# Strong convergence for certain multidimensional fraction algorithms

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Virtually in Leiden, March 27th, 2020

# Multidimensional fraction algorithms

A *Markovian multidimensional continued fraction algorithm* is specified by two piecewiese continuous maps:

$$f : [0,1]^d \to [0,1]^d;$$

and

$$A : [0,1]^d \to GL(d+1, \mathbb{Z}).$$

Notations:

- $A^{(n)}(\theta) = A(f^{n-1})(\theta)$ - $n$-th partial quotient matrix of $\theta \in [0,1]^d$;
- $C^n(\theta) = A^{(n)} \cdots A^1(\theta)$ - cocycle (*convergent matrix*).

# Convergence properties

**General purpose:** to find a diofantine approximation to a vector $\theta$; in our case it is given by the rows of $C^n(\theta)$.

$$\tilde{w}_i^{(n)} = \left( \frac{c_{i,1}^{(n)}}{c_{i,d+1}^{(n)}}, \dots, \frac{c_{i,d}^{(n)}}{c_{i,d+1}^{(n)}} \right).$$

## Definition

A MCF-algorithm is *weakly convergent* if for all *theta* $w_i^{(n)} \to \theta$ as $n \to \infty$.
A MCF-algorithm is *strongly convergent* if for all $\theta$ with rationally independent components $||c_{i,d+1}^{(n)}(\theta - w_i^{(n)})||$ tends to zero as $n \to \infty$.

# Convergent exponents

Why do people care about Lyapunov spectrum?

- *Roth exponent* $\eta(w) = -\frac{\log ||\theta - \tilde{w}||}{q(w)}$ for some given approximation $\tilde{w} = (\frac{p_1}{q}, \cdots, \frac{p_n}{q})$;
- *the best approximation exponent* $\eta(\theta) = \limsup_{q(w)\to\infty} \eta(w, \theta)$.

For a large class of MCF algorithms $\eta(\theta)$ can be expressed in terms of Lyapunov exponents of the cocycle $C$ (Baldwin, Kosygin, Lagarias).

# Simplex-splitting MCF

The parameter space is a d-dimensional simplex that is splited into a finite or countable number of subsimplices $\Delta_\alpha$, and for each point $x = (x_0, x_1, \cdots, x_d)$ in a given subsimplex $\Delta_\alpha$ the map is defined by the formula

$$f(x) = \frac{A^{-1}x}{||A^{-1}x||}.$$

Classical examples:

- ordinary continued fraction algorithm: $T(x) = \frac{1}{x}(mod 1)$. This algorithm is strongly convergent.
- Jacobi-Perron algorithm: $T(x) = (\frac{x_2}{x_1}, \cdots, \frac{x_d}{x_1}, \frac{1}{x_1})(mod 1)$; this algorithm is weakly-convergent and probably almost everywhere strongly convergent.

# Selmer and Brun

**Selmer**

$$F : (x_1, x_2, x_3) \in \mathbb{R}^3_+ \mapsto x' = (x_1', x_2', x_3'),$$

where if $\{i, j, k\} = \{1, 2, 3\}$ and $x_i \geq x_j \geq x_k$,

$$x_i' = x_i - x_k,\ x_j' = x_j,\ x_k' = x_k.$$

It can be checked that the subsimplex defined by $x_i < x_j + x_k$ for all $\{i, j, k\} = \{1, 2, 3\}$ is an invariant attractive subset of this algorithm.

**Brun:**

$$F : (x_1, x_2, x_3) \in \mathbb{R}^3_+ \mapsto x' = (x_1', x_2', x_3'),$$

where if $\{i, j, k\} = \{1, 2, 3\}$ and $x_i \geq x_j \geq x_k$,

$$x_i' = x_i - x_j,\ x_j' = x_j,\ x_k' = x_k.$$

# Lagarias conditions

- H1: ergodicity: $T$ has an absolute continuous invariant measure $d\mu$ for which $T$ is ergodic;
- H2: covering property: $T$ is piecewise continuous with nonvanishing Jacobian almost everywhere;
- H3: semi-weak convergence (mixing condition for the system $(T, A, d\mu)$);
- H4: boundedness: log-integrability of the cocycle;
- H5: partial quotient mixing: the matrices become strictly positive after a controlled number of steps.

# Lyapunov exponents vs convergence

It was shown by Lagarias that if conditions H1–H4 are satisfied, then the convergence rate of the algorithm for almost all the parameters can be estimated in the following way:

$$\eta(\theta) \geq 1 - \frac{\lambda_2}{\lambda_1}.$$

Here $\eta(\theta)$ is the best uniform approximation exponent.
Moreover, if H5 is also satisfied, then there is a set of Lebesgue measure one for which the following equality holds for the uniform approximation exponent:

$$\eta^*(\theta) = 1 - \frac{\lambda_2}{\lambda_1}.$$

Notice that these exponents are zero only if $\lambda_1 = \lambda_2$.

# Some known results - 1: Paley-Ursell and its modifications

- Paley - Ursell, 1930: strong convergence of Jacobi-Perron algorithm in dimension 2;
- modern interpretation: Schweiger, 1996;
- the same argument for Brun algorithm in dimension 2: Schratzberger' 1998;
- Broise - Guirvac'h, 2001: Jacobi-Perron algorithm, simplicity of spectrum in any dimension + convergence in dimension 2.

# Some known results - 2: other proofs

- Fujita, Ito, Keane, Otsuki: Jacobi-Perron algorithm (dimension 2);
- Hardcastle - Khanin: generalized Jacobi-Perron algorithm for higher dimension (computer assistant proof);
- Meester: Podsypanin modification of Jacobi-Perron algorithm (dimension 2).
- Avila - Delecroix: Brun and fully subtractive algorithm (dimension 2).
- Berthé - Steiner - Thuswaldner: Selmer algorithm (dimension 2).

# Counterexamples - 1

**Proposition (Perron 1907)**

*The Jacobi-Perron algorithm is not strongly convergent for $n \geq 2$.*

The proof is based on the following

**Lemma**

*If $(1, x_1, \cdots, x_n)$ is linearly dependent over $\mathbb{Q}$, then the approximation can not be uniformly convergent.*

**Proposition**

*There exists $\theta = (\theta_1, \theta_2)$ with $(1, \theta_1, \theta_2)$ rationally independent such that the Brun approximation of $\theta$ is not strongly convergent.*

# Counterexamples: Triangle sequence

Defined by T. Garrity in 2001 as an iteration of a map on a triangle which yields a sequence of nested triangles, the *homogeneous triangle sequence* is an algorithm that is almost surely defined by

$$F : (x_1, x_2, x_3) \in \mathbb{R}^3_+ \mapsto x' = (x'_1, x'_2, x'_3),$$

where if $\{i, j, k\} = \{1, 2, 3\}$ and $x_i \geq x_j \geq x_k$,

$$x'_i = x_i - x_j - bx_k, \; x'_j = x_j, \; x'_k = x_k,$$

with $b = [\frac{x_i - x_j}{x_k}]$.
The non-homogeneous triangle sequence (a.k.a. *triangle sequence*) is a renormalized version of the map $F$:

$$f(u) = \frac{F(u)}{|F(u)|}.$$

## Simplicity of spectrum

Simplicity of spectrum of a dynamical cocycle (in particular, in case of multidimensional fraction algorithms) was established in different contexts.

- products of random matrices: Furstenberg'1963, Goldsheid and Margulis'1989 and Guivarch and Raugi'1986;
- Jacobi-Perron algorithm (in any dimension): Broise and Guivarch' 2001;
- first criterion for simplicity of spectrum for dynamical cocycle: Avila and Viana'2007;
- Selmer algorithm: Herrera Torres'2014;
- Galois-version of Avila - Viana theorem: Matheus, Möller, Yoccoz'2015;
- Fougeron - S.'2019: simplicity of spectrum for triangle sequence and Cassaigne algorithm.

# Challenges

There are a few important open questions related to the strong convergence problem:

- dimension 2: is there any uniform way to prove that strong convergence holds almost everywhere? What exactly one needs to know about the algorithm?
- any dimension: is there any approach to prove rigorously strong convergence in higher dimensions?
- any dimension: is there any efficient approach to show rigorously the absence of strong convergence of any *ergodic* MCF? Can we describe explicitly the possible obstacles for the existing approaches?
- any dimension: is there any way to use the obtained results about simplicity to get a conclusion about Pisot property/strong convergence?

Stay safe!!!