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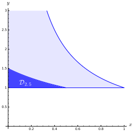
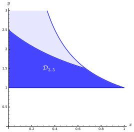


The domain $\mathcal{D}_\lambda$
$\lambda = 2.5, \quad \lambda = 3.5.$



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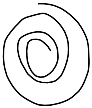
Thus fix $\alpha = \left[\underline{m_1, \ldots, m_p}\right]$, i.e., period $(m_1, \ldots, m_p)$.

Definition (ℓ-th tour)

The ℓ-th tour of α is the interval

$$\varGamma_\ell(\alpha) := \left(q_{\ell p}(\alpha), q_{(\ell+1)p}(\alpha)\right].$$



Theorem (Rescaling of the tours)

Fix $\alpha = \left[\underline{m_1, \ldots, m_p}\right]$, *i.e., period* $(m_1, \ldots, m_p)$.

Then for every fixed r *the following limit exists*

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furthermore convergence is exponential in ℓ.

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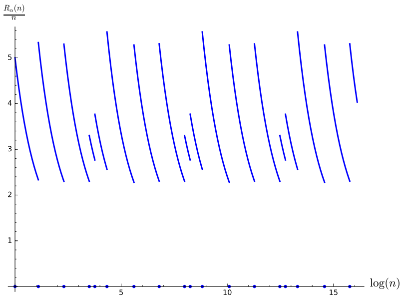

Figure: Logarithmic plot of the recurrence quotient $S(\alpha, n)$ for $\alpha = [\underline{3, 3, 3, 1, 1}] = \frac{5\sqrt{317} - 63}{86}$

Model for quadratic irrationals

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$$\mathcal{S}_D := \Big\{ \alpha \text{ quadratic irrational } : \varrho(\alpha) \leq D \Big\} ,$$

- Restriction to ℓ-th tour $\mathit{\Gamma}_\ell(\alpha)$

$$S_\ell(\alpha, n) = [\![n \in \mathit{\Gamma}_\ell(\alpha)]\!] \, S(\alpha, n) \, .$$

Main result for substitutive Sturmian words

For quadratic irrationals

probabilities are discrete and defined from

$$R_D(\ell, \lambda) := \Big\{(\alpha, n) : \alpha \in \mathcal{S}_D,\ n \in \mathit{\Gamma}_\ell(\alpha),\ S(\alpha, n) \leq \lambda\Big\},$$

Main result (R.,Vallée,19)

Limit distribution for $\alpha \mapsto S(\alpha, n)$ over quadratic irrationals

$$\lim_{D,u,\ell\to\infty} \frac{\Big|R_D(\ell,\lambda) \cap \{\frac{n}{q_{\ell p}} \in (u, \theta u)\}\Big|}{(\log D) \cdot |\mathcal{S}_D| \cdot u \cdot (\theta - 1)} = \frac{6}{\pi^2} \iint_{\mathcal{D}_\lambda} \omega(x, y) dx dy\,,$$

$\mathcal{D}_\lambda = \{(x, y) \in \mathcal{D} : f(x, y) \leq \lambda\}, \qquad \omega(x, y) = \frac{2}{y(1+x)}.$

An analytic "dictionary"

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$$g(x) := \cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_k + x}}}$$

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- and $(\mathbf{I} - \mathbf{H}_s)^{-1} = \mathbf{I} + \mathbf{H}_s + \mathbf{H}_s^2 + \ldots$ describes *all* prefixes.

Origin of the transfer operator $\mathbf{H}_s$

For the Gauss map $T\colon x = [a_1, \ldots] \mapsto \{\frac{1}{x}\} = [a_2, \ldots]$,

Question: If $f \in \mathcal{C}^0(\mathcal{I})$ were the density of $x \Longrightarrow$ density of $T(x)$?

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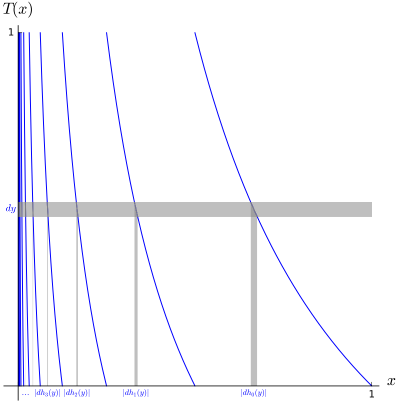

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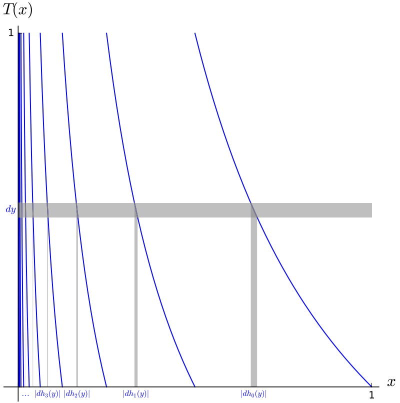

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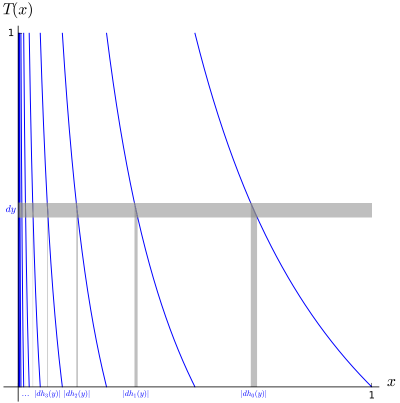

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$\Longrightarrow$ Transfer operator $\mathbf{H}_s$ extends $\mathbf{H}$, introducing a variable s

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$$G(x) := \sum \lambda_k g(\mu_k x) \implies G^*(\rho) = \left(\sum \lambda_k \mu_k^{-\rho}\right) \cdot g^*(\rho)\,.$$

Inverse branches and mirrors

For $\alpha = [m_1, m_2, \ldots]$, let $\mathsf{Seq}(\alpha)$ be the set of prefix homographies

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$$\hat{g}(0) = \tfrac{q(b(g))}{q(g)}, \qquad \hat{g}'(0) = g'(0).$$
- For $\mathcal{D}_\lambda$, set $A_g(t, n, \lambda) = \mathbf{1}_{\mathcal{D}_\lambda}\left(\hat{g}(t), \frac{|\hat{g}'(t)|^{-1/2}}{n}\right)$,
$$[\![\, \Lambda(\alpha, n) \le \lambda]\!] = \sum_{g \in \mathsf{Seq}(\alpha)} A_g(0, n, \lambda).$$

Scheme for the proof : 3 steps

1. Set up the target GFs:

$$S_\ell(s) := \sum_{h \in \mathcal{H}^+} \epsilon(h)^{-s} C_\ell(h)\,, \quad C_\ell(h) = \sum_{g : h^\ell \preceq g \prec h^{\ell+1}} A_g(0, \tfrac{q[h^\ell]}{u}, \lambda)\,,$$

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3. As $\ell \to \infty$ generating function (of Mellin transforms!) related to *trace* of operators

$$f \mapsto \mathbf{H}_{(s+\rho)/2}(I - \mathbf{H}_{(s+\rho)/2})^{-1}\left[L_{\lambda,\rho} \cdot (I - \mathbf{H}_{s/2})^{-1}[f]\right].$$

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Transform expressed in terms of *suffix* operator

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Consider the GF for $\ell = \infty$

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- Morever, for the $n \to \infty$ model, the density for α can be much more general. $\Rightarrow$ Independence between p_k/q_k and q_{k-1}/q_k.

Open questions

- Eliminate averaging of the cost?

$$\lim_{D,u,\ell\to\infty} \frac{\left| R_D(\ell,\lambda) \cap \cancel{\{\frac{n}{q_{\ell p}} \in u\cdot(1,\theta)\}} \right|}{(\log D)\cdot|\mathcal{S}_D|\cdot \cancel{u\cdot(\theta-1)}} = \frac{6}{\pi^2}\iint_{\mathcal{D}_\lambda}\omega(x,y)dxdy\,?$$

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- Possible to study generic α case with Mellin transform

  $\Rightarrow$ but we need a similar averaging !

Thank you!