Recurrence of substitutive Sturmian words

A probabilistic study

Pablo Rotondo
LITIS, Université de Rouen

Joint work with
Brigitte Vallée

Réunion du projet CODYS,
Context

- Probabilistic analysis

Object/experiment/execution?
⇒ Models, averages, distribution?
Context

- Probabilistic analysis

Object/experiment/execution?
⇒ Models, averages, distribution?

- Word Combinatorics

Study of *words*
⇒ subwords (factors), frequencies

**Thue-Morse**

\[ \sigma : 0 \mapsto 01, \, 1 \mapsto 10 \]

01101001...
Context

Probabilistic analysis

Object/experiment/execution?
⇒ Models, averages, distribution?

Word Combinatorics

Study of words
⇒ subwords (factors), frequencies

Sturmian Words
simplest not eventually periodic.
⇒ recurrence: worst case, average?

Thue-Morse
σ: 0 ↦ 01, 1 ↦ 10
0110100101101001...
Plan of the talk

1. Sturmian words
   - General Sturmian words
   - Substitutive words

2. Recurrence function
   - Definition and classical results
   - Our models and results for generic words

3. Substitutive Sturmian words
   - Model for quadratic irrationals
   - Main result

4. Toolbox for the proofs

5. Conclusion
Complexity and Sturmian words

Definition

Complexity function of an infinite word $u \in \mathcal{A}^\mathbb{N}$

$$p_u : \mathbb{N} \rightarrow \mathbb{N}, \quad p_u(n) = \#\{\text{factors of length } n \text{ in } u\}.$$
Complexity and Sturmian words

Definition

Complexity function of an infinite word \( u \in A^\mathbb{N} \)

\[ p_u : \mathbb{N} \rightarrow \mathbb{N}, \quad p_u(n) = \# \{ \text{factors of length } n \text{ in } u \}. \]

Important property

\( u \in A^\mathbb{N} \) is not eventually periodic

\[ \iff \quad p_u(n + 1) > p_u(n) \text{ for all } n \in \mathbb{N} \]
Complexity and Sturmian words

Definition

Complexity function of an infinite word $u \in \mathcal{A}^\mathbb{N}$

$$p_u : \mathbb{N} \to \mathbb{N}, \quad p_u(n) = \#\{\text{factors of length } n \text{ in } u\}.$$  

Important property

$u \in \mathcal{A}^\mathbb{N}$ is not eventually periodic

$\iff p_u(n + 1) > p_u(n)$ for all $n \in \mathbb{N}$

$\implies p_u(n) \geq n + 1$.

Sturmian words are the “simplest” that are not eventually periodic.
Complexity and Sturmian words

**Definition**

**Complexity function** of an infinite word \( u \in \mathcal{A}^\mathbb{N} \)

\[
p_u : \mathbb{N} \to \mathbb{N}, \quad p_u(n) = \#\{ \text{factors of length } n \text{ in } u \}.
\]

**Important property**

\( u \in \mathcal{A}^\mathbb{N} \) is not eventually periodic

\[\iff p_u(n + 1) > p_u(n) \text{ for all } n \in \mathbb{N} \]

\[\implies p_u(n) \geq n + 1.\]

Sturmian words are the “simplest” that are not eventually periodic.

**Definition**

\( u \in \{0, 1\}^\mathbb{N} \) is Sturmian \(\iff p_u(n) = n + 1 \) for each \( n \geq 0 \).
Explicit construction (discrete coding)

Given $\alpha, \beta \in [0, 1)$ we define

\[
\mathcal{G}_{\alpha, \beta}(n) = \lfloor (n + 1) \alpha + \beta \rfloor - \lfloor n \alpha + \beta \rfloor,
\]

\[
\overline{\mathcal{G}}_{\alpha, \beta}(n) = \lceil (n + 1) \alpha + \beta \rceil - \lceil n \alpha + \beta \rceil,
\]

for $n \geq 0$.

Figure: Sequences $\mathcal{G}_{\alpha, \beta}$ and $\overline{\mathcal{G}}_{\alpha, \beta}$ are discrete codings of $y = \alpha x + \beta$. 

Theorem [Morse & Hedlund ’40]

\[ u \text{ is Sturmian} \Leftrightarrow \text{there are } \alpha, \beta \in [0, 1), \alpha \text{ irrational, such that } u_i = \mathcal{G}_{\alpha, \beta}(i), \text{for all } i \geq 0, \text{ or } u_i = \overline{\mathcal{G}}_{\alpha, \beta}(i), \text{ for all } i \geq 0. \]
Explicit construction (discrete coding)

Given $\alpha, \beta \in [0, 1)$ we define

$$\mathcal{G}_{\alpha,\beta}(n) = \lfloor (n + 1) \alpha + \beta \rfloor - \lfloor n \alpha + \beta \rfloor,$$

$$\overline{\mathcal{G}}_{\alpha,\beta}(n) = \lceil (n + 1) \alpha + \beta \rceil - \lceil n \alpha + \beta \rceil,$$

for $n \geq 0$.

Figure: Sequences $\mathcal{G}_{\alpha,\beta}$ and $\overline{\mathcal{G}}_{\alpha,\beta}$ are discrete codings of $y = \alpha x + \beta$.

Theorem [Morse & Hedlund ’40]

$\blacktriangleright$ $u$ is Sturmian $\iff$ there are $\alpha, \beta \in [0, 1)$, $\alpha$ irrational, such that

$u_i = \mathcal{G}_{\alpha,\beta}(i)$, for all $i \geq 0$, or $u_i = \overline{\mathcal{G}}_{\alpha,\beta}(i)$, for all $i \geq 0$. 
Substitutive words

Definition (Substitutive word)
A word $u$ is substitutive iff $\sigma(u) = u$ for a primitive morphism $\sigma$.

Primitivity: $\sigma$ is primitive iff the associated matrix $M_\sigma$ is primitive

$$M_\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$
Substitutive words

Definition (Substitutive word)

A word $u$ is substitutive iff $\sigma(u) = u$ for a primitive morphism $\sigma$.

Primitivity: $\sigma$ is primitive iff the associated matrix $M_\sigma$ is primitive

$$M_\sigma = \begin{bmatrix} 0 & 1 \\ 1 & \sigma(1) \\ \sigma(0) & \sigma(0) \\ \sigma(1) & \sigma(1) \end{bmatrix}$$

Example (Fibonacci word)

Consider the primitive morphism $\sigma : 0 \mapsto 01, 1 \mapsto 0$

$$M_\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$
Substitutive words

Definition (Substitutive word)
A word $u$ is substitutive iff $\sigma(u) = u$ for a primitive morphism $\sigma$.

Primitivity: $\sigma$ is primitive iff the associated matrix $M_\sigma$ is primitive

$$M_\sigma = \begin{bmatrix} 0 & 1 \\ \sigma(0) & \sigma(1) \end{bmatrix}$$

Example (Fibonacci word)
Consider the primitive morphism $\sigma : 0 \mapsto 01, 1 \mapsto 0$

$$M_\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

its fixed point $f_\infty$ can be constructed by iteration

$f_0 = 0, f_1 = 01, f_2 = 010, f_3 = 01001, \ldots f_\infty = 0100101001001 \ldots$
Substitutive *Sturmian* words

Slope $\alpha$ of a substitutive Sturmian word is a quadratic irrational,

(1) Solution of a quadratic equation over $\mathbb{Z}$ determined by $M_\sigma$.
Substitutive *Sturmian* words

Slope $\alpha$ of a *substitutive* Sturmian word is a quadratic irrational,

1. Solution of a quadratic equation over $\mathbb{Z}$ determined by $M_\sigma$.
2. Eventually periodic *continued fraction expansion (CFE)*.
Substitutive *Sturmian* words

Slope $\alpha$ of a *substitutive* Sturmian word is a **quadratic irrational**, 

1. Solution of a quadratic equation over $\mathbb{Z}$ determined by $M_\sigma$.
2. Eventually periodic **continued fraction expansion (CFE)**.

**Reminder** for CFEs

$$\alpha = [a_1, a_2, \ldots] := \frac{1}{a_1 + \frac{1}{a_2 + \ddots}}$$

where $a_1, a_2, \ldots \in \mathbb{Z}_{>0}$ are called the **quotients**.
Substitutive *Sturmian* words

Slope $\alpha$ of a *substitutive* Sturmian word is a *quadratic irrational*,

(1) Solution of a quadratic equation over $\mathbb{Z}$ determined by $M_\sigma$.
(2) Eventually periodic *continued fraction expansion* (CFE).

**Reminder** for CFEs

$$\alpha = [a_1, a_2, \ldots] := \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$  

where $a_1, a_2, \ldots \in \mathbb{Z}_{>0}$ are called the *quotients*.

**Theorem (Characterization by continued fractions)**

*The* Sturmian word $\mathcal{S}(\alpha, \alpha)$ is *substitutive*  

$\Longleftrightarrow$  

$\alpha$ is qi and preperiod is of form given here.
Recurrence

Definition (Recurrence function)

Consider an infinite word $u$. Its recurrence function is:

$$R_u(n) = \inf \{ m \in \mathbb{N} : \text{every factor of length } m \text{ contains all the factors of length } n \}.$$
Recurrence

Definition (Recurrence function)

Consider an infinite word $u$. Its recurrence function is:

$$ R_u(n) = \inf \{ m \in \mathbb{N} : \text{every factor of length } m \text{ contains all the factors of length } n \}.$$ 

▶ Cost we have to pay to discover the factors if we start from an arbitrary point in $u = u_1u_2\ldots$
### Recurrence

#### Definition (Recurrence function)

Consider an infinite word \( u \). Its recurrence function is:

\[
R_u(n) = \inf \{ m \in \mathbb{N} : \text{every factor of length } m \text{ contains all the factors of length } n \}.
\]

- **Cost** we have to pay to discover the factors if we start from an arbitrary point in \( u = u_1 u_2 \ldots \)

- **Inequality** relating the functions,

\[
R_u(n) \geq n + p_u(n) - 1.
\]
Recurrence

Definition (Recurrence function)

Consider an infinite word $u$. Its recurrence function is:

$$R_u(n) = \inf \{ m \in \mathbb{N} : \text{every factor of length } m \text{ contains all the factors of length } n \}.$$

- **Cost** we have to pay to discover the factors if we start from an arbitrary point in $u = u_1u_2 \ldots$

- Inequality relating the functions,

$$R_u(n) \geq n + p_u(n) - 1.$$

  - for every other factor

  - first factor

  - count +1
Recurrence of Sturmian words: a link to arithmetic

Theorem (Morse, Hedlund, 1940)

The recurrence function is piecewise affine and satisfies

\[ R_\alpha(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha), \quad \text{for } q_{k-1}(\alpha) \leq n < q_k(\alpha). \]

Truncating the expansion at depth \( k \) we get a convergent

\[ \frac{p_k(\alpha)}{q_k(\alpha)} = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_k}}} = \frac{1}{1 + \frac{1}{\frac{1}{a_1} + \cdots + \frac{1}{a_k}}}. \]

The denominators \( q_k(\alpha) \) are called the continuants of \( \alpha \) and

\[ q_{k+1}(\alpha) = a_{k+1}q_k(\alpha) + q_{k-1}(\alpha). \]
Recurrence quotient

\[ S(\alpha, n) := \frac{R_\alpha(n) + 1}{n} = 1 + \frac{q_{k-1}(\alpha) + q_k(\alpha)}{n}, \quad q_{k-1}(\alpha) \leq n < q_k(\alpha). \]
Recurrence quotient

\[ S(\alpha, n) := \frac{R_\alpha(n) + 1}{n} = 1 + \frac{q_{k-1}(\alpha) + q_k(\alpha)}{n}, \quad q_{k-1}(\alpha) \leq n < q_k(\alpha). \]

Recurrence quotient \( \alpha = e^{-1} \).

Recurrence quotient \( \alpha = \phi^{-2} \).
Recurrence quotient

\[ S(\alpha, n) := \frac{R_\alpha(n) + 1}{n} = 1 + \frac{q_{k-1}(\alpha) + q_k(\alpha)}{n}, \quad q_{k-1}(\alpha) \leq n < q_k(\alpha). \]

Recurrence quotient \( \alpha = e^{-1} \).

Shape depends strongly on \( \alpha \) and position of \( n \) within \([q_{k-1}, q_k)\):

- **Worst case.** On left \( S(\alpha, q_{k-1}) = 2 + a_k + O(1/a_k) \).
- **Best case.** On right \( S(\alpha, q_k - 1) = 2 + O(1/a_k) \).
Studies of the recurrence function

- Previous studies of $R_\alpha(n)$ give
  - information about extreme cases.
  - results for almost all $\alpha$. 
Studies of the recurrence function

- Previous studies of $R_\alpha(n)$ give
  - information about extreme cases.
  - results for almost all $\alpha$.

**Thm** (Morse & Hedlund ’40) $\forall \epsilon > 0$, for almost every $\alpha$

$$\limsup_{n \to \infty} \frac{S(\alpha, n)}{\log n} = \infty, \quad \lim_{n \to \infty} \frac{S(\alpha, n)}{(\log n)^{1+\epsilon}} = 0.$$
Studies of the recurrence function

- Previous studies of $R_\alpha(n)$ give
  - information about extreme cases.
  - results for almost all $\alpha$.

**Thm** (Morse & Hedlund ’40) $\forall \epsilon > 0$, for almost every $\alpha$

$$
\limsup_{n \to \infty} \frac{S(\alpha, n)}{\log n} = \infty, \quad \lim_{n \to \infty} \frac{S(\alpha, n)}{(\log n)^{1+\epsilon}} = 0.
$$

- In our probabilistic setting we
  - fix an integer $n$ (we want $n \to \infty$ ...)
  - pick an irrational $\alpha$ uniformly at random from
    (1) the “generic” reals from $[0, 1]$
    (2) quadratic irrationals of “size” $\leq D$ and let $D \to \infty$.
  - study expectations $\mathbb{E}_\alpha[S(\alpha, n)]$, distributions $\mathbb{P}_\alpha(S(\alpha, n) \leq \lambda)$
Theorem (uniform $\alpha \in (0, 1)$, [R., Vallée, 17])

The random variable $\alpha \mapsto S(\alpha, n)$ admits a limiting distribution

$$\lim_{n \to \infty} P(\alpha : S(\alpha, n) \leq \lambda) = \int_{[2, \lambda]} g(y) dy,$$

for $\lambda \geq 2$ (and 0 otherwise), where the density $g$ equals

$$g(\lambda) = \begin{cases} 
\frac{12}{\pi^2} \frac{1}{\lambda-1} \log(1 + \frac{\lambda-2}{1}) & \text{if } \lambda \in [2, 3] \\
\frac{12}{\pi^2} \frac{1}{\lambda-1} \log(1 + \frac{1}{\lambda-2}) & \text{if } \lambda \in [3, \infty) \end{cases}.$$

Figure: Histogram with $\epsilon = 1/n$.  
Figure: Limit density.
For $q_{k-1}(\alpha) \leq n < q_k(\alpha)$,

$$S(\alpha, n) = 1 + \frac{q_{k-1}(\alpha)+q_k(\alpha)}{n} = 1 + \frac{q_k(\alpha)}{n} \left( \frac{q_{k-1}(\alpha)}{q_k(\alpha)} + 1 \right) = f \left( \frac{q_{k-1}(\alpha)}{q_k(\alpha)}, \frac{q_k(\alpha)}{n} \right),$$

with

$$f(x, y) = 1 + y(1 + x), \ (x, y) \in \mathcal{D} := \{(x, y) \in \mathbb{R}_{\geq 0} : xy \leq 1 < y\}.$$
For $q_{k-1}(\alpha) \leq n < q_k(\alpha)$,

$$S(\alpha, n) = 1 + \frac{q_{k-1}(\alpha) + q_k(\alpha)}{n} = 1 + \frac{q_k(\alpha)}{n} \left( \frac{q_{k-1}(\alpha)}{q_k(\alpha)} + 1 \right) = f \left( \frac{q_{k-1}(\alpha)}{q_k(\alpha)}, \frac{q_k(\alpha)}{n} \right),$$

with

$$f(x, y) = 1 + y(1 + x), \ (x, y) \in D := \{ (x, y) \in \mathbb{R}_{\geq 0} : xy \leq 1 < y \}.$$ 

**Theorem (uniform $\alpha \in (0, 1)$, [R., Vallée, 17])**

*Limit distribution for $\alpha \mapsto S(\alpha, n)$ (+ more general class) given by*

$$\lim_{n \to \infty} \mathbb{P}(\alpha : S(\alpha, n) \leq \lambda) = \frac{6}{\pi^2} \iint_{D_\lambda} \omega(x, y) dx dy,$$

where

$$D_\lambda = \{ (x, y) \in D : f(x, y) \leq \lambda \}, \quad \omega(x, y) = \frac{2}{y(1+x)}.$$ 

The domain $D_\lambda$

$\lambda = 2.5, \ \lambda = 3.5.$
Simplifying assumptions for the talk.

- Slopes $\alpha$ that are reduced quadratic irrationals, i.e., corresponding to purely periodic expansions.
- Periods may be *primitive* or not. Here we omit this detail.
Simplifying assumptions for the talk.

- Slopes \( \alpha \) that are reduced quadratic irrationals, i.e., corresponding to purely periodic expansions.

- Periods may be primitive or not. Here we omit this detail.

Thus fix \( \alpha = \left[ m_1, \ldots, m_p \right] \), i.e., period \( (m_1, \ldots, m_p) \).
Simplifying assumptions for the talk.

- **Slopes** $\alpha$ that are reduced quadratic irrationals, i.e., corresponding to purely periodic expansions.
- Periods may be *primitive* or not. Here we omit this detail.

Thus fix $\alpha = \left[ m_1, \ldots, m_p \right]$, i.e., period $(m_1, \ldots, m_p)$.

**Definition ($\ell$-th tour)**

The $\ell$-th tour of $\alpha$ is the interval

$$\Gamma_\ell(\alpha) := \left( q_\ell p(\alpha), q_{\ell+1} p(\alpha) \right).$$
Theorem (Rescaling of the tours)

Fix \( \alpha = \left[ m_1, \ldots, m_p \right] \), i.e., period \( (m_1, \ldots, m_p) \).

Then for every fixed \( r \) the following limit exists

\[
Q_r(\alpha) := \lim_{\ell \to \infty} \frac{q_{\ell p+r}(\alpha)}{q_{\ell p}(\alpha)},
\]

furthermore convergence is exponential in \( \ell \).
Theorem (Rescaling of the tours)

Fix $\alpha = \left[ m_1, \ldots, m_p \right]$, i.e., period $(m_1, \ldots, m_p)$.

Then for every fixed $r$ the following limit exists

$$Q_r(\alpha) := \lim_{\ell \to \infty} \frac{q_{\ell p + r}(\alpha)}{q_{\ell p}(\alpha)},$$

furthermore convergence is exponential in $\ell$.

**Figure:** Logarithmic plot of the recurrence quotient $S(\alpha, n)$ for $\alpha = [3, 3, 3, 1, 1] = \frac{5 \sqrt{317} - 63}{86}$
Model for quadratic irrationals

Quadratic irrationals present two \textit{striking features}

- \textbf{Countable} and dense subset of \([0, 1]\).
- \textbf{Periodic structure} (after re-scaling) with respect to tours.
Model for quadratic irrationals

Quadratic irrationals present two striking features

- **Countable** and dense subset of $[0, 1]$.
- **Periodic structure** (after re-scaling) with respect to tours.

Model takes these into account

- **Pick $\alpha$ uniformly at random** from the finite

\[
S_D := \left\{ \alpha \text{ quadratic irrational} : \varrho(\alpha) \leq D \right\},
\]
Model for quadratic irrationals

Quadratic irrationals present two striking features

- Countable and dense subset of \([0, 1]\).
- Periodic structure (after re-scaling) with respect to tours.

Model takes these into account

- Pick \(\alpha\) uniformly at random from the finite

\[
S_D := \left\{ \alpha \text{ quadratic irrational} : \rho(\alpha) \leq D \right\},
\]

- Restriction to \(\ell\)-th tour \(\Gamma_\ell(\alpha)\)

\[
S_\ell(\alpha, n) = \left\lfloor n \in \Gamma_\ell(\alpha) \right\rfloor S(\alpha, n).
\]
Main result for substitutive Sturmian words

For quadratic irrationals, probabilities are discrete and defined from

\[ R_D(\ell, \lambda) := \left\{ (\alpha, n) : \alpha \in S_D, n \in \Gamma_\ell(\alpha), S(\alpha, n) \leq \lambda \right\}, \]

Main result (R., Vallée, 19)

Limit distribution for \( \alpha \mapsto S(\alpha, n) \) over quadratic irrationals

\[
\lim_{D, u, \ell \to \infty} \frac{\left| R_D(\ell, \lambda) \cap \left\{ \frac{n}{q_{\ell p}} \in (u, \theta u) \right\} \right|}{(\log D) \cdot |S_D| \cdot u \cdot (\theta - 1)} = \frac{6}{\pi^2} \int \int_{D_\lambda} \omega(x, y) dxdy, \\
D_\lambda = \{(x, y) \in D : f(x, y) \leq \lambda\}, \quad \omega(x, y) = \frac{2}{y(1+x)}.
\]
An analytic “dictionary”

A prefix \((m_1, \ldots, m_k)\) of the CFE defines an homography \(g \in \mathcal{H}^k\)

\[
g(x) := \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_k + x}_k}}
\]

associated with an operator, its generating function,

\[
\mathcal{H}_{[g],s}[f](x) := |g'(x)|^{s/2} f(g(x)).
\]
An analytic “dictionary”

A prefix \((m_1, \ldots, m_k)\) of the CFE defines an homography \(g \in \mathcal{H}^k\)

\[
g(x) := \frac{1}{m_1 + \frac{1}{1} + \frac{1}{m_k + x}}
\]

associated with an operator, its generating function,

\[
H_{[g],s}[f](x) := \left|g'(x)\right|^{s/2} f(g(x)).
\]

Generating functions.

\(\mathbf{H}_s := \sum_{g \in \mathcal{H}} H_{[g],s}\) describes all prefixes of depth 1.
An analytic “dictionary”

A prefix \((m_1, \ldots, m_k)\) of the CFE defines an homography \(g \in \mathcal{H}^k\)

\[
g(x) := \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_k + x}}}\]

associated with an operator, its generating function,

\[
H_{[g],s}[f](x) := |g'(x)|^{s/2} f(g(x)) .
\]

Generating functions.

\(H_s := \sum_{g \in \mathcal{H}} H_{[g],s}\) describes all prefixes of depth 1.

\(H^k_s = H_s \circ \cdots \circ H_s\) describes all prefixes of depth \(k\).
An analytic “dictionary”

A prefix \((m_1, \ldots, m_k)\) of the CFE defines an homography \(g \in \mathcal{H}^k\)

\[
g(x) := \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_k + x}}}
\]

associated with an operator, its generating function,

\[
H[g, s][f](x) := |g'(x)|^{s/2} f(g(x)).
\]

Generating functions.

\(\quad\quad\quad\)

\[\begin{align*}
\text{\(H_s := \sum_{g \in \mathcal{H}} H[g, s]\) describes all prefixes of depth 1.}\n\text{\(H_k^s = H_s \circ \cdots \circ H_s\) describes all prefixes of depth \(k\).}\n\text{\(and \((I - H_s)^{-1} = I + H_s + H_s^2 + \ldots\) describes all prefixes.}\n\end{align*}\]
Origin of the transfer operator $H_s$

For the Gauss map $T: x = [a_1, \ldots] \mapsto \{\frac{1}{x}\} = [a_2, \ldots]$.

Question: If $f \in C^0(\mathcal{I})$ were the density of $x \mapsto$ density of $T(x)$?
Origin of the transfer operator $H_s$

For the Gauss map $T: x = [a_1, \ldots] \mapsto \{\frac{1}{x}\} = [a_2, \ldots]$.

Question: If $f \in C^0(\mathcal{I})$ were the density of $x \mapsto$ density of $T(x)$?

Answer: The density is $H[f](x) = \sum_{g \in H} \left| g'(x) \right| s f(g(x)).$
Origin of the transfer operator $H_s$

For the Gauss map $T: x = [a_1, \ldots] \mapsto \{\frac{1}{x}\} = [a_2, \ldots].$

**Question:** If $f \in C^0(I)$ were the density of $x \mapsto$ density of $T(x)$?

**Answer:** The density is

$$H[f](x) = \sum_{g \in \mathcal{H}} |g'(x)| \cdot f(g(x))$$

$$= \sum_{m \geq 0} \frac{1}{(m + x)^2} \cdot f \left( \frac{1}{m + x} \right).$$
Origin of the transfer operator $H_s$

For the Gauss map $T: x = [a_1, \ldots] \mapsto \{\frac{1}{x}\} = [a_2, \ldots]$, 

**Question:** If $f \in C^0(I)$ were the density of $x \mapsto$ density of $T(x)$?

**Answer:** The density is

$$H[f](x) = \sum_{g \in \mathcal{H}} |g'(x)| \; f(g(x))$$

$$= \sum_{m \geq 0} \frac{1}{(m + x)^2} f \left( \frac{1}{m + x} \right).$$

$\implies$ Transfer operator $H_s$ extends $H$, introducing a variable $s$

$$H_s[f](x) = \sum_{g \in \mathcal{H}} |g'(x)|^s \; f(g(x)).$$
Principles of the proofs

Mellin transform of $f : [0, \infty) \rightarrow \mathbb{C}$ is defined by

$$f^*(\rho) := \int_0^\infty f(u)u^{\rho-1}du.$$
Principles of the proofs

**Mellin transform** of \( f : [0, \infty) \rightarrow \mathbb{C} \) is defined by

\[
f^*(\rho) := \int_0^\infty f(u)u^{\rho-1}du.
\]

**Key properties:**

- **Singularities** of Mellin transform transfer into limits.

- Harmonic sums produce generating functions of Dirichlet type
  \[
  G(x) := \sum \lambda_k g(\mu_k x) = \Rightarrow G^*(\rho) = \left(\sum \lambda_k \mu_k - \rho_k\right) \cdot g^*(\rho).
  \]
Principles of the proofs

**Mellin transform** of \( f : [0, \infty) \to \mathbb{C} \) is defined by

\[
f^*(\rho) := \int_0^\infty f(u) u^{\rho-1} \, du.
\]

**Key properties:**

- **Singularities** of Mellin transform transfer into limits.

\[
f(u) \sim \text{Res}[f^*(\rho); s = \sigma] \cdot u^{-\sigma} (1 + O(u^a)) \quad u \to 0^+.
\]
Principles of the proofs

**Mellin transform** of \( f : [0, \infty) \to \mathbb{C} \) is defined by

\[
f^*(\rho) := \int_0^\infty f(u) u^{\rho-1} \, du.
\]

**Key properties:**

▶ **Singularities** of Mellin transform transfer into limits.

\[
f(u) \sim \text{Res}[f^*(\rho); s = \sigma] \cdot u^{-\sigma} \left(1 + O(u^a)\right) , \quad u \to 0^+.
\]

▶ **Harmonic sums** produce generating functions of **Dirichlet type**
Principles of the proofs

Mellin transform of $f : [0, \infty) \to \mathbb{C}$ is defined by

$$f^{\ast}(\rho) := \int_{0}^{\infty} f(u) u^{\rho-1} du.$$  

Key properties:

- **Singularities** of Mellin transform transfer into limits.

  $$f(u) \sim \text{Res}[f^{\ast}(\rho); s = \sigma] \cdot u^{-\sigma} (1 + O(u^{a})) \quad u \to 0^+.$$  

- **Harmonic sums** produce generating functions of *Dirichlet* type

  $$G(x) := \sum \lambda_k g(\mu_k x) \implies G^{\ast}(\rho) = \left( \sum \lambda_k \mu_k^{-\rho} \right) \cdot g^{\ast}(\rho).$$
Inverse branches and mirrors

For \( \alpha = [m_1, m_2, \ldots] \), let \( \text{Seq}(\alpha) \) be the set of prefix homographies

\[
\begin{align*}
g(x) & := \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_k + x}}} , \\
b(g)(x) & := \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_{k-1} + x}}}
\end{align*}
\]

Important properties.

▶ Continuant \( q(g) := q_k(\alpha) \) is just \( |g'(0)| - \frac{1}{2} \).

▶ Mirror \( \hat{g} \) satisfies \( \hat{g}(0) = q(b(g)) q(g) \), \( \hat{g}'(0) = g'(0) \).

▶ For \( D_\lambda \), set \( A_g(t,n,\lambda) = \frac{1}{D_{\hat{g}(t)}(\hat{g}(t), |\hat{g}'(t)| - \frac{1}{2} n)} \) \( \Lambda(\alpha, n) \leq \lambda \) = \( \sum_{g \in \text{Seq}(\alpha)} A_g(0,n,\lambda) \).
Inverse branches and mirrors

For $\alpha = [m_1, m_2, \ldots]$, let $\text{Seq}(\alpha)$ be the set of prefix homographies

$$
g(x) := \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_k + x}}}, \quad b(g)(x) := \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_{k-1} + x}}}.
$$

Important properties.

- **Continuant** $q(g) := q_k(\alpha)$ is just
  $$|g'(0)|^{-1/2}.$$
Inverse branches and mirrors

For $\alpha = [m_1, m_2, \ldots]$, let $\text{Seq}(\alpha)$ be the set of prefix homographies

$$
g(x) := \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_k + x}}} \quad \text{and} \quad b(g)(x) := \frac{1}{m_1 + \frac{1}{\cdots + \frac{1}{m_k - 1 + x}}}
$$

Important properties.

- **Continuant** $q(g) := q_k(\alpha)$ is just
  $$
  |g'(0)|^{-1/2}.
  $$

- **Mirror** $\hat{g}$ satisfies
  $$
  \hat{g}(0) = \frac{q(b(g))}{q(g)} \quad \text{and} \quad \hat{g}'(0) = g'(0).
  $$
Inverse branches and mirrors

For \( \alpha = [m_1, m_2, \ldots] \), let \( \text{Seq}(\alpha) \) be the set of prefix homographies

\[
    g(x) := \frac{1}{m_1 + \frac{1}{m_2 + \frac{1}{\ldots + \frac{1}{m_k + x}}}}, \quad b(g)(x) := \frac{1}{m_1 + \frac{1}{m_2 + \frac{1}{\ldots + \frac{1}{m_{k-1} + x}}}}.
\]

**Important properties.**

- **Continuant** \( q(g) := q_k(\alpha) \) is just

  \[
  |g'(0)|^{-1/2}.
  \]

- **Mirror** \( \hat{g} \) satisfies

  \[
  \hat{g}(0) = \frac{q(b(g))}{q(g)} , \quad \hat{g}'(0) = g'(0).
  \]

- For \( \mathcal{D}_\lambda \), set

  \[
  A_g(t, n, \lambda) = \mathbf{1}_{\mathcal{D}_\lambda} \left( \hat{g}(t), \frac{\hat{g}'(t)|^{-1/2}}{n} \right),
  \]

  \[
  \left[ \Lambda(\alpha, n) \leq \lambda \right] = \sum_{g \in \text{Seq}(\alpha)} A_g(0, n, \lambda).
  \]
Scheme for the proof: 3 steps

1. Set up the target GFs:

\[ S_\ell(s) := \sum_{h \in \mathcal{H}^+} \epsilon(h)^{-s} C_\ell(h), \quad C_\ell(h) = \sum_{g : h^\ell \preceq g \prec h^{\ell+1}} A_g(0, \frac{q[h\ell]}{u}, \lambda), \]

where \( \epsilon(h) := |h'(h^*)|^{-1/2} \) is size of \( h^* = [m_1, \ldots, m_p] \).
Scheme for the proof: 3 steps

1. Set up the target GFs:

\[ S_\ell(s) := \sum_{h \in H^+} \epsilon(h)^{-s} C_\ell(h), \quad C_\ell(h) = \sum_{g : h^\ell \preceq_g \prec h^{\ell+1}} A_g(0, \frac{q[h^\ell]}{u}, \lambda), \]

where \( \epsilon(h) := |h'(h^*)|^{-1/2} \) is size of \( h^* = [m_1, \ldots, m_p] \).

2. To study limit \( u \to 0 \), take Mellin transforms

\[ S_\ell(s, \rho) := \sum_{h \in H^+} \epsilon(h)^{-s} \langle u \mapsto C_\ell(h) \rangle_\rho, \]

the “taking out the harmonics” comes in handy.
Scheme for the proof : 3 steps

1. Set up the target GFs:

\[ S_\ell(s) := \sum_{h \in \mathcal{H}^+} \epsilon(h)^{-s} C_\ell(h), \quad C_\ell(h) = \sum_{g : h^{\ell} \preceq g < h^{\ell+1}} A_g(0, \frac{q[h^{\ell}]}{u}, \lambda), \]

where \( \epsilon(h) := |h'(h^*)|^{-1/2} \) is size of \( h^* = [m_1, \ldots, m_p] \).

2. To study limit \( u \to 0 \), take Mellin transforms

\[ S_\ell(s, \rho) := \sum_{h \in \mathcal{H}^+} \epsilon(h)^{-s} \langle u \mapsto C_\ell(h) \rangle_\rho, \]

the “taking out the harmonics” comes in handy.

3. As \( \ell \to \infty \) generating function (of Mellin transforms!)

related to trace of operators

\[ f \mapsto H_{(s+\rho)/2}(I - H_{(s+\rho)/2})^{-1} \left[ L_{\lambda, \rho} \cdot (I - H_{s/2})^{-1}[f] \right]. \]
Step 2 : Mellin transforms!

Given our cost

\[ C_\ell(h) = \sum_{g: h_\ell \preceq g < h_{\ell+1}} A_g(0, \frac{q[h_\ell]}{u}, \lambda), \]

convenient to write \( g = h_\ell \circ g \), with \( g < h \),

\( \Rightarrow \) Note. \( \hat{h}_\ell(0) \rightarrow \hat{h}_\star \) and \( \epsilon(h) = \epsilon(\hat{h}) \).
Step 2: Mellin transforms!

Given our cost

\[ C_\ell(h) = \sum_{g : h^\ell \preceq g < h^{\ell+1}} A_g(0, \frac{q[h^\ell]}{u}, \lambda), \]

convenient to write \( g = h^\ell \circ g \), with \( g < h \), then

\[ A_g \left( 0, \frac{q[h^\ell]}{u}, \lambda \right) = A_{\hat{g}} \left( \hat{h}^\ell(0), \frac{1}{u}, \lambda \right). \]
Step 2: Mellin transforms!

Given our cost

\[ C_\ell(h) = \sum_{g : h_\ell \preceq g < h_{\ell+1}} A_g(0, \frac{q[h_\ell]}{u}, \lambda), \]

convenient to write \( g = h_\ell \circ g \), with \( g < h \), then

\[ A_g \left( 0, \frac{q[h_\ell]}{u}, \lambda \right) = \hat{A}_\hat{g} \left( \hat{h}_\ell(0), \frac{1}{u}, \lambda \right). \]

Transform expressed in terms of suffix operator

\[ \langle C_\ell(h) \rangle_\rho = \sum_{v \text{ suffix of } \hat{h}} \sum_{v \in \mathcal{H}^+} \left\langle u \mapsto A_v(\hat{h}_\ell(0), \frac{1}{u}, \lambda) \right\rangle_\rho \]

\[ = \sum_{v \text{ suffix of } \hat{h}} |v'(y)|^{\rho/2} L_{\lambda, \rho}(y) = G_{\hat{h}, \rho/2}[L_{\lambda, \rho}](y), \]

where \( y = \hat{h}_\ell(0) \).
Step 2 : Mellin transforms!

Given our cost

\[ C_\ell(h) = \sum_{g: h_\ell \leq g < h_\ell + 1} A_g(0, \frac{q[h_\ell]}{u}, \lambda), \]

convenient to write \( g = h_\ell \circ g, \) with \( g < h, \) then

\[ A_g \left( 0, \frac{q[h_\ell]}{u}, \lambda \right) = A_{\hat{g}} \left( \hat{h}_\ell(0), \frac{1}{u}, \lambda \right). \]

Transform expressed in terms of suffix operator

\[
\langle C_\ell(h) \rangle_\rho = \sum_{v \text{ suffix of } \hat{h}} \langle u \mapsto A_v(\hat{h}_\ell(0), \frac{1}{u}, \lambda) \rangle_\rho = \sum_{v \text{ suffix of } \hat{h}} |v'(y)|^{\rho/2} L_{\lambda,\rho}(y) = G_{[\hat{h}],\rho/2}[L_{\lambda,\rho}](y),
\]

where \( y = \hat{h}_\ell(0). \) \Rightarrow Note. \( \hat{h}_\ell(0) \to \hat{h}^* \) and \( \epsilon(h) = \epsilon(\hat{h}). \)
Consider the GF for $\ell = \infty$

$$S_\infty(s, \rho) := \sum_{h \in \mathcal{H}^+} \epsilon(h)^{-s} G_{[h], \rho/2}[L\lambda, \rho](h^*)$$

$$= \sum_{h \in \mathcal{H}^+} \left[ H_{[h], s/2}[1] \cdot G_{[h], \rho/2}[L\lambda, \rho] \right](h^*) .$$
Step 3 : transfer operators

Consider the GF for $\ell = \infty$

$$S_{\infty}(s, \rho) := \sum_{h \in \mathcal{H}^+} \epsilon(h)^{-s} G_{[h], \rho/2}[L_\lambda, \rho](h^*)$$

$$= \sum_{h \in \mathcal{H}^+} \left[ H_{[h], s/2}[1] \cdot G_{[h], \rho/2}[L_\lambda, \rho] \right](h^*).$$

The GF shares the dominant singularities with trace of

$$\Psi_{s, \rho} : F \mapsto H_{(s+\rho)/2}(I - H_{(s+\rho)/2})^{-1} \left[ L_\lambda, \rho \cdot (I - H_{s/2})^{-1} \left[ F \right] \right].$$
Step 3: transfer operators

Consider the GF for $\ell = \infty$

$$S_\infty(s, \rho) := \sum_{h \in H^+} \epsilon(h)^{-s} G_{[h],\rho/2}[L_\lambda,\rho](h^*)$$

$$= \sum_{h \in H^+} \left[ H_{[h],s/2}[1] \cdot G_{[h],\rho/2}[L_\lambda,\rho] \right](h^*) .$$

The GF shares the dominant singularities with trace of

$$\Psi_{s,\rho}: F \mapsto H_{(s+\rho)/2}(I - H_{(s+\rho)/2})^{-1} \left[ L_\lambda,\rho \cdot (I - H_{s/2})^{-1} [F] \right] .$$

Analytical study.

- Two functional spaces: one for trace, one for Mellin.
Step 3: transfer operators

Consider the GF for $\ell = \infty$

$$S_{\infty}(s, \rho) := \sum_{h \in \mathcal{H}^+} \epsilon(h)^{-s} G_{[h], \rho/2}[L_{\lambda, \rho}](h^*)$$

$$= \sum_{h \in \mathcal{H}^+} \left[H_{[h], s/2}[1] \cdot G_{[h], \rho/2}[L_{\lambda, \rho}]\right](h^*).$$

The GF shares the dominant singularities with trace of

$$\Psi_{s, \rho}: F \mapsto H_{(s+\rho)/2}(I - H_{(s+\rho)/2})^{-1}\left[L_{\lambda, \rho} \cdot (I - H_{s/2})^{-1}[F]\right].$$

Analytical study.

- Two functional spaces: one for trace, one for Mellin.
- Estimates $(I - H_{s/2})^{-1}[F](t) \sim \frac{2}{\epsilon} \frac{1}{s-2} \psi(t) I[F]$. 
Step 3: transfer operators

Consider the GF for $\ell = \infty$

$$S_\infty(s, \rho) := \sum_{h \in \mathcal{H}^+} \epsilon(h)^{-s} G_{[h], \rho/2} [L_{\lambda, \rho}](h^*)$$

$$= \sum_{h \in \mathcal{H}^+} \left[ H_{[h], s/2}[1] \cdot G_{[h], \rho/2} [L_{\lambda, \rho}] \right](h^*) .$$

The GF shares the dominant singularities with trace of

$$\Psi_{s, \rho}: F \mapsto H_{(s+\rho)/2} (I - H_{(s+\rho)/2})^{-1} \left[ L_{\lambda, \rho} \cdot (I - H_{s/2})^{-1} [F] \right].$$

Analytical study.

- Two functional spaces: one for trace, one for Mellin.
- Estimates $$(I - H_{s/2})^{-1} [F](t) \sim \frac{2}{\varepsilon} \frac{1}{s-2} \psi(t) I[F].$$
- Dolgopyat-Baladi-Vallée estimates for left of $s = 2$. 
Final remarks

- Strong parallels between the respective models and methods.
Final remarks

- Strong parallels between the respective models and methods.

- Results hold more generally for what we call $Q$-functions. We say $\Lambda(\alpha, n)$ is a $Q$-function associated with $f$ when

$$\Lambda(\alpha, n) = f \left( \frac{q_{k-1}(\alpha)}{q_k(\alpha)}, \frac{q_k(\alpha)}{n} \right)$$

for $k = k(\alpha, n)$ such that $q_{k-1}(\alpha) \leq n < q_k(\alpha)$. 
Final remarks

- Strong parallels between the respective models and methods.

- Results hold more generally for what we call \( Q \)-functions. We say \( \Lambda(\alpha, n) \) is a \( Q \)-function associated with \( f \) when

\[
\Lambda(\alpha, n) = f \left( \frac{q_{k-1}(\alpha)}{q_k(\alpha)}, \frac{q_k(\alpha)}{n} \right)
\]

for \( k = k(\alpha, n) \) such that \( q_{k-1}(\alpha) \leq n < q_k(\alpha) \).

- Moreover, for the \( n \to \infty \) model, the density for \( \alpha \) can be much more general.
Final remarks

▶ Strong parallels between the respective models and methods.

▶ Results hold more generally for what we call \( Q \)-functions. We say \( \Lambda(\alpha, n) \) is a \( Q \)-function associated with \( f \) when

\[
\Lambda(\alpha, n) = f \left( \frac{q_{k-1}(\alpha)}{q_k(\alpha)}, \frac{q_k(\alpha)}{n} \right)
\]

for \( k = k(\alpha, n) \) such that \( q_{k-1}(\alpha) \leq n < q_k(\alpha) \).

▶ Moreover, for the \( n \to \infty \) model, the density for \( \alpha \) can be much more general. \( \Rightarrow \) Independence between \( p_k/q_k \) and \( q_{k-1}/q_k \).
Open questions

- **Eliminate** averaging of the cost?

\[
\lim_{D,u,\ell \to \infty} \frac{|R_D(\ell, \lambda) \cap \left\{ \frac{n}{q_{\ell p}} \in u \cdot (1, \theta) \right\}|}{(\log D) \cdot |S_D| \cdot u \cdot (\theta < 1)} = \frac{6}{\pi^2} \int_{D_\lambda} \int \omega(x, y) dx dy?
\]
Open questions

- **Eliminate** averaging of the cost?

  $$\lim_{D,u,\ell \to \infty} \frac{\left| R_D(\ell, \lambda) \cap \left\{ \frac{n}{q_{\ell \perp}} \in u \cdot (1, \theta) \right\} \right|}{(\log D) \cdot |S_D| \cdot u \cdot (\theta - 1)} = \frac{6}{\pi^2} \int\int_{D \lambda} \omega(x, y) dxdy?$$

- **Make the parameter** $\lambda$ **vary with** $n$ ?

  In our previous work (generic $\alpha$) we proved

  $$\mathbb{E}_{\alpha}[S(\alpha, n) | x(\alpha, n) \geq \epsilon(n)] \sim \frac{12}{\pi^2} |\log \epsilon(n)|,$$

  where $x(\alpha, n) = q_{k-1}(\alpha)/q_k(\alpha)$, $q_{k-1}(\alpha) \leq n < q_k(\alpha)$. 
Open questions

- **Eliminate** averaging of the cost?

\[
\lim_{D,u,\ell \to \infty} \left| \frac{R_D(\ell, \lambda) \cap \left\{ \frac{n}{q_{\ell p}} \in u \cdot (1, \theta) \right\}}{(\log D) \cdot |S_D| \cdot u \cdot (\theta - 1)} \right| = \frac{6}{\pi^2} \int \int_{D \lambda} \omega(x, y) dx dy?
\]

- **Make the parameter** \( \lambda \) **vary with** \( n \) ?

  In our previous work (generic \( \alpha \)) we proved

  \[
  \mathbb{E}_\alpha[S(\alpha, n) | x(\alpha, n) \geq \epsilon(n)] \sim \frac{12}{\pi^2} |\log \epsilon(n)|,
  \]

  where \( x(\alpha, n) = q_{k-1}(\alpha)/q_k(\alpha), q_{k-1}(\alpha) \leq n < q_k(\alpha) \).

- **Possible to study** generic \( \alpha \) case with Mellin transform

  \( \Rightarrow \) but we need a similar **averaging** !
Thank you!