ANR CODYS

Orbites des systèmes dynamiques discrets en informatique

http://www.irif.fr/~codys

Journées de lancement-Novembre 2018

CODYS

- Appel à projets générique 2018
- Date de démarrage : 1/11/2018
- Durée : 48 mois
- Fin : 31/10/2022
- Projet ANR-18-CE40-0007
- 2 partenaires
  - IRIF, P7, V. Berthé
  - LABRI, Univ. Bordeaux S. Labbé

Tasks

- GEN: Genericity and randomness.
- PER: study of finite and periodic orbits, it describes their characteristics, and compares them to the orbits of the initial system.
- DIS: discretization with a view toward simulation.
- SIM: simulation directed toward numerical computing but also toward the study of dynamical systems as computing models.
- SPE: algorithmic checkability of properties, together with the construction of special orbits and invariants for dynamical systems.

IRIF

- Valérie Berthé Coordinator All tasks Arithmetic dynamics, continued fractions.
- Olivier Carton Task GEN, SIM, SPE Algorithmic randomness, normal numbers.
- Paulina Cecchi Bernales Task GEN, PER Ergodic theory and symbolic dynamics.
- Pierre-Antoine Guihéneuf Task GEN, PER, DIS, SIM Discretization of dynamical systems, mathematical aspects of visualisation.
- Peter Habermehl Task GEN, SIM, SPE Program verification, invariant synthesis.
- Pierre Ohlmann Task GEN, SPE Invariant synthesis for linear dynamical systems.
- Loïck Lhote Task GEN, PER, DIS, SIM, SPE Probabilistic analysis of algorithms, analytical combinatorics, generating functions, dynamical analysis.

IRIF

- Frédéric Paccaut Task GEN, PER, DIS Discrete stochastic processes, analysis of random data structures, stochastic modelling.
- Amaury Pouly Task GEN, SIM, SPE Decision, control, and synthesis problems for continuous and discrete linear DS.
- Wolfgang Steiner Task GEN, PER, DIS Numeration dynamics, continued fractions.
- Brigitte Vallée Task GEN, PER, DIS, SPE Analytical combinatorics, dynamical analysis of algorithms, transfer operators.
- Postdoc Task GEN, SIM, PER 24 mois

LABRI

- Sébastien Labbé Coordinator All tasks Symbolic dynamics, discrete geometry, Sage implementation.
- Vincent Delecroix Task GEN, PER, DIS, SIM, SPE Dynamical systems and ergodic theory.
- Nathanaël Fijalkow Task GEN, SIM, SPE Probabilistic computation models around automata theory, invariant synthesis.
- Doctorant.e Task GEN, SIM, SPE

CODYS

• Dynamical systems model physical processes but also numerous phenomena from the digital world (execution of an algorithm, a loop in a program as the action of a multidimensional linear map).

• A discrete-time dynamical system (X, T) is defined as the action of a map T acting on a space X (usually assumed to be compact).

• Orbit/trajectory $\mathcal{O}(x) = \{T^n x,\ n \in \mathbb{Z}\}$

• Discrete orbits orbits of discrete-time dynamical systems that are relevant in computer science for computer simulations and for computational models.

Discrete orbits

Consider the simulation of a dynamical system, e.g. in floating-point arithmetic. The number of floating points being finite, then all the points have (ultimately) periodic orbits. The same applies more generally when the number of points of a system is finite.

Discrete orbits

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We thus consider two main types of discrete orbits, namely

- finite and periodic orbits; these orbits have usually an arithmetic meaning for the systems we consider and form a countable set;
- orbits for discretizations of dynamical systems with the discretization usually being performed with respect to a finite space.

Reachability and statistical properties of orbits

The first question we can ask about a trajectory is

- whether it will enter (once or infinitely often) a given subregion Z of X or even reach a given point z. This is a reachability problem. It will be handled by focusing on the construction of special invariants and orbits.
- We can also ask about its recurrence and long-term behavior: how long will it stay in the subregion Z? This type of questions will be handled through the use of ergodic theory and probabilistic methods.

Reachability problems and invariants

- Reachability problems often lead to undecidable problems.
- One strategy to tackle them is to consider invariant sets, instead of orbits (verification or control theory): subsets invariant under the action of T that contain a point x and not y.
- The viewpoint we develop here is mainly the synthesis of invariants.
- We are thus interested here in exact properties of the total orbits and their relations with invariants.
- Affine and algebraic invariants, as well as polyhedral invariants. We focus here on the case where the system (X, T) is given by a linear map with rational entries.

From discretization to simulation

• The computer simulation of a dynamical system is inherently partial.

• How distorted is the qualitative behavior of the system (X, T) once discretized? What happens when the number of points of the discretization space tends to infinity? To what extent does the dynamics of the discretized application reflect the dynamics of the map T?

• We associate with the dynamical system (X, T) a sequence of discretizations $(X_N, T_N)_N$ which depend on a parameter N (related to the number of points of the discretization space).
Is it true that this asymptotic behavior will be close to the one of the initial system, and reminiscent of it? or, does it completely forget the initial system and behaves as a random finite dynamical system?

CODYS

- We focus on the study and comparison of discrete orbits (finite, periodic, and orbits of discretizations).
- We address the question of the relevance of ergodic and dynamical methods for their study, and more generally, for simulation.
- What are the limits of the ergodic theorem? Are methods from dynamical systems (ergodic and probabilistic) relevant for the study of discrete orbits and discretizations of discrete-time dynamical systems?
- We want to develop both a theoretical and practical study of discretization and simulation for simple systems like expanding maps of the interval or linear maps.
- We want to construct in an effective way special orbits and invariants, and to study the asymptotic statistical behavior of a dynamical system, from a computability and effective perspective.

A trajectory for $T : X \to X$

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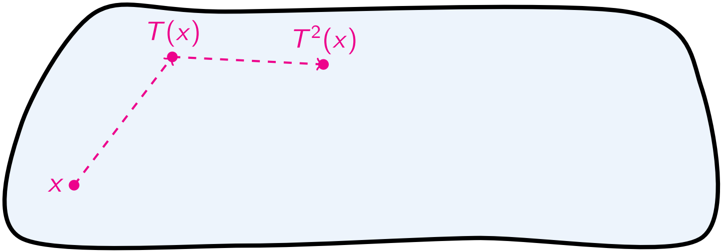

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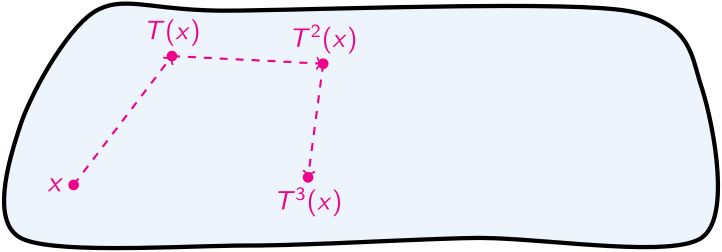

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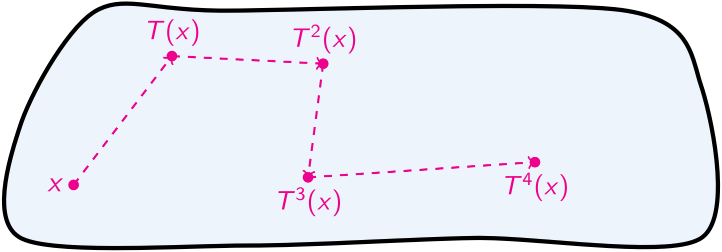

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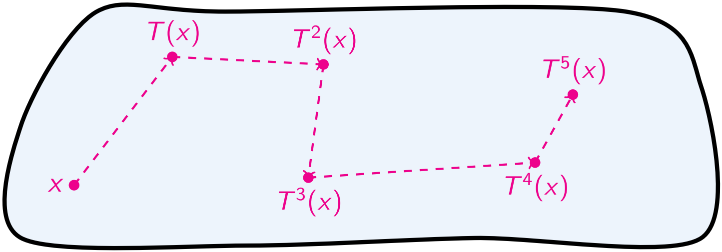

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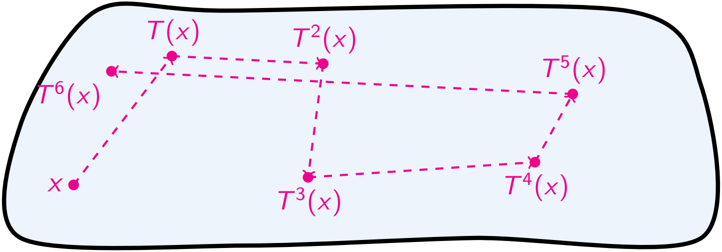

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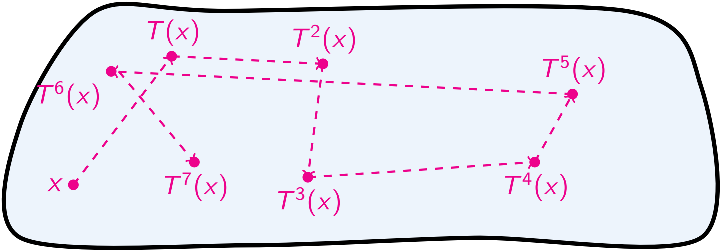
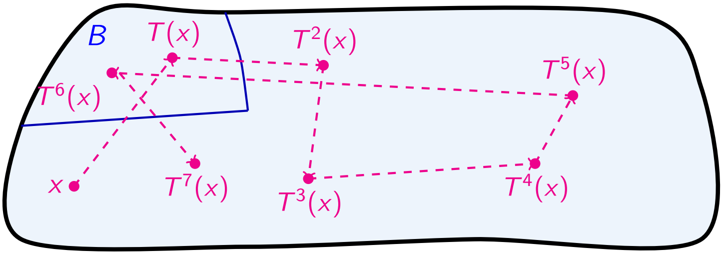
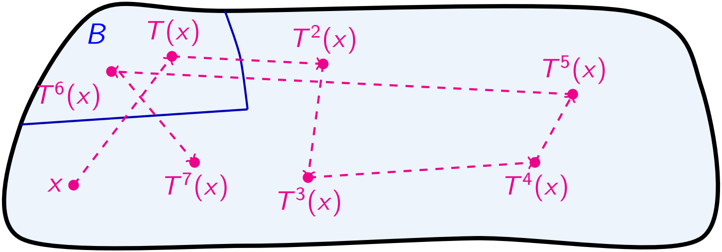


Ergodic theorem



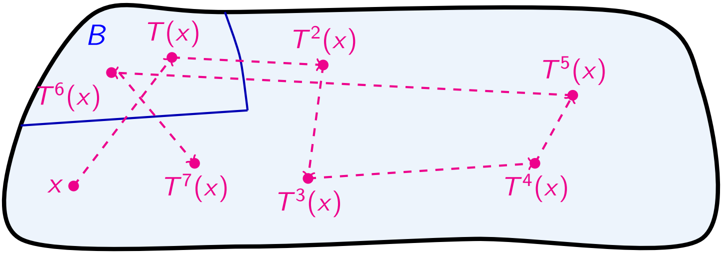
Ergodic theorem



Among the first N points of the orbit of x, how many of them enter B?

How often do they visit B?

Ergodic theorem



Let 1_B be the characteristic function of B

Among the first N points of the orbit of x, how many of them enter B? $\sum_{0 \leqslant n < N} 1_B(T^n x)$

How often do they visit B? $\lim_{N\to\infty} \frac{1}{N} \sum_{0 \leqslant n < N} 1_B(T^n x)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \leqslant n < N} 1_B(T^n x) = \mu(P) \quad \text{a.e. } x$$

Discrete dynamical system

We are given a dynamical system

$$T : X \to X$$

- Topological dynamics describes the qualitative/topological asymptotic behaviour of trajectories/orbits
  The map T is continuous and the space X is compact
- Ergodicity describes the long term statistical behaviour of orbits
  The space X is endowed with a probability measure and T is measurable $(X, T, \mathcal{B}, \mu)$

How well are the orbits distributed?

According to which measure?

Ergodic theorem

We are given a dynamical system $(X, T, \mathcal{B}, \mu)$ with $T\colon X \to X$
An ergodic system is such that the time spent by a system in some region is proportional to the volume of this region, it has the same behavior averaged over time as averaged over all the space

- Average time values: one particle over the long term
- Average space values: all particles at a particular instant

Ergodicity

$$\mu(B) = \mu(T^{-1}B) \quad \text{T-invariance}$$

$$T^{-1}B = B \implies \mu(B) = 0 \text{ or } 1 \quad \text{ergodicity}$$

Ergodic theorem space mean= average mean

$$\text{If } f \in L_1(\mu), \quad \lim_N \frac{1}{N} \sum_{0 \leqslant n < N} f(T^n x) = \int f \, d\mu \quad \text{a.e. } x$$

Numeration dynamics

Numeration dynamical systems are simple algorithms that produce digits in classical representation systems

- Decimal expansions

$$T\colon [0,1] \to [0,1],\ x \mapsto 10x - [10x] = \{10x\}$$

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$$T : [0,1] \to [0,1],\ x \mapsto 10x - [10x] = \{10x\}$$

$$x_1 = T(x) = 10x - [10x] = 10x - a_1$$

$$x = \frac{a_1}{10} + \frac{x_1}{10}$$

$$x_2 = T(x_1) = T^2(x) \qquad a_2 = \lfloor 10T(x) \rfloor$$

$$x = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{x_2}{10^2} = \sum_{i=1}^{\infty} a_i 10^{-i}$$

Numeration dynamics

Numeration dynamical systems are simple algorithms that produce digits in classical representation systems

- Decimal expansions

$$T\colon [0,1] \to [0,1],\ x \mapsto 10x - [10x] = \{10x\}$$

The map T produces the digits

$$a_n = \lfloor 10T^{n-1}(x) \rfloor$$

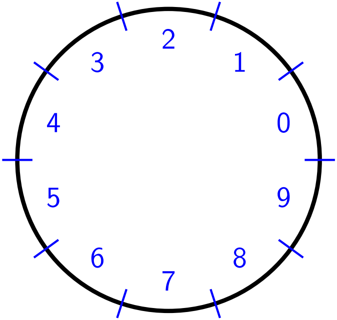
The action of T can be seen as a shift on the sequence of digits

$$x \sim a_1 a_2 a_3 a_4 \cdots \qquad T(x) \sim a_2 a_3 a_4 \cdots$$

Multiplication by 10 on $[0, 1]$

$X = [0, 1] \qquad T : x \mapsto 10\,x \pmod 1$

$\mathcal{P} = \left\{ \left[\frac{i}{10}, \frac{i+1}{10} \right[: 0 \leqslant i \leqslant 9 \right\}$



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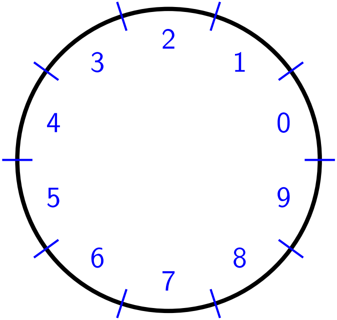

Orbit of $\pi - 3$

Multiplication by 10 on $[0,1]$

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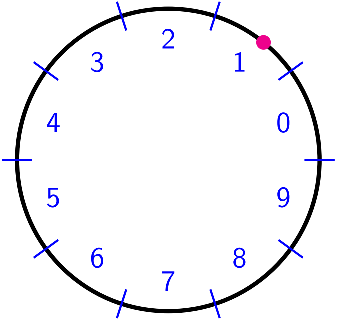

Orbit of $\pi - 3$

0.1

Multiplication by 10 on $[0, 1]$

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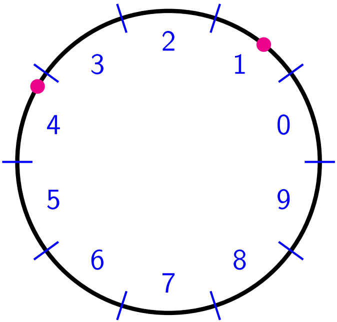

Orbit of $\pi - 3$

0.14

Multiplication by 10 on $[0, 1]$

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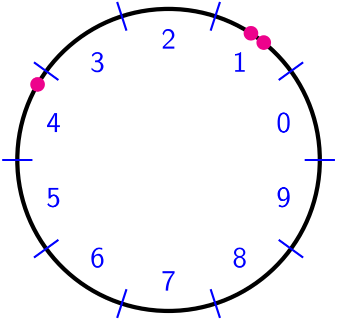

Orbit of $\pi - 3$

0.141

Multiplication by 10 on $[0, 1]$

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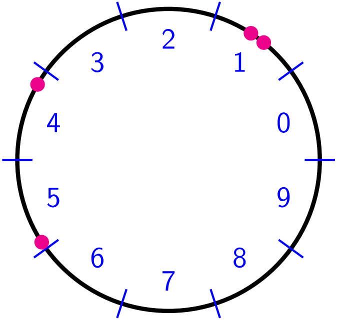

Orbit of $\pi - 3$

0.1415

Multiplication by 10 on $[0, 1]$

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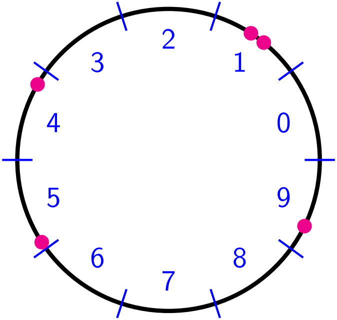

Orbit of $\pi - 3$

0.14159

Multiplication by 10 on $[0, 1]$

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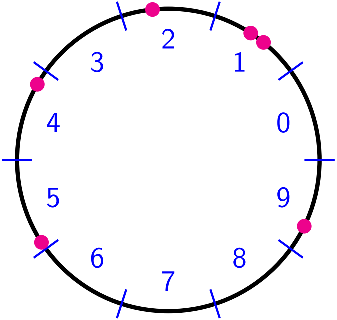

Orbit of $\pi - 3$

0.141592

Multiplication by 10 on $[0, 1]$

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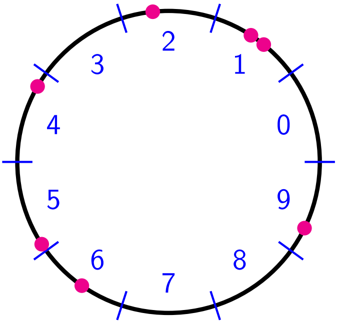

Orbit of $\pi - 3$

0.1415926

Multiplication by 10 on $[0, 1]$

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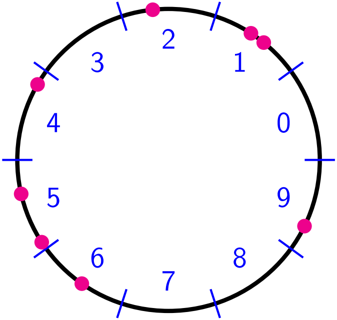

Orbit of $\pi - 3$

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Multiplication by 10 on $[0,1]$

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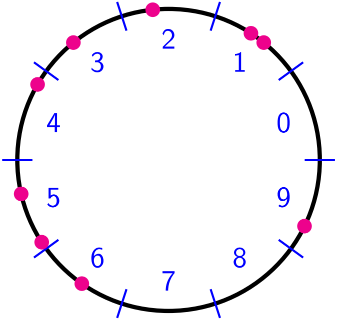

Orbit of $\pi - 3$
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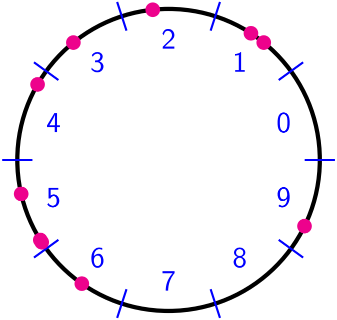

Orbit of $\pi - 3$

0.1415926535

Multiplication by 10 on $[0, 1]$

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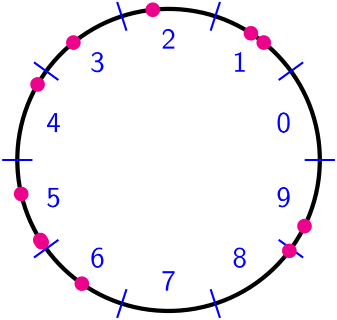

Orbit of $\pi - 3$

0.14159265358

Multiplication by 10 on $[0, 1]$

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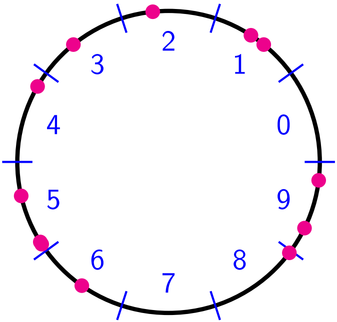

Orbit of $\pi - 3$

0.141592653589

Multiplication by 10 on $[0, 1]$

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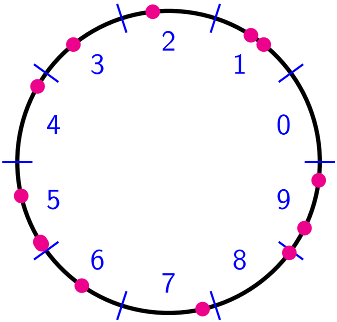

Orbit of $\pi - 3$

0.1415926535897

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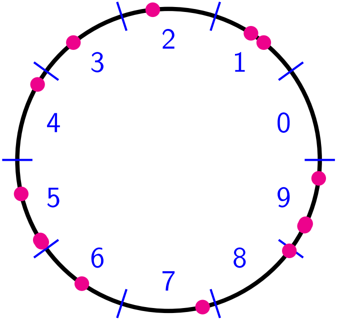

Orbit of $\pi - 3$

$0.14159265358979312\cdots$

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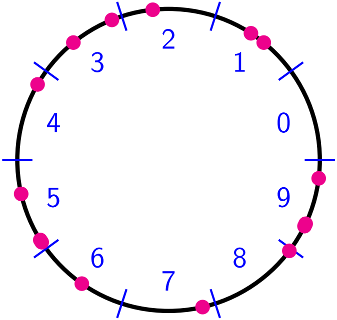

Orbit of $\pi - 3$

$0.14159265358979312\cdots$

Codings $\Longleftrightarrow$ decimal expansions

Ergodic theorem

We are given a dynamical system (X, T, μ)

$$\mu(B) = \mu(T^{-1}B) \quad T\text{-invariance}$$

$$T^{-1}B = B \implies \mu(B) = 0 \text{ or } 1 \quad \text{ergodicity}$$

Ergodic theorem Let $f\colon X \to X$ be in $L^1(\mu)$, and μ be T-invariant. If (X, T, μ) is ergodic then, for μ-almost all x in X

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(T^i(x)) = \int_X f(x) d\mu$$

Unique ergodicity (X, T) is uniquely ergodic if it admits a unique invariant measure μ.

Theorem If (X, T) is uniquely ergodic and f and T are continuous, then the convergence is valid for every x.

Numeration dynamical systems

- Numeration $T\colon [0,1] \to [0,1],\ x \mapsto 10x - [10x] = \{10x\}$
- Beta-transformation $T\colon [0,1] \to [0,1],\ x \mapsto \{\beta x\}$
- Continued fractions $T\colon [0,1] \to [0,1],\ x \mapsto \{1/x\}$
- Kronecker dynamics $R_\alpha\colon x \mapsto \alpha + x \mod 1$ unique ergodicity

From numeration dynamics to symbolic dynamics

- Decimal expansion $T\colon [0,1] \to [0,1],\ x \mapsto \{10x\}$
- Beta-transformation $T\colon [0,1] \to [0,1],\ x \mapsto \{\beta x\}$
- Continued fractions $T\colon [0,1] \to [0,1],\ x \mapsto \{1/x\}$

From numeration dynamics to symbolic dynamics

- Decimal expansion $T\colon [0,1] \to [0,1],\ x \mapsto \{10x\}$
- Beta-transformation $T\colon [0,1] \to [0,1],\ x \mapsto \{\beta x\}$

$$\beta > 1 \qquad x = \sum_{i=1}^{\infty} a_i \beta^{-i}$$

- Continued fractions $T\colon [0,1] \to [0,1],\ x \mapsto \{1/x\}$

$$x = \frac{1}{a_1 + x_1} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

Tasks

- GEN: Genericity and randomness.
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- DIS: discretization with a view toward simulation.
- SIM: simulation directed toward numerical computing but also toward the study of dynamical systems as computing models.
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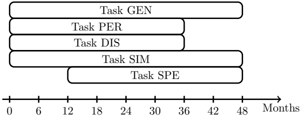
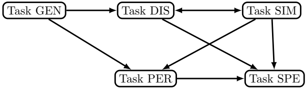
Tasks

Task GEN runs during all the project. This is the place where a common tool and culture is developed.
Task SIM also runs during the whole project: simulations are needed for all the other tasks, and it will be nourished by theoretical results from Task DIS.
Task PER and DIS have to start from the beginning of the project.
Task SPE will be started once a common culture will have been developed in particular through Task GEN.
During year 2 and 3 all the tasks will be conducted.





Task GEN

- What is a generic trajectory?

Genericity = most of the initial conditions = a typical orbit, whose initial condition satisfies the ergodic theorem.

- But since we restrict ourselves to discrete trajectories (i.e., periodic orbits of (X, T), and orbits of the discretization (X_N, T_N)), a specific notion of genericity has to be formulated. How to adapt the definition of genericity (defined for continuous systems) to a discrete genericity?

- Since we work with finite sets X_N, reasonable notions of genericity rely on a notion of convergence (when N tends to infinity) toward a standard notion of genericity issued from the continuous world: for all N, the property has to hold on X_N, out of an exceptional set of small size, which tends to 0 with N, and in a rather fast way.

- What is a generic dynamical system (generic in a class of dynamical systems having some prescribed property)?

Genericity

- Measure-theoretic genericity Ergodic genericity: an initial point whose orbit satisfies Birkhoff's ergodic theorem; it is a typical orbit. This requires in particular to understand with respect to which invariant measure.
- Topological genericity in the sense of Baire. A property is generic if it is satisfied (at least) on a countable intersection of dense open sets.
- Computational genericity will be well suited to the construction of effective orbits. It refers to algorithmic randomness (Martin-Löf randomness, Schnorr randomness, etc.) and to exact computation models of effective analysis where topological notions are revisited in an effective way.

Task PER-Finite and periodic orbits for the original system (X, T)- V. Berthé

- What are the finite and the periodic expansions of the dynamical system (X, T)? How to characterize them?
- Long-term behavior of finite and periodic orbits. Do finite and periodic expansions for (X, T) have a typical behavior? With respect to which invariant measure?

Toy example: the case of the β-numeration with β being the golden ratio.

- Transfer operator with the question of the extension of the application field (which is a priori Euclidean dynamics) for the methodology of dynamical analysis of algorithms. Is it possible to give to the transfer operator a generating role even when the dynamical system is not of homographic type?

Task DIS-Discretization: study of the system (X_N, T_N)-P.-A. Guihéneuf

• Which discretizations to consider?

uniform vs. non-uniform (floating-point arithmetics); finite state machine simulations of dynamical systems; symbolic dynamical systems.

• What dynamical properties of T can be read on the (global) dynamics of the discretizations T_N?

• What are the finite or the periodic expansions of the dynamical system (X_N, T_N)? Do the discretizations (X_N, T_N) asymptotically behave as random maps (when the parameter N becomes large)?

• Long-term behavior of orbits of the discretized system (X_N, T_N). Action of the discretization on the invariant measures. Does discretization detect typical behavior? What is the spatial distribution of periodic cycles for (X_N, T_N)?

• Non-stationary dynamics: the map T can be changed with respect to time. Ex: (generic) compositions of linear maps, which is useful for the study of $\mathcal{C}^1$-diffeomorphisms.

Task SIM- Simulation-S. Labbé

- Which properties of the original system can be checked by a computer? What can computers simulate?
- Conceptual aspects of discrete machines: computational power.
- Rounding and truncation errors. What can be said concerning the roundoff errors when simulating trajectories?
- Convergence of the sequence of discretized systems (X_N, T_N) toward (X, T). For each dynamical feature of the homeomorphism, do its discrete analogues appear on an infinite number of discretizations? Are periodic orbits of (X, T) shadowed by periodic orbits of (X_N, T_N) with the same period for an infinite number of discretizations?
- Simulation of $\mathbb{Z}^d$-actions.
- Numerical simulation for dynamical parameters. Ex: Lyapunov exponents, and invariant measures.

Task SPE - Construction of special orbits and invariants-N. Fijalkow

Effective constructions of generic and special orbits together with the question of the existence and algorithmic synthesis of invariants, and their related decision problems.

• Invariants for linear dynamical systems. Going beyond the case of linear dynamical systems on one hand, and beyond semi-algebraic sets for the invariants on the other hand.

• Constructions of normal points.

• Random generation of generic objects: probabilistic study of truncations.