(1)

Sobolev spaces $\leftrightarrow$ transfer operators

Bob Zeitzuel-

piecewise expanding maps

rate of mixing

spectral gap.

Why Sobolev is better for piecewise expanding maps

Peut le faire avec des applications hyperboliques

I. Expanding maps of C^1 spaces → last talk.

$C^1 \to$ Sobolev

$T: S^1 \circlearrowleft$ $C^{1+\varepsilon}$ expanding

orientation preserving $T' \geq \lambda > 1$

Thm $\exists!$ absolutely continuous invariant measure μ for T, μ has C^{ε} density and μ is exponentially mixing for Hölder observables.

I will assume $\varepsilon = 1$. T C^2

Ruelle's proof $\Rightarrow$ expanding $\to$ subshift.

uses transfer operator

on symbolic dynamical systems

Proof $\mathcal{L}\varphi(x) = \sum_{Ty=x} \frac{1}{T'(y)} \varphi(y) \quad x \in S^1$

$\varphi: S^1 \to \mathbb{C}$

orientation preserving.

$\int \varphi \cdot \psi \circ T \, dx = \int \mathcal{L}\varphi \cdot \psi \, dx.$

If $\mathcal{L}\rho = \rho$ then $\mu = \rho \, dx$ is an inv. μ for T

Ruelle proof – key point.

Lasota Yorke inequality.

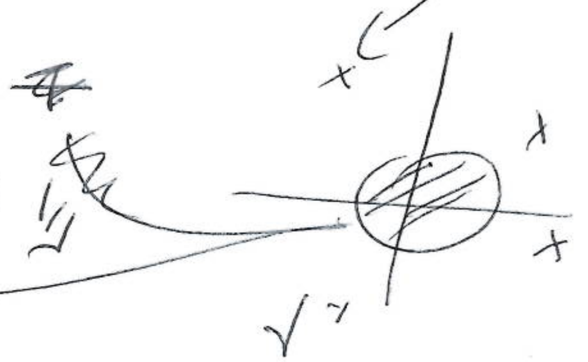
$\mathcal{L}: C^1 \circlearrowleft$ is bounded & there is constant $C \geq 0$ such that for every $n \in \mathbb{N}$ there is constant $C_n \geq 0$ s.t. for every $\varphi \in C^1$,

$$\|\mathcal{L}^n \varphi\|_{C^1} \leq C \lambda^{-n} \|\varphi\|_{C^1} + C_n \|\varphi\|_{C^0}$$

derivative tends to flatten

main point: the injection $C^1 \subseteq C^0$ is compact.

spectrum of Compact operator.



isolated eigenvalues with finite multiplicity

+ spectre $\subseteq$

$C\alpha^{-n} \|\varphi\|_{C^1}$

Hennion's theorem Let E, F Banach spaces with Compact inclusion $F \subseteq E$

$\mathcal{L} : F \circlearrowleft$ bounded

let $C, r > 0$ be such that for every $n \in \mathbb{N}$

$\exists C_n > 0$ st. $\forall \varphi \in F$.

$$\|\mathcal{L}^n \varphi\|_F \leq C\rho^n \|\varphi\|_F + C_n \|\varphi\|_E$$

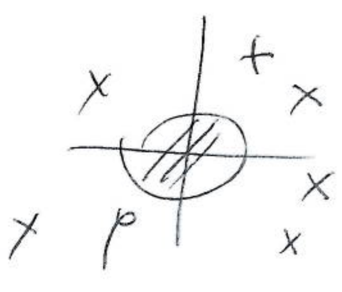
↳ important for Blocker case that we do not control the dependence / n

(mais dans C on depend pas de n)

then

$\rho_{ess}(\mathcal{L}) \leq \rho$

essential radius.



④ Proof of Lasota Yorke

Recall $\|\varphi\|_{C^1} = \|\varphi'\|_\infty + \|\varphi\|_\infty$.

$$\mathcal{L}^n\varphi(x) = \sum_{T^n y = x} \frac{1}{|(T^n)'(y)|} \varphi(y).$$

$$\|\mathcal{L}^n\varphi(x)\| \leq \|\mathcal{L}^n 1\|_\infty \|\varphi\|_\infty.$$

$$(\mathcal{L}^n\varphi)'(x) = \sum_{T^n y = x} \frac{1}{(T^n)'(y)} \left[\frac{1}{|(T^n)'(y)|} \varphi'(y) - \underbrace{\frac{\varepsilon}{\text{sign}} \frac{(T^n)''(y)\,\varphi(y)}{|(T^n)'(y)|^2}}_{\lesssim \|\varphi\|_\infty} \right]$$

$= 1$ orientation preserving

$T^n y = x$

$\frac{dy}{dx}(T^n)'(y) = 1$

to bound.

$$\sum_{T^n y = x} \frac{1}{|(T^n)'(y)|^2} \varphi'(y)$$

$$\leq \lambda^{-n} \underbrace{\sum_{T^n y = x} \frac{1}{|(T^n)'(y)|}}_{\mathcal{L}^n 1} \|\varphi\|_{C^1}$$

(5) $\int \mathcal{L}^n 1 \, dx = 1$

bounded distortion argument

$\Downarrow$

$\exists C > 0 \quad \mathcal{L}^n 1(x) \leq C \, \mathcal{L}^n 1(y) \quad \forall x, y \in \$$

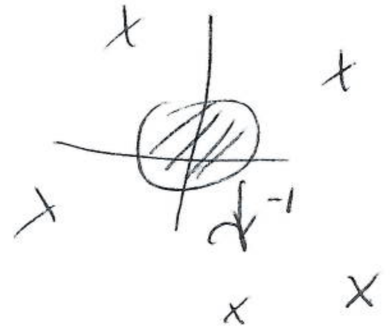
integrate / y

$\Rightarrow \quad \| \mathcal{L}^n 1 \|_\infty \leq C$

+ topological mixing to get rid of eigenvalues $|\;| = 1$.

Lasota Yorke : bounded variation. $\square$

Exercise



e - eigenvalues.

1 eigenvalue

If $m \in (C^1)^*$ is defined C^1 $\quad \varphi$ eigenvector

$m(\varphi) = 1 \qquad m(\varphi) = \int \varphi \, dx \qquad m(\varphi) = 1.$

Then $\mathcal{L}^* m = m \qquad \underset{\Downarrow}{} \quad 1 \in \sigma(\mathcal{L}) \qquad 1 \in \sigma(\mathcal{L})$

spectrum not injective $\rightarrow 1 \in \sigma(\mathcal{L})$ ~~$\$$~~ 1 is an eigenvalue

(6)

$\mathcal{L}$ has no eigenvalues of modulus ≥ 1 but 1

1 simple eigenvalue

$\exists$ an associated eigenvector $\varphi_{\mathcal{L}_0}$.

(expanding connected manifold) $\varphi_{1+\varepsilon}$

Hint for the proof see p. 7

piecewise expanding

$\Downarrow$

on s'attend à ce qu'on ait des multiplicités at 1 + root of unity

Thomine

no eigenvalue of $|\ | $ root of unity

topological mixing forbids roots of unity

uniqueness of the physical measure

ici opérateur de transfert

[pas d'$\Leftrightarrow$ avec ergodicité de la mesure cin.

mais vrai avec Koopman (dual)]

pas d'unicité de la mesure physique

+ physical measures are not mixing.

$\Rightarrow$ iterates produce mixing.

Bounded distortion is a property of the map

(7)

$\varphi : S^1 \to \mathbb{C}$ Hölder

$y \in B_n(x, \varepsilon)$

Bounded distortion hyperbolic dynamics

dynamical Ball

$$\sum_{k=0}^{n-1} \varphi(T^k y) - \sum_{k=0}^{n-1} \varphi(T^k x) \le C$$

By derivation. $\Longrightarrow \mathcal{L}^n 1(x) \le C \mathcal{L}^n 1(y)$

Hint duite Apply the positivity of $\mathcal{L}$

& the fact that

Kato perturbation theory for linear operators

$\forall x \in S^1$ $\bigcup_{n \in \mathbb{N}} T^{-n}\{x\}$ is dense.

∃ spectral decomposition $\Longleftarrow$ If we separate the spectrum.

$\mathcal{L} = \rho \otimes m + Q$

Commute (of Dunford)

$\mathcal{Q} = \mathcal{L}\Pi$ Cauchy

Supplementary subspaces

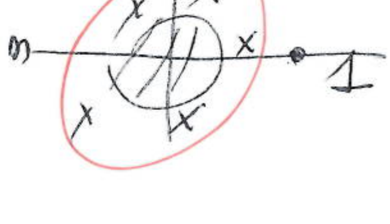
1 eigenvalue $\Pi = \frac{1}{2i\pi} \int_\gamma (z - \mathcal{L})^{-1} dz$

mv. spectral radius < 1

$\rho \otimes m(\varphi) = m(\varphi)\rho$

spectral radius of $Q < \eta$

$\exists\, \eta \in]0,1[$; $\|Q^n\| \le C\eta^n$



⑧

$$\int_{S^1} \varphi \circ \psi \circ T^n \rho \, dx = \int_{S^1} \mathcal{L}^n(\varphi_\rho)\psi \, dx$$

$\psi \in$ dual C^1

$$\mathcal{L}^n = \rho \otimes m + Q^n$$

$$\int_{S^1} \varphi \; \psi \circ T^n d\mu = \int_{S^1} \varphi \, \psi \circ T^n \rho \, dx = \int_{S^1} \mathcal{L}^n(\varphi_\rho)\psi \, dx$$

$$= \underbrace{\int_{S^1} m(\varphi_\rho)\rho\psi \, dx}_{= \mu(\varphi)\, \mu(\psi)} + \int_{S^1} Q^n(\varphi_\rho)\psi \, dx$$

$$\leq C \eta^n \|\varphi_\rho\|_{C^1} = O(\eta^n)$$

exponential mixing for C^1 observables

$\Downarrow$

approximation $\Rightarrow$ Hölder observables

This terminates the C^1 case.

about spectral decomposition.

II Espaces de Sobolev

a) On $\mathbb{R}^d$

If $p \in [1, +\infty]$ & $t \in \mathbb{N}$, $t \geq 1$, we may define

$H_p^t(\mathbb{R}^d)$ (or $W^{t,p}(\mathbb{R}^d)$) as the space

of distributions $u \in \mathcal{D}'(\mathbb{R}^d)$ such that
$$\cup$$
$$(\mathcal{S}'(\mathbb{R}^d))$$

$$\forall \alpha \in \mathbb{N}^d$$

with $|\alpha| \leq t$ $\partial^\alpha u \in L^p$

$$|\alpha| = \sum \alpha_i$$

(10) Distributions linear functional

$\mathcal{D}'(\mathbb{R}^d)$ = Continuous linear functional on Compactly supported $C_c^\infty(\mathbb{R}^d)$ ← not Fréchet, inductive limit of Fréchet.

We will work with the "(messy space) nicest messy space"

Schwartz space

$S(\mathbb{R}^d)$ rapidly decaying functions + derivatives also

$$S(\mathbb{R}^d) := \{\varphi : \mathbb{R}^d \xrightarrow{C^\infty} \mathbb{C} \;.\; \forall k \in \mathbb{N} \quad \forall \alpha \in \mathbb{N}^d$$

$\|\varphi\|_{k,\alpha}$ — $$\sup_{x \in \mathbb{R}^d} (1+|x|^k) \, |\partial^\alpha \varphi(x)| < +\infty \}$$

↑ polynomially

$$\alpha = (\alpha_1, \dots, \alpha_d)$$

$$\partial^\alpha \varphi(x) := \frac{\partial^{\alpha_1 + \dots \alpha_d}}{\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_d}_{x_d}} \varphi(x)$$

topological dual

semi-norms

Complete Fréchet

Fénod

locally Convex.

Dual $S'(\mathbb{R}^d)$ = tempered distributions on $\mathbb{R}^d$

Continuous functional

(11) $u : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C}$ tempered distribution.

linear belongs to $\mathcal{S}'(\mathbb{R}^d)$ if. $\exists C, k$
$\forall \varphi \in \mathcal{S}(\mathbb{R}^d)$ $\alpha_1 \dots \alpha_N$

$$|u(\varphi)| \leq \|\varphi\|_{k,\alpha} \leq C \left(\|\varphi\|_{k,\alpha_1} + \dots + \|\varphi\|_{k,\alpha_N} \right)$$

---

a tempered distribution / function.

example $u : \mathbb{R}^d \to \mathbb{C}$

grows at most polynomially
and is measurable., we identify it with.

$$u(\varphi) = \int_{\mathbb{R}^d} u \varphi \, dx$$

we identify it with a tempered distribution

Q. Tempered distribution is identified as a function.

(12) Other example

$$\delta_0 \in \mathcal{S}'(\mathbb{R}^d)$$

$\delta_0(\varphi) = \varphi(0)$ controlled by $\|\varphi\|_{0,0}$

$L^p \to$ tempered

But

H_p^t $\quad \mathcal{L}_p$

$t \in \mathbb{N}$ $\quad$ + t derivatives in $\mathcal{L}_p$.

How to define a notion of derivative for functions in $\mathcal{L}^p$

If $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$

We define $\partial^\alpha u$ as

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d),\ \partial^\alpha u(\varphi) = (-1)^{|\alpha|}\, u(\partial^\alpha \varphi)$$

$$\text{"}\ \partial^\alpha u(\varphi) = \int \partial^\alpha u\, \varphi\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} u\, \partial^\alpha \varphi\, dx\ \text{"}$$

decaying (pas de boundary term) $= (-1)^{|\alpha|}\, u(\partial^\alpha \varphi)$

(13) If $u \in L^p$

and $\varphi \in S(\mathbb{R}^d)$ then $\varphi \in L^q$ $1/p + 1/q = 1$

Hence $u . \varphi \in L^1$ (Young inequality)

Def $H_p^t(\mathbb{R}^d)$

$H_p^t(\mathbb{R}^d) = \{ u \in S'(\mathbb{R}^d) ; \ \forall \alpha \in \mathbb{N}^d$
$|\alpha| \leq t \quad \partial^\alpha u \in L^p \}$

$t \in \mathbb{N}$

$|\alpha| = \sum \alpha_i$

Norme Banach

$$\|u\|_{H_p^t} = \left(\sum_{|\alpha| \leq t} \| \partial^\alpha u \|_{L^p}^p \right)^{1/p}$$

$p \neq 1$
∞

hemy

$p = 2$
Hilbert OK.

---

piecewise expandus $t < 1/p$.

$t \cancel{>} 1/p$ dim. 1

$H_p^t(\mathbb{R}^d) \Rightarrow$ Hölder.

Continuous

We need $t < 1/p < 1$ to get discontinuous functions!

$t < 1$!

(14) We need to have. t Aon entier.

$\Rightarrow$ Fourier transform on $S(\mathbb{R}^d)$ $\forall u \in S(\mathbb{R}^d)$

define $\hat{u} = \mathcal{F}u : \mathbb{R}^d \to \mathbb{C}$ by

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\,dx.$$

Basic Facts $\mathcal{F} : S(\mathbb{R}^d)$ continuous automorphism.

integration by part $\to$ decays / polynomial

Stable by $\mathcal{F}u$ + inversion formula.

$$\mathcal{F}^{-1}u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} u(\xi)\,d\xi$$

exact formula.

$$\mathcal{F}^{-1} = \frac{1}{(2\pi)^d} \mathcal{F}^*$$

[L^2 adjoint of $\mathcal{F}$..

(15) It implies that we may extend. $\mathcal{F}$ to $S'(\mathbb{R}^d)$

If $u \in S'(\mathbb{R}^d)$ and $\varphi \in S(\mathbb{R}^d)$ we define

$$\mathcal{F}u(\varphi) = u(\overline{\mathcal{F}^* \overline{\varphi}})$$

L^2 adjoint

Hilbert

Fourier transform on tempered distribution

$$u(\varphi) = \text{"} \int_{\mathbb{R}^d} u\, \varphi \, du \text{"}$$

- $\mathcal{F}$ intertwines differentiation and multiplication.

$$\forall \alpha \in \mathbb{N}^d, \ \partial^\alpha \mathcal{F}u = \mathcal{F}((-ix)^\alpha u)$$

$$x^\alpha := x_1^{\alpha_1} \dots x_d^{\alpha_d}$$

(∞)

$$(i\xi)^\alpha \ \mathcal{F}u(\xi) = \mathcal{F}(\partial^\alpha u)(\xi)$$

intégration par partie.

$u \to$ Fourier transform

$\to \times x^\alpha$.

$\to$ de Fourier.

(b) Ψu cannot ~~tempered~~ distribution by a bounded function

but C^∞ + derivatives pow at most polynomial.

$$\rho u \; (\varphi) = u\,(\psi\,\varphi)$$
$\uparrow$ tempered

Def

If $\psi : \mathbb{R}^d \xrightarrow{C^\infty} \mathbb{C}$ and its derivatives grow at most polynomially we define the Fourier multiplyer $\psi(D)$ by: $\mathcal{F}u \in S'(\mathbb{R}^d)$ tempered distribution.

$$\psi(D)\,u = \mathcal{F}^{-1}\left(\psi \cdot \mathcal{F}u\right)$$
$\uparrow$ notation

$$D = \begin{pmatrix} \frac{1}{i}\,\partial/\partial x_1 \\ \vdots \\ \frac{1}{i}\,\partial/\partial x_d \end{pmatrix} \qquad \psi(\xi) = (i\xi)^\alpha$$

$$\left(\psi(D) = \partial^\alpha\right)$$

(*)

explicit spectral theorem

$$\psi(D)\,\varphi(D) = (\psi\,\varphi)\,(D)$$

$\checkmark$ functional calculus

(17) not exactly differential operator.

for differential operator: Compactly supported is supported

not true for Fourier multiplier

pseudo locality.

$\Psi(D) \quad \to$ when $\psi(\xi) = (i\xi)^\alpha = (i\xi_1)^{\alpha_1} \cdots (i\xi_d)^{\alpha_d}$

then $\Psi(D) = \partial^\alpha = \left(i \cdot \frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(i \times \frac{1}{i}\frac{\partial}{\partial x_d}\right)^{\alpha_d}$

Definition of Sobolev space

$p \in]1, +\infty[\qquad t \in \mathbb{R}$

1, $+\infty$ noisy Define $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$

smooth $\neq 0$ (of norm)

Japanese bracket.

$u \in H_p^t(\mathbb{R}^d)$ if $\langle D \rangle^t u \in L^p$ $(\Psi(D))\, u$

$(\Psi = \langle \xi \rangle^t)$ for $\Psi = \langle \xi \rangle^t$

The norm: $\|u\|_{H_p^t} = \| \langle D \rangle^t u \|_{L^p}$

18

If $t' \leq t$

$$H_p^t \subseteq H_p^{t'}$$

$t' < t$ in $\mathbb{R}^n$ not compact

Compact manifold — Compact inclusion.

$$\langle D \rangle^t u = \mathcal{F}^{-1}\left(\langle \xi \rangle^t \mathcal{F}u \right)$$

$$\langle \xi \rangle = \left(1 + |\xi|^2\right)^{1/2}$$

$t \in \mathbb{R}$ $\quad \|u\|_{H_p^t}$ $\neq$ norm but equivalent norm.

We have the notion of Sobolev ~~difference~~.

$t > 0$ $\quad H_p^t \subseteq L^p$

$t = 0$ $\quad L^p$

($t < 0$ possible)

we will work mostly with functions $\in L^p$.

(19)

Theorem Marcinkiewicz multiplier theorem

$\mathbb{R}^d$

let $p \in]1, +\infty[$ Then there is $C > 0$

st. if φ is in $C^\infty(\mathbb{R}^d)$ and

there is B st.

$\forall \xi \in \mathbb{R}^d$: $|\xi^\gamma| \, |\partial^\gamma \varphi(\xi)| \leq B$

for any $(\gamma_1 \dots \gamma_d)$ with B does not depend on γ

nbr = fini

$|\gamma| \leq \frac{d}{2} + 1$

$\gamma_j \in \{0, 1\}$

we need to check for a finite set of γ

C does not depend on φ.

then the Fourier multiplier $\varphi(D)$ is bounded on L^p and

$$\| \varphi(D) \|_{L^p \to} \leq CB$$

↳ norm of the operator.

$p=2$ we do not need condition on derivatives

Application. • t positive integer.

We recover the ~~kne~~ space.

that we have

We can then put that derivatives are Fourier X

• If $\langle D\rangle^t u \in L^p$ and $|\alpha| \leq t$

then $\partial^\alpha u = (-i)^\alpha D^\alpha u$.

$$= (-i)^\alpha \frac{\partial^\alpha}{\langle D\rangle^t} \underbrace{\langle D\rangle^t u}_{\in L^p}$$

bounded on L^p

$\varphi(D)$ with $\varphi(\xi) = \dfrac{\xi^\alpha}{\langle \xi\rangle^t}$

$\Rightarrow$ $V=0$ inéquality OK.

Rem $(\langle D\rangle^t)^{-1} = \langle D\rangle^{-t}$

→ By hand, differentiate

→ or apply Cauchy formula
analytic extension

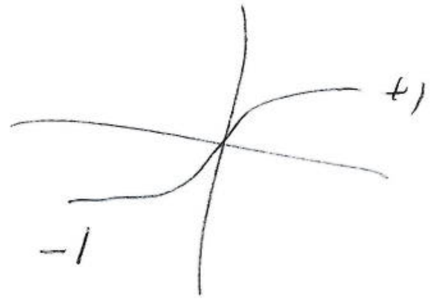
$\Rightarrow \partial^\alpha u \quad L^p$

(4) • If $\partial^\alpha u \in L^p \ \forall\ |\alpha| \le t$

$$1 + \sum_{j=1}^{d} \xi_j^t \qquad \langle \xi \rangle^t$$

t even OK.

t odd



positive terms

$$1 + \sum_{j=1}^{d} \xi_j^t \, \psi(\xi_j) \simeq \langle \xi \rangle^t$$

$$\langle D \rangle^t u = \frac{\langle D \rangle^t}{1 + \sum_{j=1}^{d} D_j^t \, \psi(D_j)} \left(u + \sum \psi(D_j) \, D_j^t u\right)$$

$\in L^p$

bounded on L^p

bounded on L^p ~~ath~~ as

asymptotic

Marcinkiewicz

Bounded operators build up from differential operators.

Norms are $\sim$

$\langle \ \rangle$ smooth at 0 + does not vanish

Bracket interest

We have 2 definitions of Sobolev spaces

- $t \in \mathbb{R}$
- $t \in \mathbb{N}$ easier to prove results. We can compute derivatives

The idea is to deduce results from any t by results for $t \in \mathbb{N}$.

Prop $p_0, p_1 \in]1, +\infty[$ $t_0, t_1 \in \mathbb{R}$.

$\theta \in [0,1]$ $t_\theta := (1-\theta) t_0 + \theta t_1$

interpolation

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

If A is a ~~bounded~~ linear operator bounded from $H_{p_0}^{t_0}$ ↺ to itself and from $H_{p_1}^{t_1}$ ↺

then A is bounded from $H_{p_\theta}^{t_\theta}$

Moreover $\| A \|_{H^{t_\theta}_{p_\theta}} \leq \| A \|^{1-\theta}_{H^{t_0}_{p_0} ↺} \| A \|^{\theta}_{H^{t_1}_{p_1} ↺}$

Rem C^∞ [illegible] supported are dense in H_p^t

Autre définition Paley Littlewood de Composition

$f \rightarrow$ cut it scale of frequences

cf Vivian's book partition of unity on the Fourier side

↳ she wants more than Lasota-Yorke nuclear power de Composition & zeta

Tonine Lasota-Yorke local result.

dans le Gaus.

b) Cas des manifolds

Lemme $p \in]1, +\infty[$, $t \in \mathbb{R}$, $r > |t|$

If $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a C^r map ($r \in \mathbb{N}$)

and $\chi : \mathbb{R}^d \rightarrow \mathbb{C}$ compactly supported C^r function

Then the operator $u \longmapsto (\chi u) \circ F$

is bounded on $H_p^t(\mathbb{R}^d)$

(via interpolation)

$\Rightarrow$ cut out and look at it in charts

Def M a compact manifold.

We cover M by domains of charts

We choose a partition of unity $(\theta_\omega)_{\omega\in\Omega}$

$$(U_\omega, K_\omega)_{\omega\in\Omega}$$

adapted to $(U_\omega)_{\omega\in\Omega}$

Then we say that $u \in \mathcal{D}'(M)$

belongs to $H_p^t(M)$ if

$\forall \omega \in \Omega$

$$(\theta_\omega u)\circ K_\omega^{-1} \in H_p^t(\mathbb{R}^d)$$

(Schwartz space with $\| \ \|_\infty$ pas besoin de de Leray) $\in L^p$)

Norm $\displaystyle \Big[\sum_{\omega\in\Omega} \| \theta_\omega u \circ K_\omega^{-1} \|^p_{H_p^t(\mathbb{R}^d)}\Big]^{1/p}$

norme avec N.

- Banach
- does not depend on the choice of charts and partition of unity thanks to the lemma.

(25)

Rem C^∞ dense
- fractional power of laplacian can also be used to define them

Theorem Sobolev inequality - Rellich-Kondrachov theorem

Let $1 < p, q < \infty$ and $s, t \in \mathbb{R}$

Assume $s > t$

$$s - \frac{d}{p} > t - \frac{d}{q}$$

d = dimension = $\dim M$

Then there is a compact inclusion of $H_p^s(M) \subseteq H_q^t(M)$

+ Compact manifold

We need $>$ $\Rightarrow$ for compactness

true for $M = \mathbb{R}^d$

$\geq$ without compact inclusion just inclusion.

Classical Application
Poincaré inequality

M Riemanian

There is a constant C such that

If $u \in H_2^1(M)$ and $\int_M u\,dx = 0$

then $\|u\|_{L^2} \leq C \|Du\|_{L^2}$

$\Rightarrow$ spectrum of the laplacian

$\hookrightarrow$ gradient

(26) ex Torus . Algebraic manifolds

If Λ = torus Fourier series $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$

$$f : \mathbb{T}^d \to \mathbb{C}$$

$$C_k(f) = \int_{\mathbb{T}^d} e^{-2i\pi k x} f(x)\,dx \qquad k \in \mathbb{Z}^d$$

Analogous of $\langle D \rangle^t$ is

$$f \mapsto \underbrace{\sum_{k \in \mathbb{Z}^d} C_k(f)\, (1 + |k|^2)^{t/2} e^{2i\pi k x}}$$

$$f \in H_p^t \iff \in L^p$$

- $\neq [0,1]^d$
- for t large continuous function

$s > d/p \Rightarrow$ continuous the $q = \infty$. + Hölder continuous

t small. $\longleftrightarrow$ Caractéristique function

OK.

~~Hölder~~ Hölder regularity

(27) toward Lasota - Yorke inequality

III ~~Sobolev spaces of expanding maps~~

Lemma Lasota Yorke inequality

$\varepsilon \in]0,1[$

let T is $C^{1+\varepsilon}$ expanding map on M

let $t \in]0,\varepsilon[$ and $1 < p < +\infty$

then $\mathcal{L}$ is bounded on $H_p^t(M)$

there is $\lambda > 0$ st. for every $n \in \mathbb{N}$,
$\lambda > 1$

there is C_n st for every $u \in H_p^t(M)$

λ expanding factor

$$\| \mathcal{L}^n u \|_{H_p^t} \leq C \lambda^{-nt} \|u\|_{H_p^t} + C_n \|u\|_{L^p}$$

Proof 1) local version on charts

cf Thomine's paper

Lemma Local Lasota Yorke

let $F : \mathbb{R}^d \to \mathbb{R}^d$ a C^1 diffeomorphism and
$A \in GL_d(\mathbb{R})$. Assume that (F is close to be differentiable)

(28)

Assume that for all $x \in \mathbb{R}^d$

We want $\mathcal{C}^1$ + F and A not to far.

$$\| A^{-1} \circ DF(x) \| \leq 2$$

$$\& \quad \| DF(x) \circ A^{-1} \| \leq 2$$

"uniformly $\mathcal{C}^1$"

Control the modulus of continuity for DF

Then for every $t \in [0,1]$

every $p \in]1, +\infty[$ there is a

Constant $C = C(t, p)$ (does not depend on F nor on A)

st $\forall u \in H_p^t(\mathbb{R}^d)$

$$\| u \circ F \| \leq C \, |\det A|^{-1/p} \, \| A \|^t$$

↳ Jabbien

$$+ C \, |\det A|^{-1/p} \, \| u \|_{L^p}$$

(Compact inclusion) on M

F inverse branche for T

A differentiable for F.

$A \simeq d^{-nt}$

Take small ball Chart

F close to be differentiable

(2) Proof

Write $u \circ F = u \circ A \circ (A^{-1} \circ F)$

$$\overset{\parallel}{\tilde{F}}$$

We shall prove $\|u \circ \tilde{F}\|_{H_p^t} \leq C \|u\|_{H_p^t}$

1

C does not depend on A nor F.

and $\|u \circ A\|_{H_p^t} \leq C |\det A|^{-1/p} \|A\|^t \|u\|_{H_p^t}$

2

$+ C |\det A^{-1/p}| \|u\|_{L^p}$

replace F by A.

1 $t=0$ $\|u \circ \tilde{F}\|_{L^p} \leq d_p |\det D\tilde{F}|^{-1/p} \|u\|_{L^p}$

$\leq 2^{d/p} \|u\|_{L^p}$ this is 1

$t=1$ $\|u\|_{H_p^1} \simeq \|u\|_{L^p} + \|Du\|_{L^p}$

$D(u \circ \tilde{F}) = Du \circ D\tilde{F}$ this gives 1

We can compute the derivative

1 holds

$\forall t \in [0,1]$ 1

Interpolation

30

2

$$\mathcal{F}(u \circ A)(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\, u(Ax)\, dx$$

$$= \int_{\mathbb{R}^d} e^{-iy\cdot({}^TA^{-1}\xi)}\, u(y)\, \frac{dy}{|\det A|}$$

$$= \frac{\mathcal{F}u({}^TA^{-1}\cdot\xi)}{|\det A|}$$

For a Fourier multiplier $\Psi(D)$

$$\Psi(D)\,(u \circ A) = \left[(\Psi \circ {}^TA)(D)\, u\right] \circ A$$

$$\text{let } \Psi(\xi) = \frac{(1 + |{}^TA\,\xi|^2)^{t/2}}{1 + \|A\|^t \langle \xi \rangle^t}$$

— main ingredient —

!

this will be bounded

- If $|\xi| > \frac{1}{\|A\|}$, then

$$1 \le |\xi|\,\|A\| \qquad |\Psi(\xi)| \le \frac{2^{t/2}\, \|A\|^t |\xi|^t}{1 + \|A\|^t \langle \xi \rangle^t}$$

$$< 2^{t/2}$$

31

- If $|\xi| \leq \frac{1}{\|A\|}$ $|\Psi(\xi)| \leq 2^{t/2}$

→ either Cauchy

→ or Marcinkiewicz multiplier theorem

$\|\Psi(D)\|_{L^p \circlearrowleft} \leq C$ uniform.

Back to 2 $\qquad$ Ψ bounded

$$\|u \circ A\|_{H_p^t} = \| \langle D\rangle^t (u \circ A)\|_{L^p}$$

$$= \| (\langle {}^T\!A . D\rangle^t u) \circ A\|_{L^p}$$

$$\leq |\det A|^{-1/p} \| \langle {}^T\!A\, D\rangle^t u\|_{L^p}$$

Notation $\langle {}^T\!A . D\rangle^t = \Psi(D)$ for $\Psi(\xi) = \langle {}^T\!A . \xi\rangle^t$

$$\leq |\det A|^{-1/p} \| \Psi(D)(1 + \|A\|^t \langle D\rangle^t) u\|_{L^p}$$

Bounded uniformly C

$$\leq C\, |\det A|^{-1/p} \| (1 + \|A\|^t \langle D\rangle^t) u\|_{L^p}$$

$$\leq C\,|\det A|^{-1/p}\left(\|u\|_{L^p} + \|A\|^t\,\|u\|_{H_p^t}\right)$$

bis

local $\leftrightarrow$ global. $\square$

IV Piecewise expanding maps

Def Thomine

Compact. M Riemannian manifold.

For $\alpha > 0$. $T : M \circlearrowleft$ is a $C^{1+\alpha}$ piecewise expanding map if

- $\exists$ disjoint open subsets $O_1 \dots O_N$ that cover Lebesgue a.e. M whose boundaries are C^1 hypersurfaces with boundaries.

ex polygons in $\mathbb{T}^d$

- for $i \in \{1, \dots, N\}$ there is a $C^{1+\alpha}$ map T_i defined on a neighbourhood of $\overline{O_i}$ and a diffeomorphism on its image such that $T|_{O_i} = T_i|_{O_i}$

- For any $x \in M$ whenever $T'(x)$ defined

$$d(x) := \inf_{\substack{v \in T_x M \\ v \neq 0}} \frac{|D_x T v|}{|v|} > 1$$

32 We can get a Lasota Yorke inequality

(~~ratio~~ < 1 ~~~~ boldly beats the Complexity)

Transfer operator

$$\mathcal{L}u(x) = \sum_{i=1}^{N} 1_{T_i(a)}(x) \frac{u \circ T_i^{-1}(x)}{|\det DT_i|}$$

Bounded Variation. $\dim = 1$.

H_1^1 closure of C^∞ in BV. Compact support.

L^1 function $\Rightarrow$ Complex measure ~~total~~ Variation $= L^1$ norm.

C^∞ not dense in BV

Bound BV $d \geq 2$.

L^1 closed subspace Complex measure

C^∞ compact support. L^1.

$C^{1+\text{Lipschitz}}$

$C^{1+1/2}$ works for Sobolev.

Proposition Strichartz

Let $0 < t < 1/p < 1$. Then there is $C = C(t,p)$ such that for every integer $d \geq 1$ for each measurable set $\Omega \subset \mathbb{R}^d$ such that the intersection of Ω

33)

with almost every line parallel to some coordinate axis has at most L connected components for every $u \in H_p^t(\mathbb{R}^d)$

$$\| 1_Q u \|_{H_p^t} \leq C L \|u\|_{H_p^t}$$

dobel lasota york

+ Position Strichartz

Notion de Complexity

If $\bar{i} = (i_0 \dots i_{n-1}) \in [1, N]$

define ~~Oi~~ $O_{\bar{i}}$

$$O(i_0) = O_{i_0}$$

$$O(i_0 \dots i_{n-1}) = \{ x \in O_{i_0} \; \overline{T_{i_0} x} \in O(i_1 \dots i_{n-1}) \}$$

you may define

$$T^n_{\bar{i}} = T_{i_{n-1}} \circ \dots \circ T_{i_0}$$

on a neighbourhood of $\overline{O_{\bar{i}}}$

Complexity of the beginning

$$D_n^b = \max_{x \in M} \# \{ \bar{i} \in [1, N]^n ; x \in \overline{O_{\bar{i}}} \}$$

(34) Complexity at the end

$$D_n^e = \max_{x \in M} \{ i \in [\![1, N]\!] ;\ x \in \overline{T_i^{n}(O_i)} \}$$

ex $2x \bmod 1$ exponential doubling map.

$x \in O_i$

$$\lambda_n(x) = \inf_{\substack{v \in T_x M \\ v \neq 0}} \frac{|D T_i^n v|}{|v|} > 1$$

Ratio in LY lasota Yorke on $H_p^t(M)$

$t < \min(\alpha, 1/p)$

strictement

$$\rho = \lim_{n \to \infty} (D_n^b)^{1/n \, 1/p} (D_n^e)^{1/n (1 - 1/p)} \times \| |\det DT^n|^{1/p - 1} \lambda_n^{-t} \|_\infty^{1/n}$$

upper bound

choice of p depends on the size of D_n^b or D_n^e

choisir p près de 1 permet de tuer l'effet de D_n^e

35 $e = 1$ D_m^b does not matter.

You expect D_m^b small

BV

Full Branches D_m^c large.

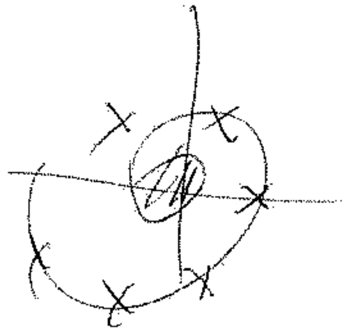
take p close to 1.

$e < 1$ ergodic
finitely many ~~cases~~ physical measures.

- If $p < 1$. mixing?

1 multiplicity?
Several physical measures

~~they~~



mixing $\not\Rightarrow$
exponential mixing.

Full Branches not Markov Case.

Markov $\Rightarrow$ finitely many pieces.

(36) beyond Markov case

Stabilizer is used to

{ . to reduce regularity

Introduced for

piecewise hyperbolic Baladi Gouëzel.

⇒ tout était de faire les bilans

Dolgopyat

Liverani Baladi

piecewise hyperbolic flow

exponential mixing

hyperbolic → anisotropic spaces to

deal with.