On the expansion of Monadic Second-Order Logic with
- Cantor Bendixson Rank, and
- Order type

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One slide overview

« Can we extend the expressiveness of MSO while retaining its wonderful decidability properties ? »

RESULTS: We consider:

- MSO + $\mathrm{otp}_\alpha(X)$ over countable ordinals chains
- MSO + $\mathrm{CBrank}_\beta(X)$ over binary trees

In all cases,

SAT of $\mathcal{L}$ is decidable if and only if $\mathcal{L}$ = MSO.

Monadic Second-Order Logic (MSO)

MSO = FO + set quantifiers + membership predicate

Example: There exists a path from x to y in a directed graph:

For all sets X,
if X contains x and is closed under edge successors,
then y belongs to X.

MSO has a wonderful theory:

Over finite chains/words	MSO = automata = monoids = …
Over finite trees	MSO = automata = …
Over infinite trees	MSO = automata = mu-calculus = …
Over countable linear orders	MSO definable = o-monoids = …

Everything is effective.

SAT is decidable all these cases.

MSO over countable ordinals

Properties expressible in MSO:

- $(L,\leq)$ is a total order (FO)
- $(L,\leq)$ is well-ordered = isomorphic to an ordinal
- $(L,\leq)$ is a countable ordinal
- $(L,\leq)$ is a finite ordinal
- $(L,\leq)$ is isomorphic to ω
- $(L,\leq)$ is isomorphic ω^2; to ω^k; to α for any $\alpha < \omega^\omega$

Property NOT expressible in MSO:

- $(L,\leq)$ is isomorphic to ω^ω, and more generally to some countable $\alpha \geq \omega^\omega$

Some extensions of MSO over ω

MSO + U [Bojanczyck 03]

set variable

$UX.\ \phi(X)$ « there exist arbitrarily large sets X satisfying $\phi(X)$ »

Decidability:

- SAT of ω-chains Bool(MSO + U^+) **[BC17]**
  (U appears positively only)
- SAT of WMSO + U over infinite trees

Undecidability:

- MSO+U undecidable over ω **[BPT16]**

BMSO [Blumensath, Carton, C. 14]

Models are sequences of numbers: $f : \omega \to \mathbb{N}$.

$B(X)$ « f is bounded over X »

BMSO = MSO + B(-)

Decidability:

- Some fragments.
- Many open questions.

Undecidability:

- SAT of BMSO undecidable over ω
  (by (bi-)reduction to MSO+U) **[BCC14]**

MSO + order type

We work over ordinals, ie the signature of order.

Fix a countable ordinal α.

$\mathrm{otp}_\alpha(X)$ means « the order type of X is α »
= « the linear order restricted to X is isomorphic to α »

Theorem: For all ordinals α, the following properties are equivalent:

- $\alpha < \omega^\omega$,
- α is MSO-axiomatizable,
- MSO + otp_α = MSO,
- MSO + otp_α is decidable over countable ordinals,
- MSO + otp_α is decidable over α.

Main lemma: For all countable ordinals $\alpha \geq \omega^\omega$, MSO + otp_α is undecidable over α.

MSO + otp_α

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Main lemma: the case $\alpha = \omega^\omega$

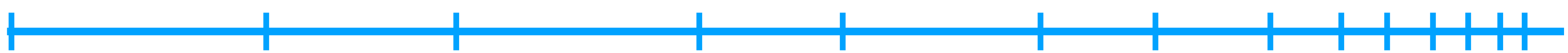
For $\beta < \omega^\omega$, let level(β) be n such that $\omega^n \leq \beta < \omega^{n+1}$

Remark: Given a sequence $(\beta_m)_{m\in\omega}$ of ordinals $< \omega^\omega$ then

$\beta_0 + \beta_1 + \beta_1 + \ldots = \omega^\omega$ if and only if level(β_n) is not bounded

Let ϕ be a BMSO sentence, we construct an MSO + $\mathrm{otp}_{\omega^\omega}(X)$ sentence ϕ^*.

Step 1. Guess ω disjoint non-empty intervals covering ω^ω (doable in MSO).



Step 2. Then check ϕ after transforming it as follows:

- first-order variables are constrained to be minimal elements of an interval
- monadic variables are constrained to be union of intervals
- B(X) become $\neg\mathrm{otp}_{\omega^\omega}(X)$

Lemma: $\omega^\omega \models \phi^*$ if and only if $f \models \phi$ for some $f : \omega \to \mathbf{N}$.

Corollary: MSO + $\mathrm{otp}_{\omega^\omega}(X)$ is undecidable.

Remark: It works for all $\alpha = \omega^\lambda$ for λ a countable limit ordinal.

MSO + otp$_\alpha$

Main lemma: For all countable ordinals $\alpha \geq \omega^\omega$, MSO + otp$_\alpha$ is undecidable over α.

We have seen the undecidability of $\alpha = \omega^\lambda$ for all countable limit ordinal λ.

Main lemma: the case $\alpha = \omega^{\lambda+1}$ (1)

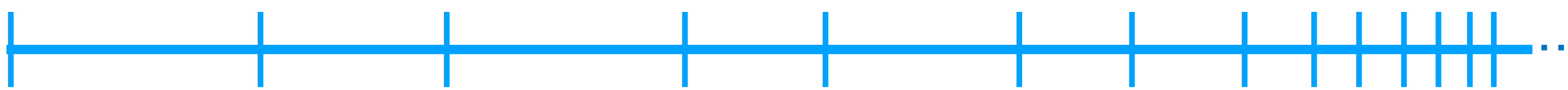
Assume $MSO + otp_{\omega^\lambda}$ is known to be undecidable over ω^λ.

We want to show that $MSO + otp_{\omega^{\lambda+1}}$ is undecidable over $\omega^{\lambda+1}$.

Note that $\omega^{\lambda+1} = (\omega^\lambda).\omega$, ie is a concatenation of ω block of otp ω^λ.

Step 1: decompose $\omega^{\lambda+1}$ into ω intervals (almost all) of order type ω^λ.

Consider a decomposition of $\omega^{\lambda+1}$ into ω intervals $(J_i)_{I\in\omega}$ such that



(A) Such that every infinite union of suffixes of intervals has order type $\omega^{\lambda+1}$.

=> almost all intervals has order type of the form $\beta + \omega^\lambda$.

(B) There is no way to split infinitely many intervals in while keeping (A) true.

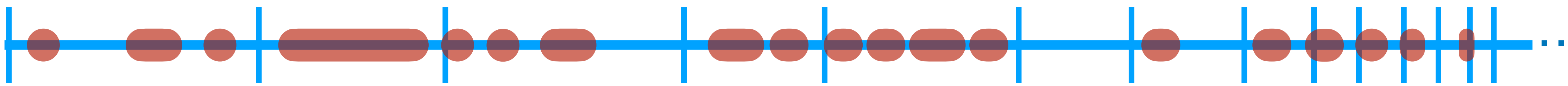
=> almost all intervals have order type ω^λ.

Call it a **good** decomposition.

Lemma: The exists an $MSO + otp_{\omega^{\lambda+1}}$ formula which checks that a decomposition of the input $\omega^{\lambda+1}$ into ω intervals is **good**.

Main lemma: the case $\alpha = \omega^{\lambda+1}$ (2)

Lemma: The exists an MSO + $otp_{\omega^{\lambda+1}}$ formula which checks that a decomposition of the input into ω intervals is good.



Remark: For J a **good** decomposition, and $X \subseteq \omega^{\lambda+1}$, then the order type of X is $\omega^{\lambda+1}$ if and only if $J_n \cap X$ is of order type ω^λ for almost all $n \in \omega$.

Given an MSO + otp_{ω^λ} formula ϕ
we build an MSO + $otp_{\omega^{\lambda+1}}$ formula ϕ^*
such that:

$(F_n)_{n\in\omega}$ range over infinite subsequences of a fixed **good** decomposition.

$$\omega^{\lambda+1} \models \phi^*(F, X_1, \ldots, X_k)$$

if and only if $\quad \omega^\lambda \models \phi(X_1 \cap F_n, \ldots, X_k \cap F_n)$ for almost all $n \in \omega$.

(Un)boundedness case: $(\neg B(X))^*(F,X) := otp_{\omega^{\lambda+1}}(X)$

Negation case: $(\neg\phi)^*(F,X) := \forall\, F' \subseteq F\ .\ \neg(\phi^*(F',X))$

Consequence: If MSO + otp_{ω^λ} is undecidable, then MSO + $otp_{\omega^{\lambda+1}}$ is undecidable.

MSO + otp_α

Main lemma: For all countable ordinals $\alpha \geq \omega^\omega$, MSO + otp_α is undecidable over α.

We have seen the undecidability of $\alpha = \omega^\lambda$ for all countable limit ordinals λ.

We have seen the undecidability of $\alpha = \omega^{\lambda+1}$ assuming undecidability for ω^λ.

Hence we have undecidability for $\alpha = \omega^\lambda$ for all countable ordinals λ.

With a bit of work we have undecidability for all countable $\alpha \geq \omega^\omega$.

Theorem: For a countable ordinal α, the following properties are equivalent:

- $\alpha < \omega^\omega$,
- α is MSO-axiomatizable,
- MSO + otp_α = MSO,
- MSO + otp_α is decidable over α,
- MSO + otp_α is decidable over countable ordinals.

MSO + Cantor-Bendixson rank

We work over binary trees.

Fix a countable ordinal α.

$\mathrm{CBrank}_\alpha(X)$ means « the cantor-Bendixson rank of X is α »

Intuition: T has CBrank $\leq \alpha$ if when we remove all nodes that are root of subtrees of CBrank $< \alpha$, only finitely many infinite branches remain.

Theorem: For all ordinals α, the following properties are equivalent:

- $\alpha < \omega$,
- $\mathrm{CBrank}_\alpha(-)$ is MSO-expressible,
- MSO + CBrank_α = MSO (over binary trees)
- MSO + CBrank_α is decidable over binary trees,
- MSO + CBrank_α is decidable over the complete binary tree.

Remark: A tree T has a CBrank if and only if the nodes can be reoriented (swapping left and right), such that the lexicographic ordering is a well-order.

Remark: And T has CBrank α if the lex order type belongs to $[\omega^\alpha,\omega^{\alpha+1})$.

Conclusion

RESULTS: We consider:

- MSO + $otp_{\alpha}(X)$ over countable ordinals chains
- MSO + $CBrank_{\beta}(X)$ over binary trees

In all cases,

SAT of $\mathcal{L}$ is decidable if and only if $\mathcal{L}$ = MSO.

Is there still some hope to extend the expressiveness of MSO while retaining its wonderful decidability properties ?