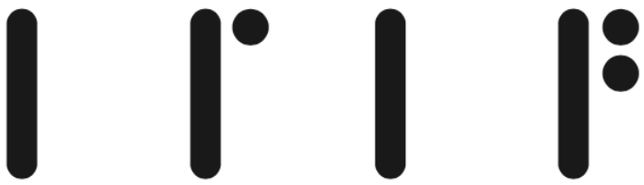
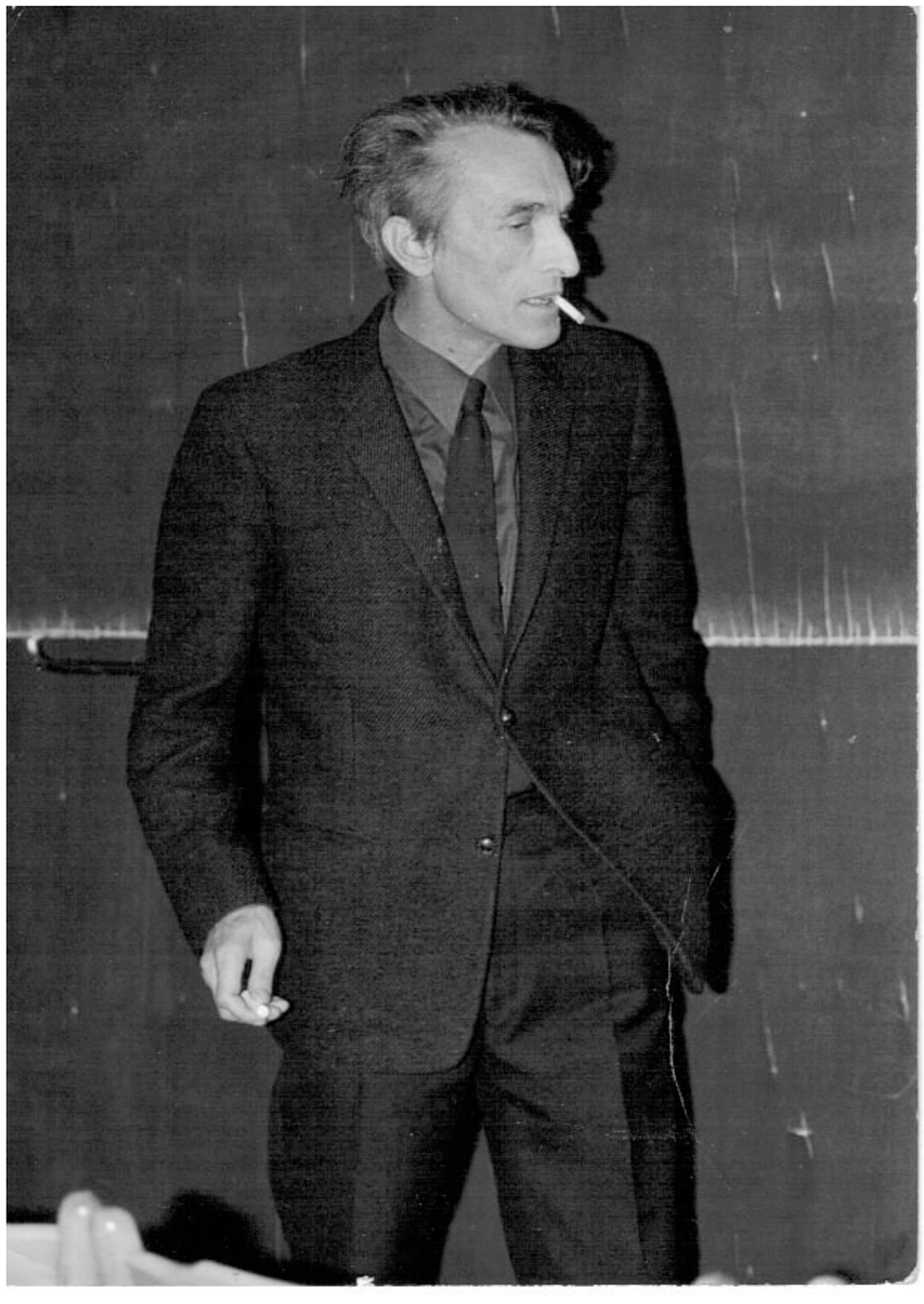
Regular cost functions

Conference dedicated to the scientific legacy
of Marcel-Paul Schützenberger

Thomas Colcombet

23 Mars 2016





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- machines: non-deterministic, alternating automata (…)
- expressions: rational expressions, generalized versions, …
- algebra: recognizability by monoid, deterministic automata, …
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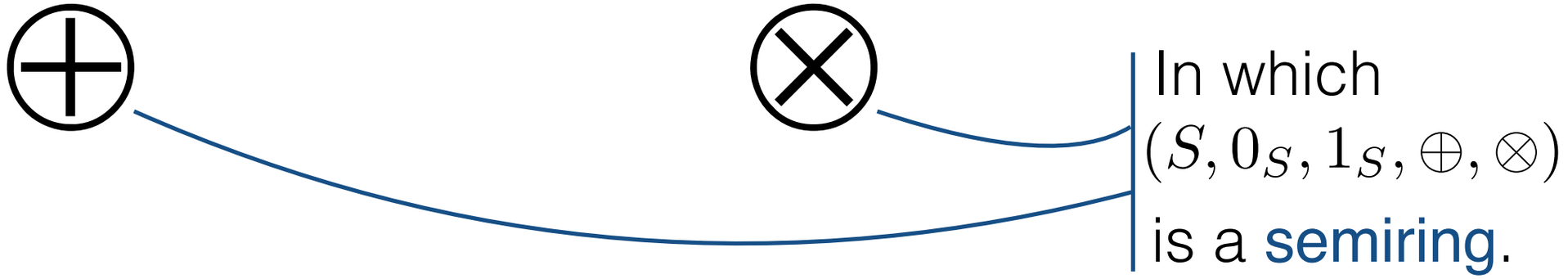

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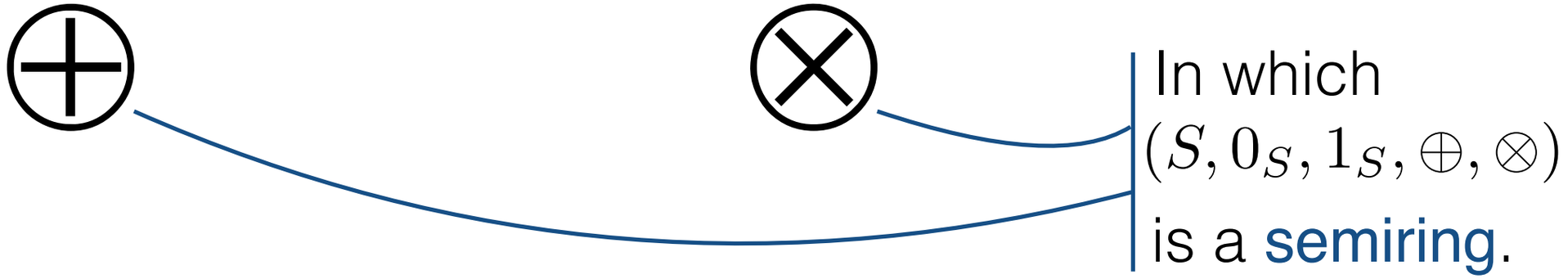

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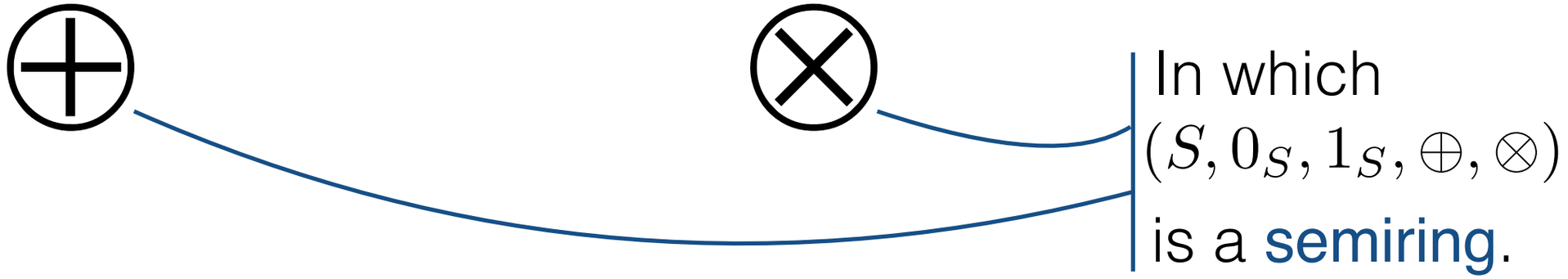

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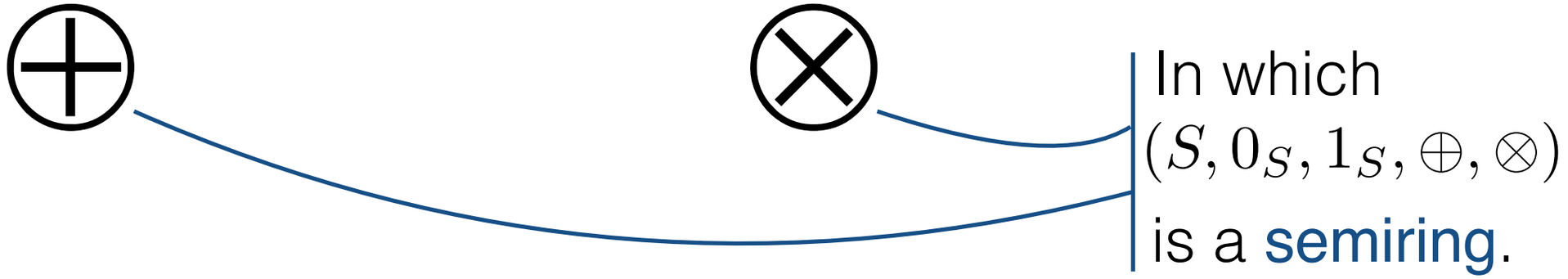

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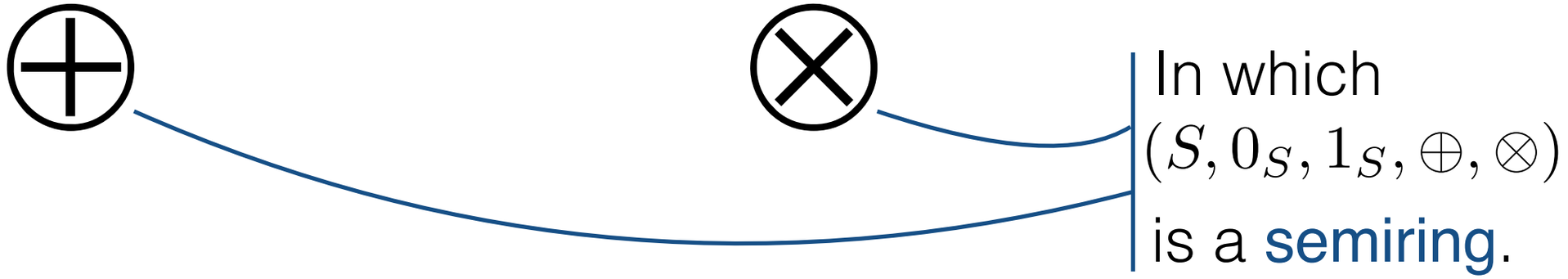

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Tropical automata: $(\mathbb{R} \cup \{-\infty\}, 0, -\infty, \max, +)$ $(\mathbb{R} \cup \{\infty\}, 0, \infty, \min, +)$

Max,+ automata Min,+ automata

Distance automata

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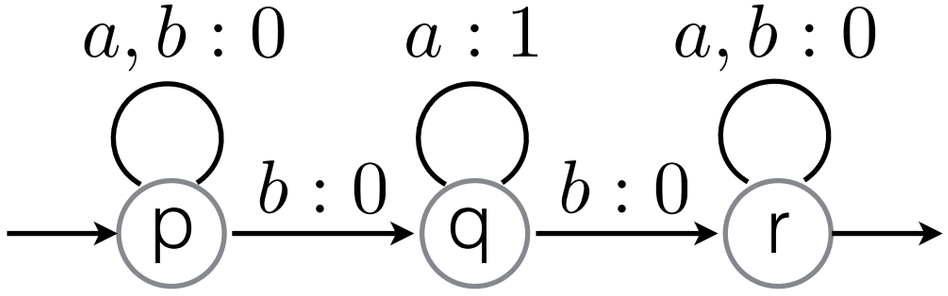

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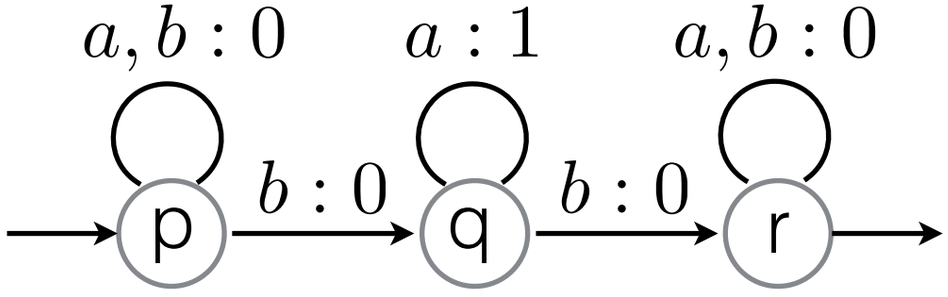

A distance automaton computes:

$$[\![\mathcal{A}]\!] : A^* \to \mathbb{N} \cup \{\infty\}$$

$u \mapsto$ the infimum over all accepting runs of the sum of weights.

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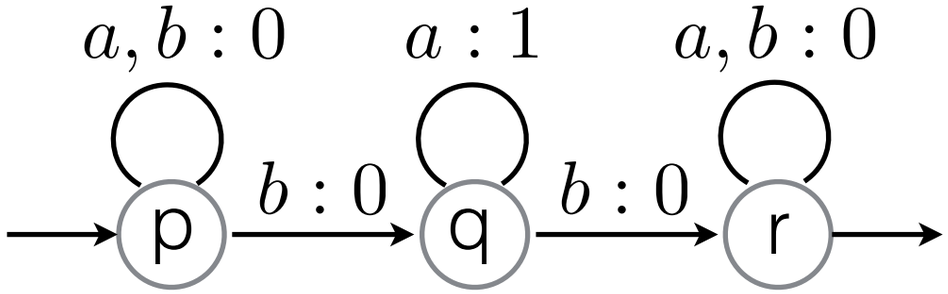

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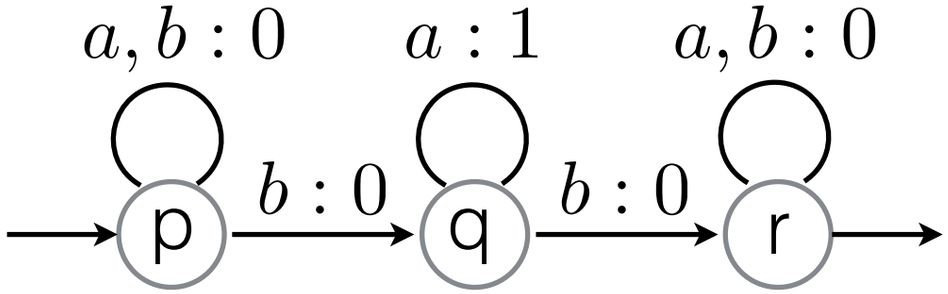

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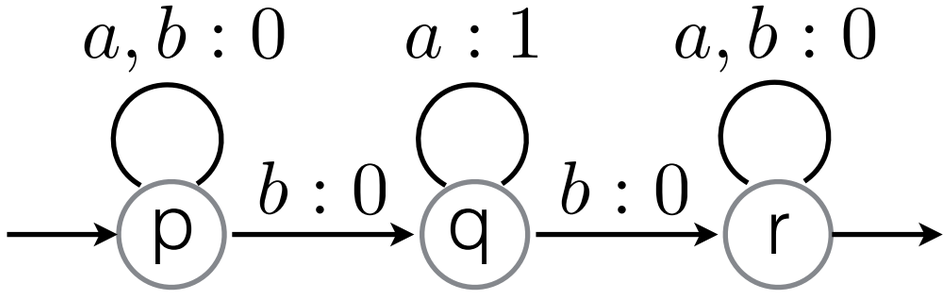

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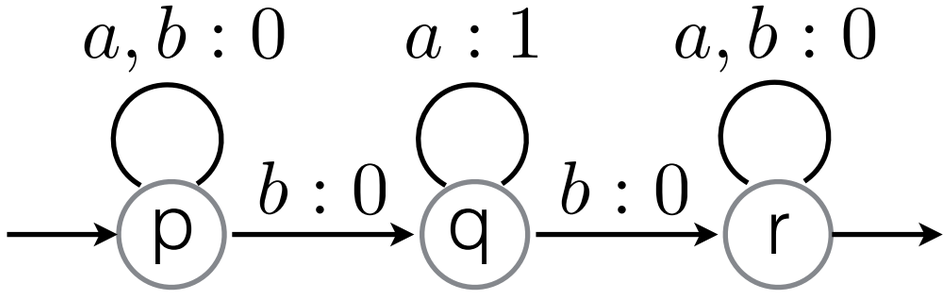

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A cost function is an equivalence class for the relation $\approx$:

$f \preccurlyeq g$ (f is dominated by g) if

for all sets X (f|X is bounded implies g|X is bounded)

Boundedness equivalence

$f \preccurlyeq g$ (f is dominated by g) if for all sets X⊆A*
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f≈g if f≼g and g≼f

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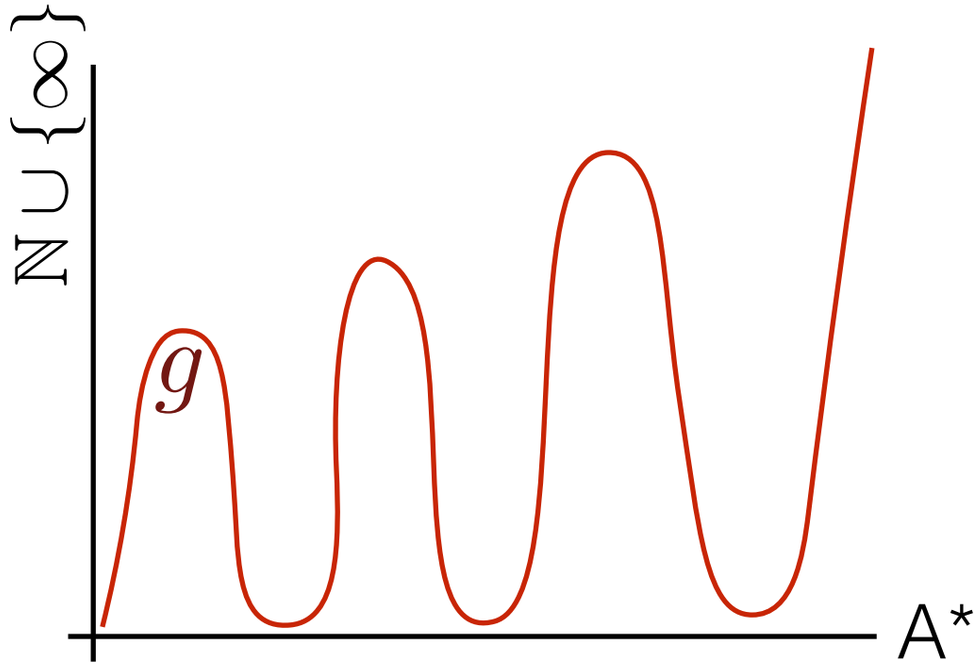

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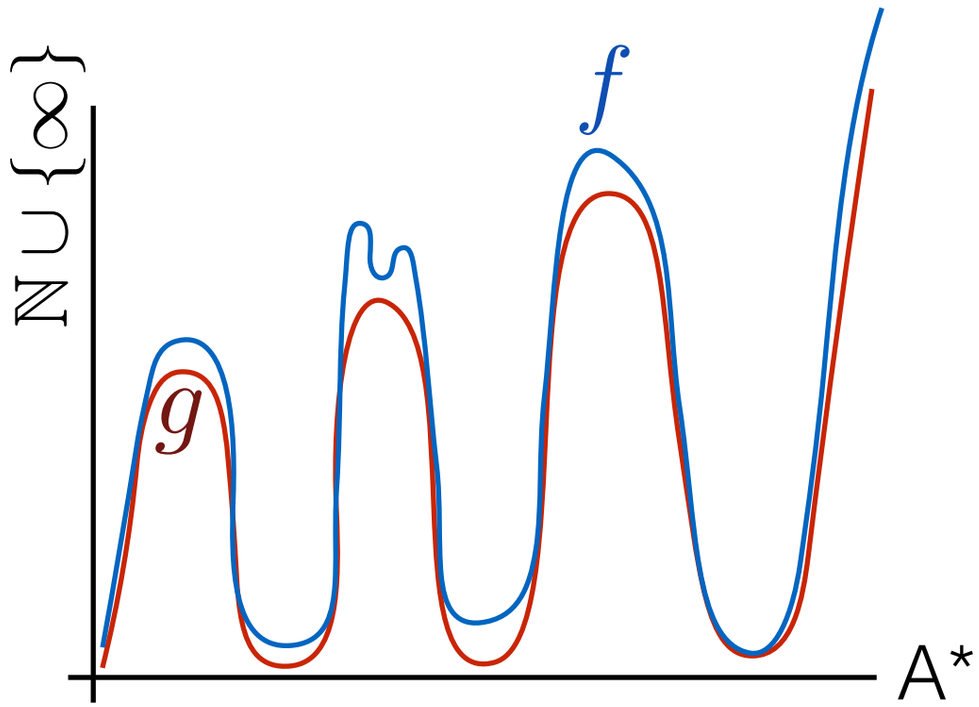

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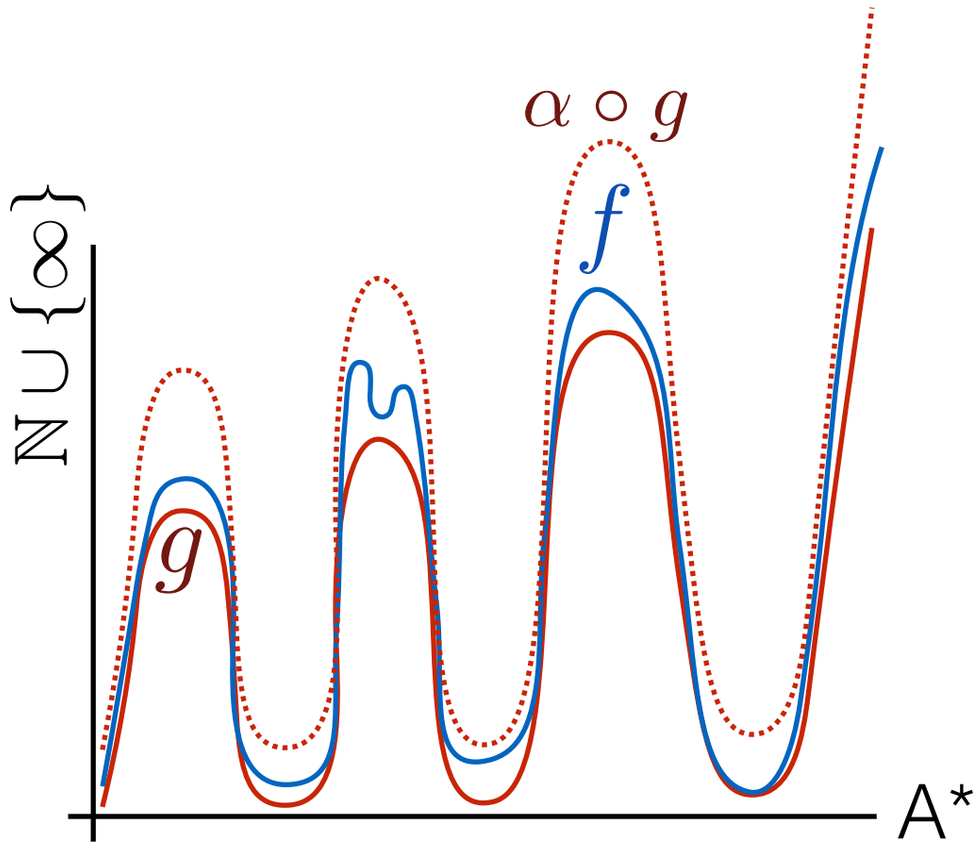

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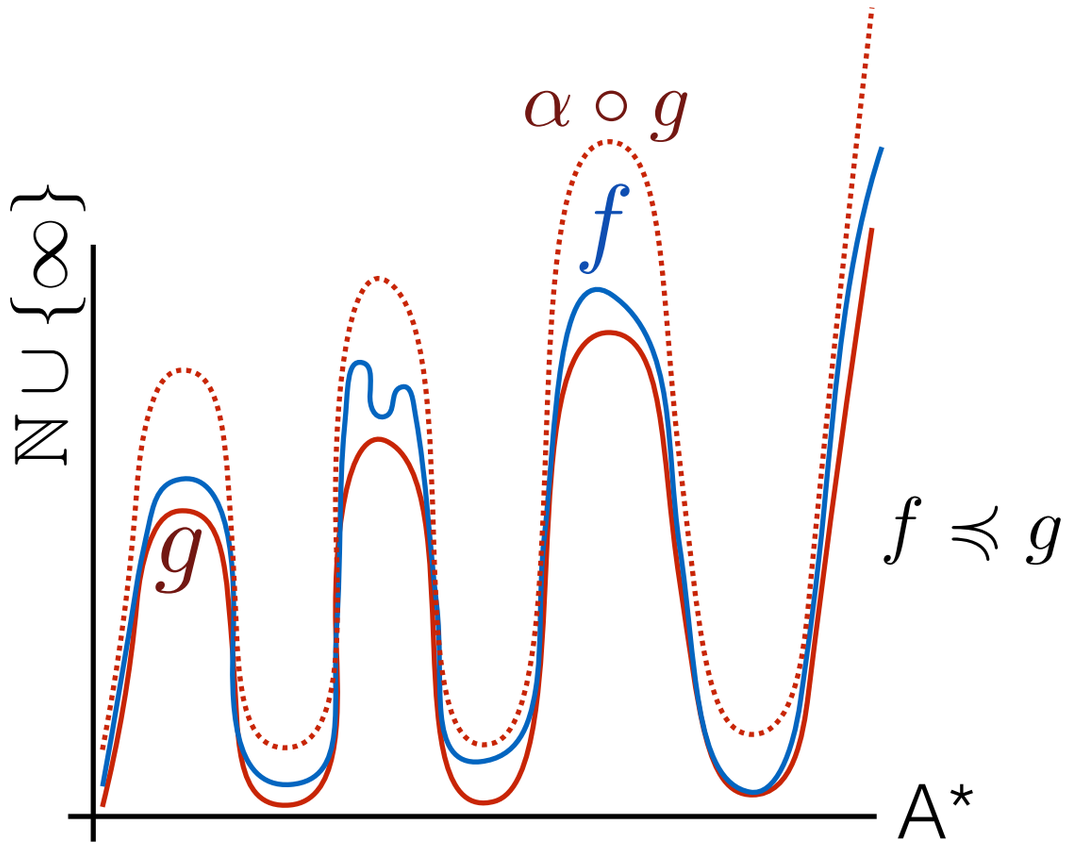

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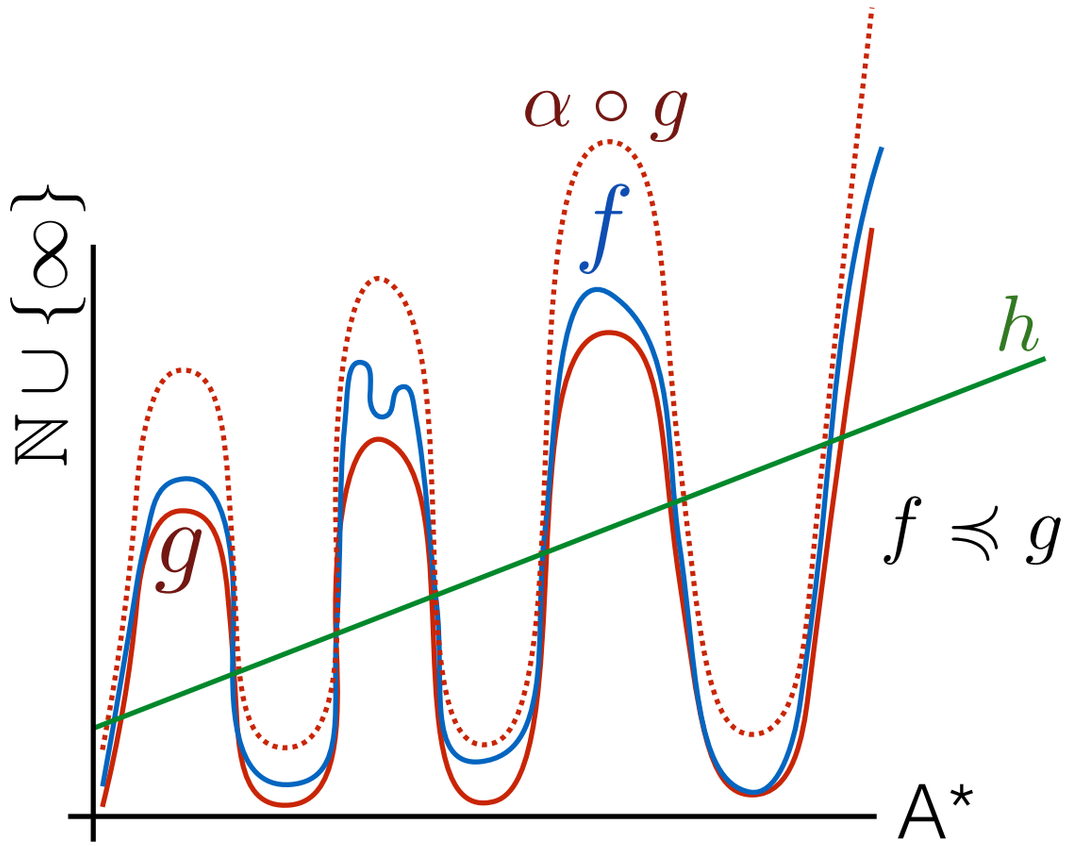

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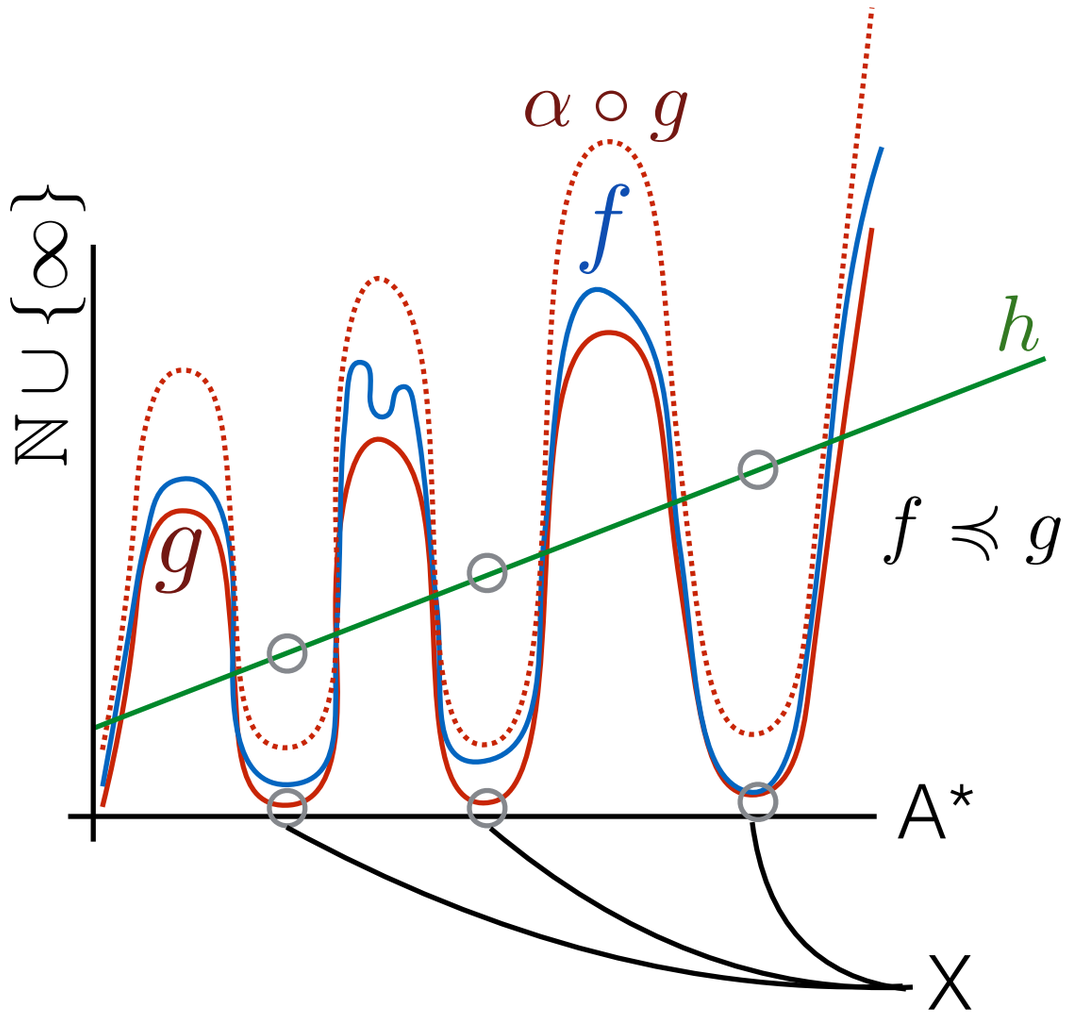

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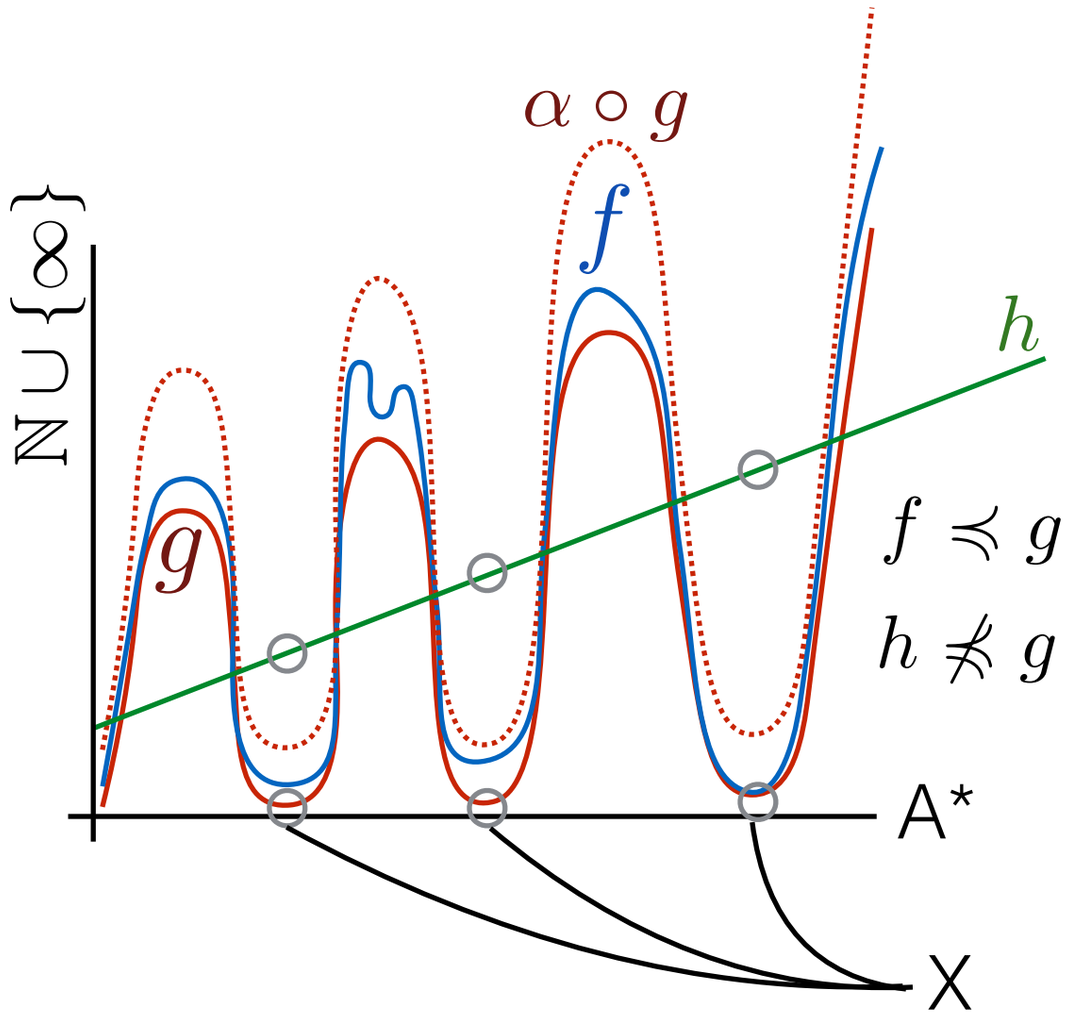

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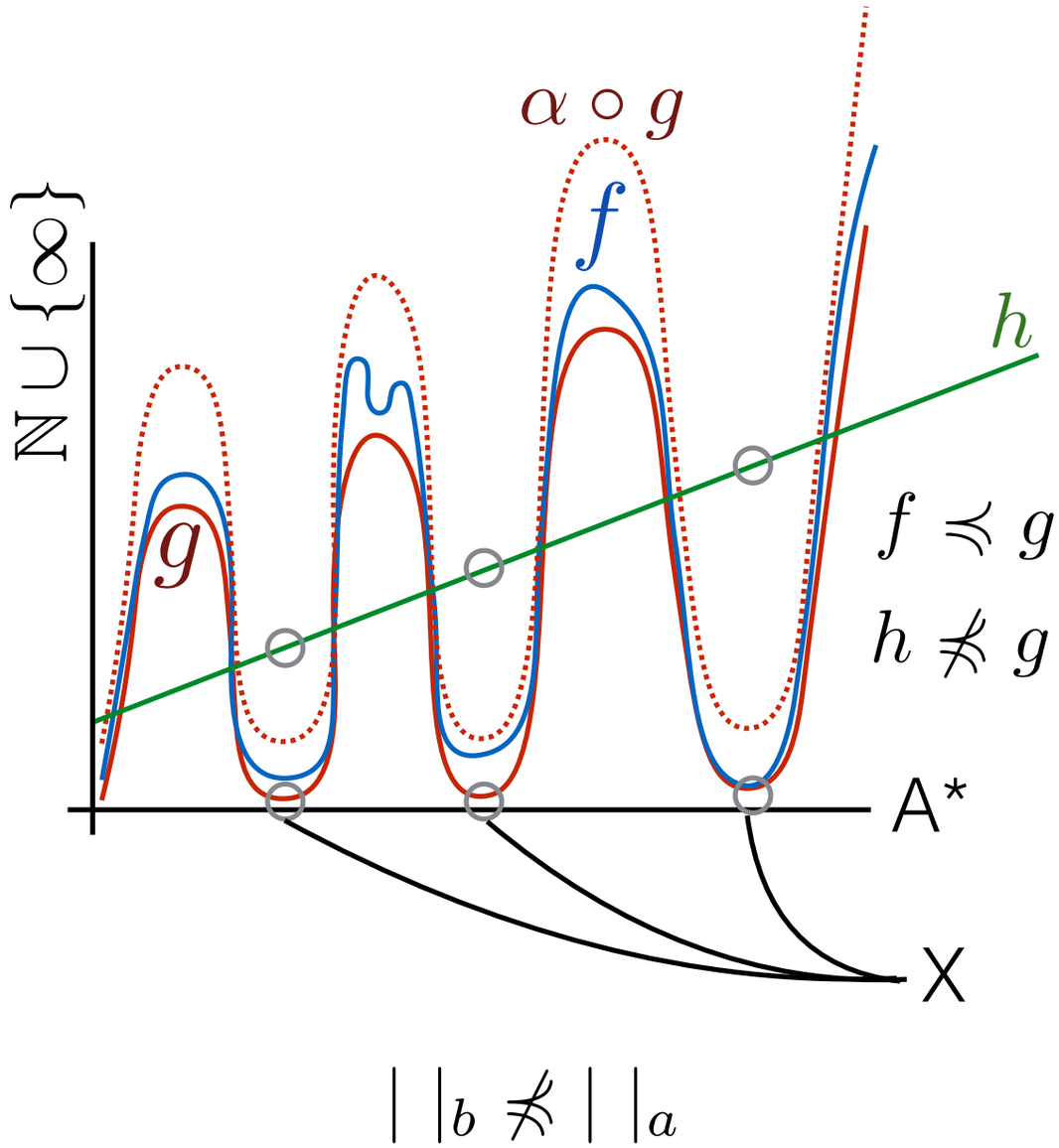

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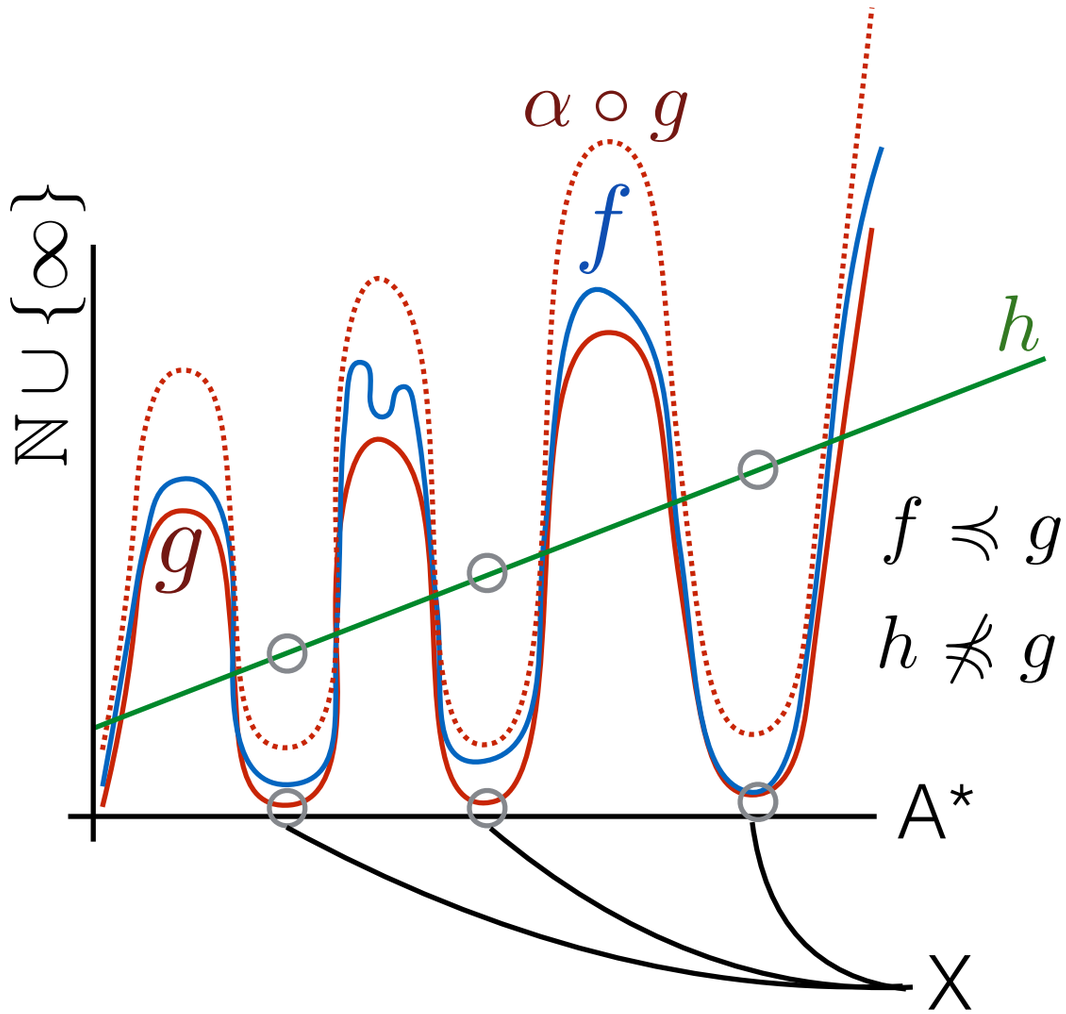

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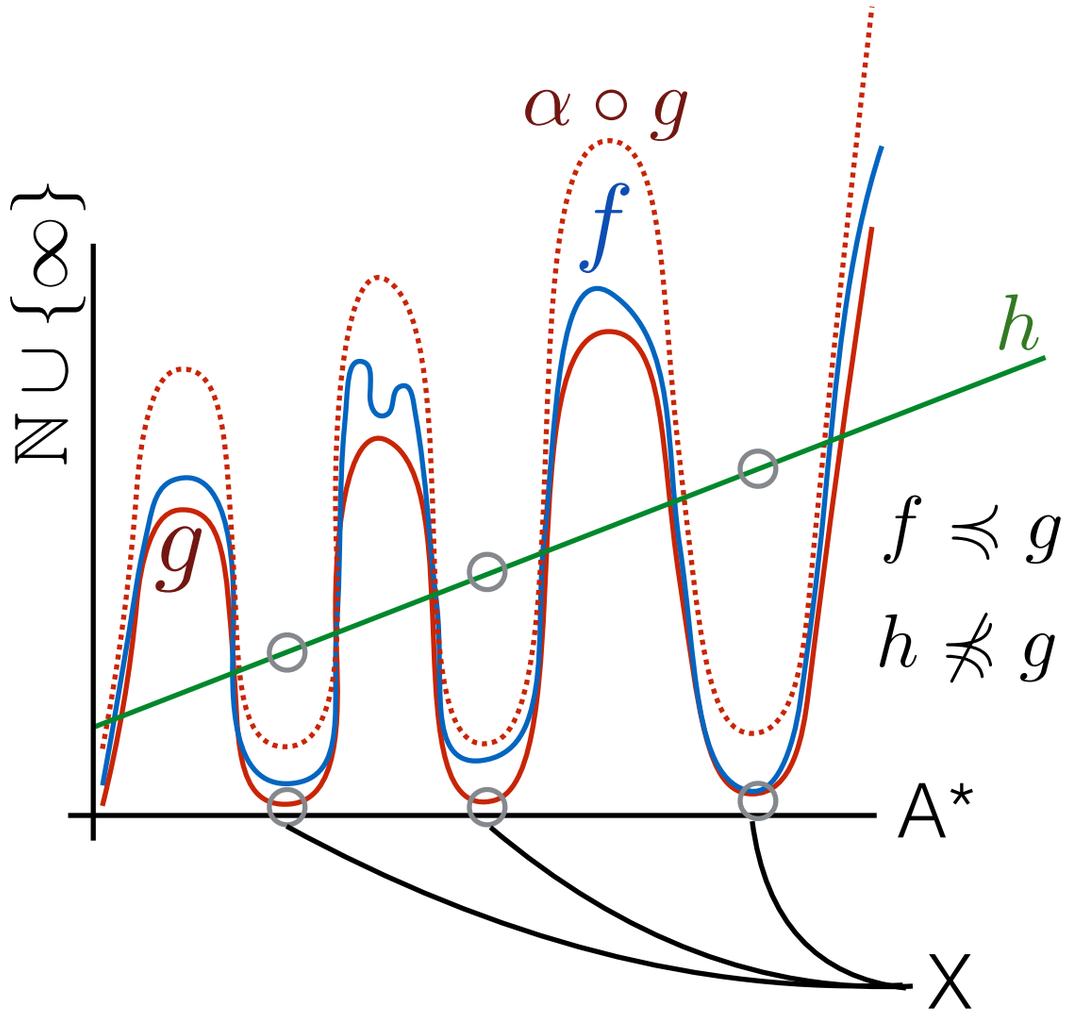

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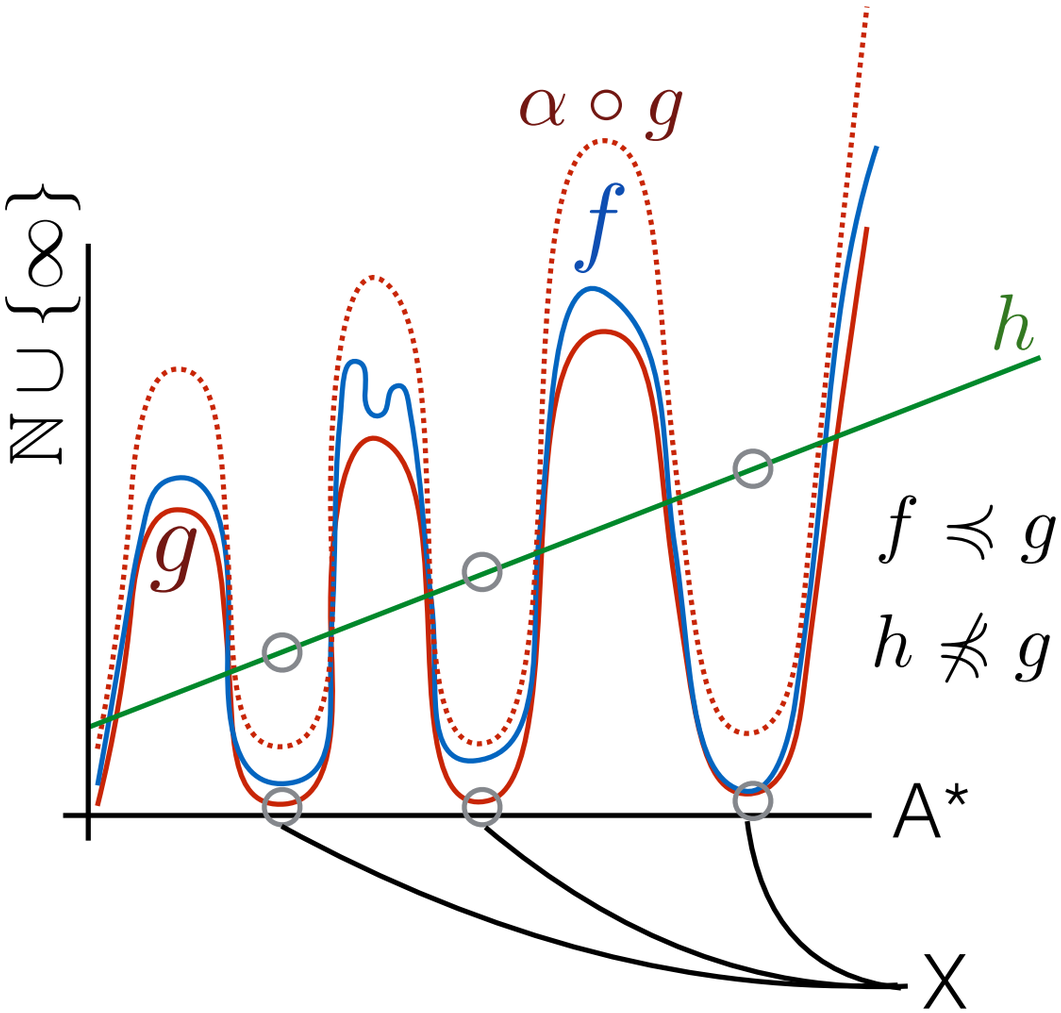

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- $F(f)$ is a filter of countable basis.

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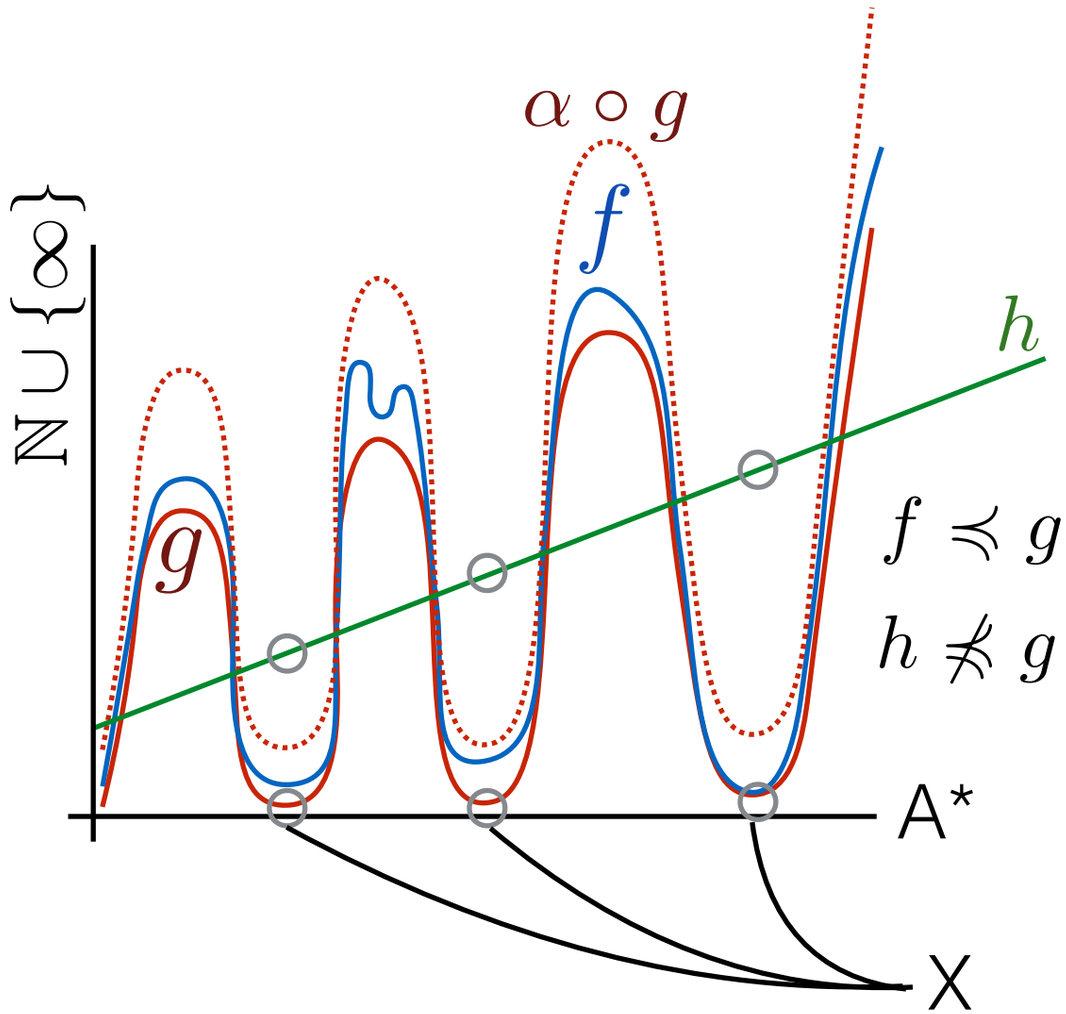

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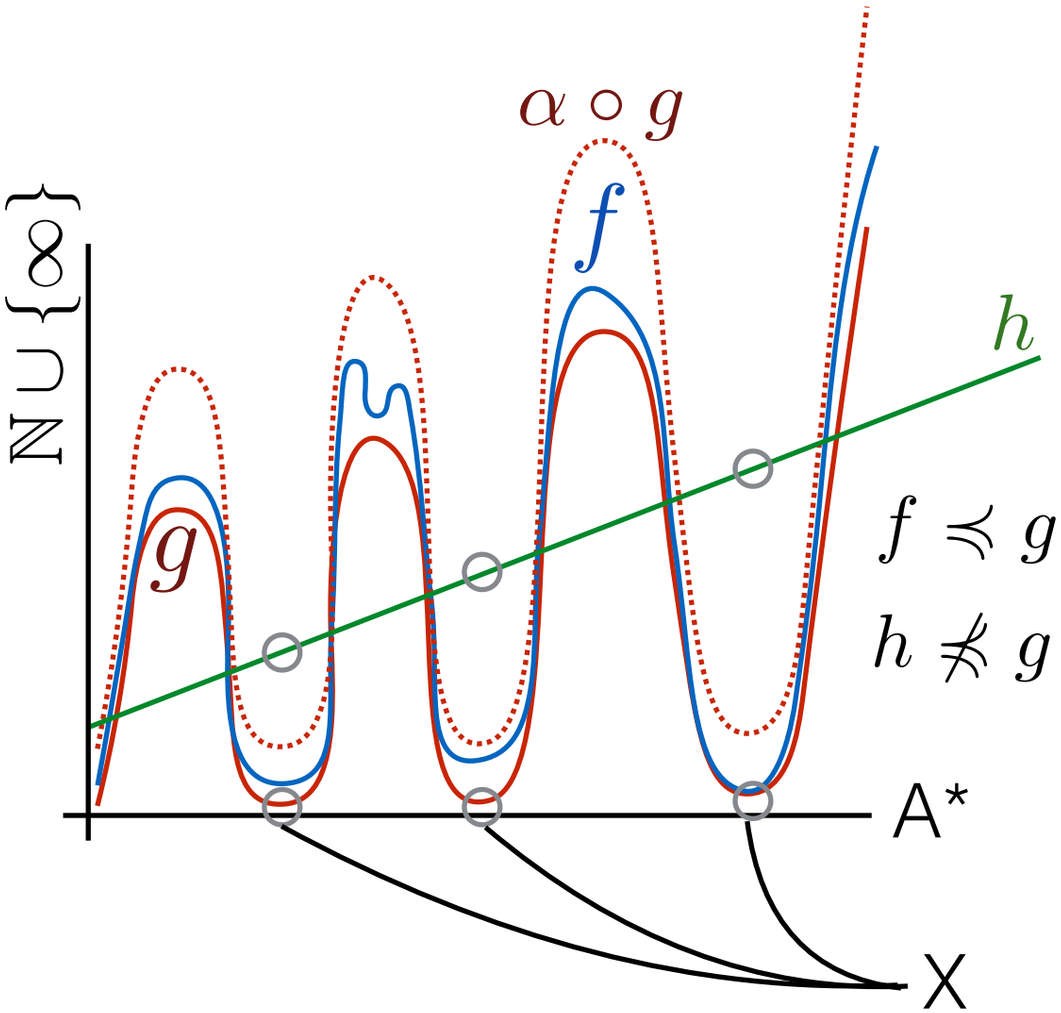

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Regular cost functions over finite words

Cost functions are effectively equivalently described by

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Over words, $|\ |_a$, $\mathrm{maxblock}$, $\mathrm{minblock}$, distance automata...

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These results (in essence) extend regular languages.

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Similar relation with all classical objects exist.

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Regular cost functions = Toolbox of results for deciding boundedness questions.

Some application of regular cost functions

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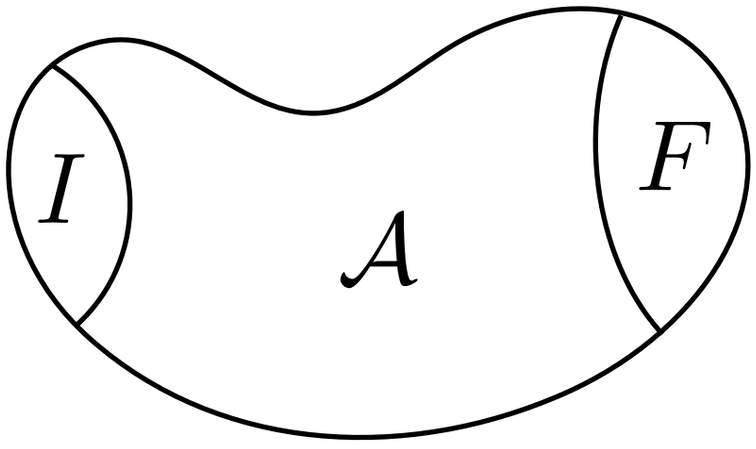

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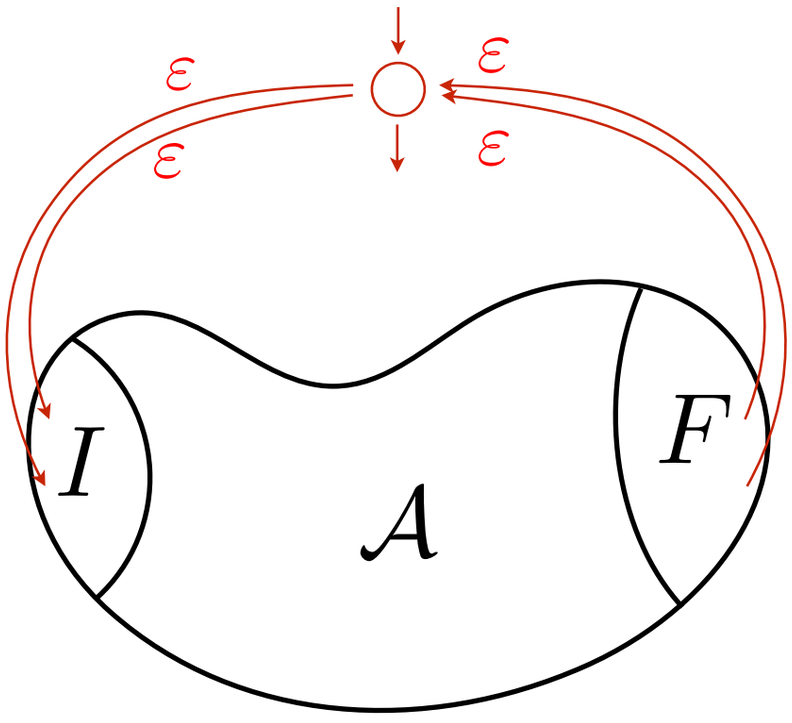

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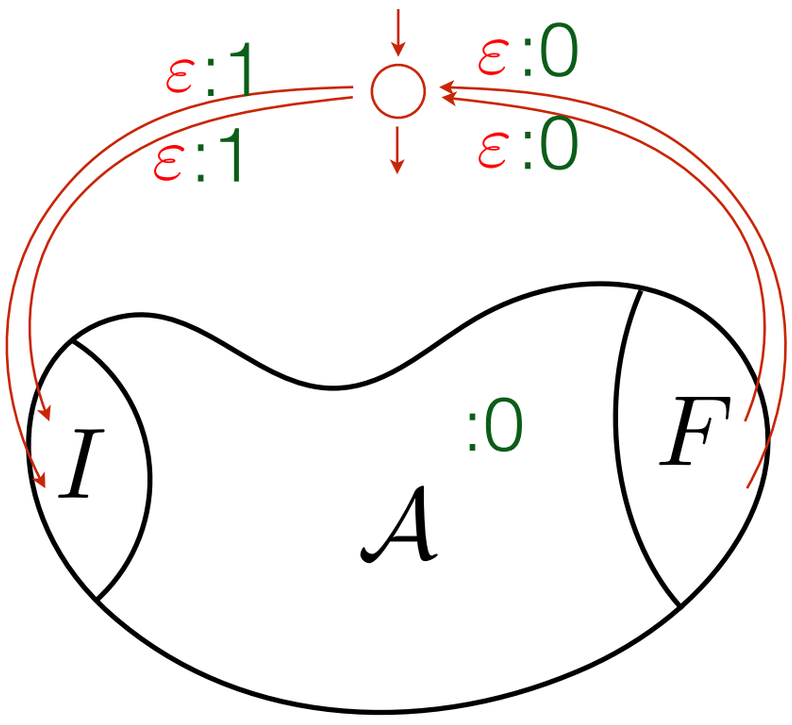

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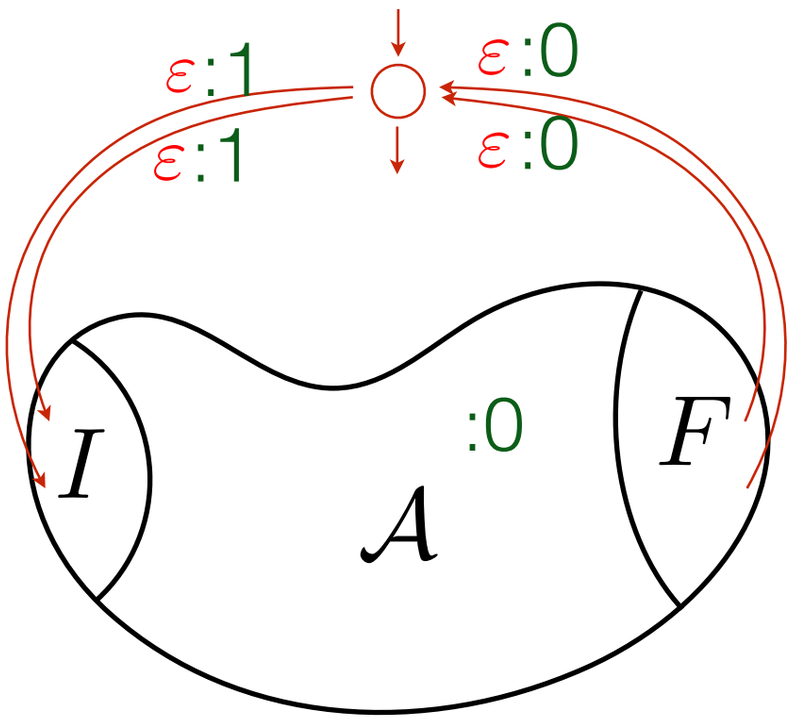

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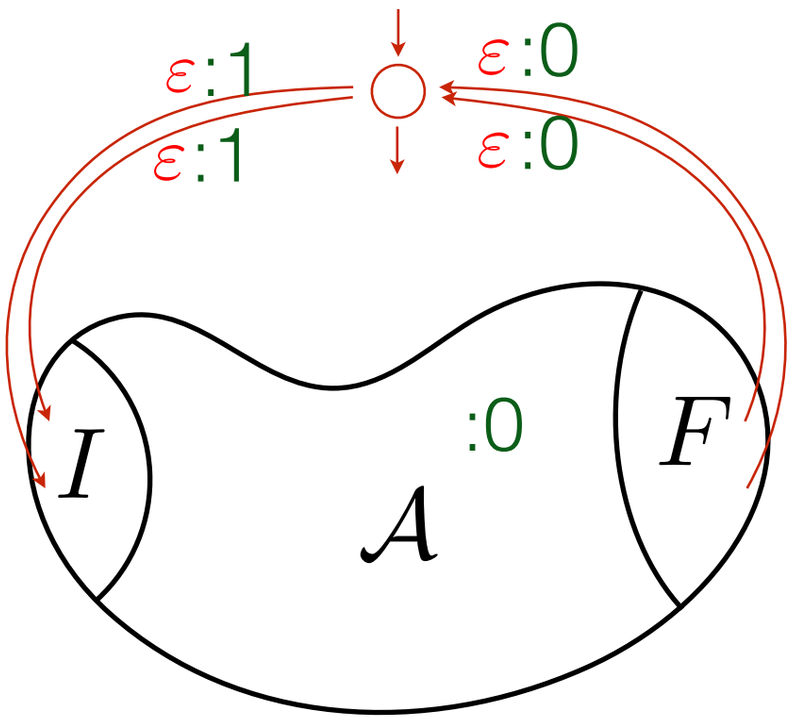

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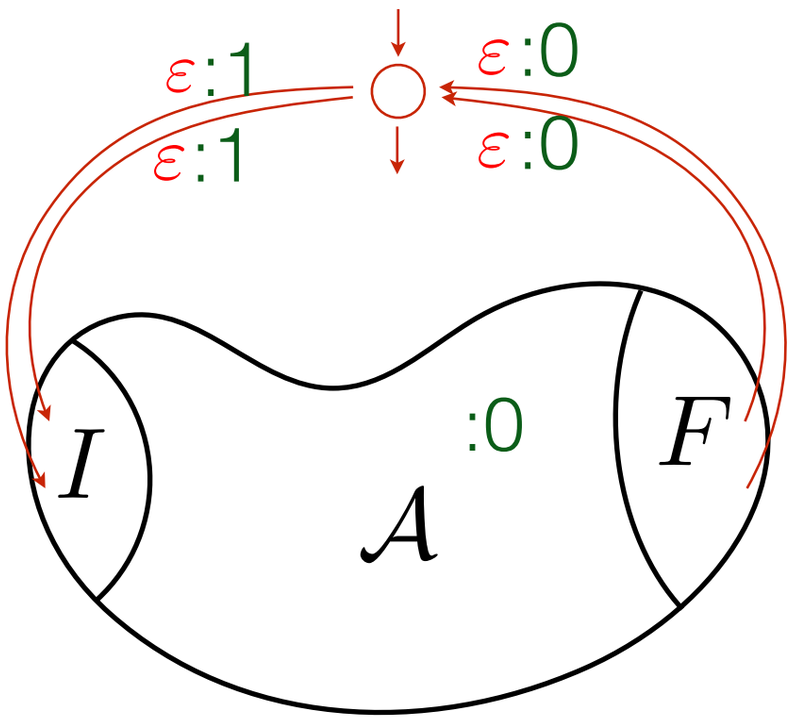

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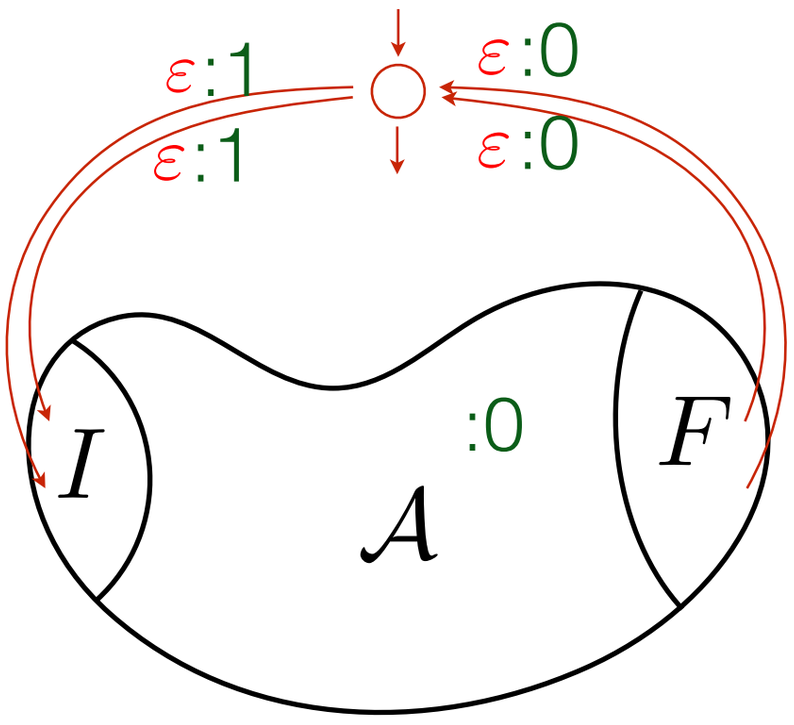

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[Simon78] The finite power property is decidable.

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Does there exist $n \in \mathbb{N}$ such that $F^n_{\mathfrak{A}}(\emptyset) = \mathit{fixpoint}(F_{\mathfrak{A}})$ for all $\mathfrak{A}$?

The map $(t, x) \mapsto \inf\{n \mid x \in F^n_t(\emptyset)\}$ for x a node of an infinite tree t is computed by an alternating two-way distance parity automaton running over t starting from x.

Boundedness of such automata is decidable.

[Blumensath&Otto&Weyer12] The boundedness of monadic second-order logic fixpoints is decidable over infinite trees.

Membership to (non-deterministic) classes

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Many other classes known, over finite words,

Σ_1 FO_2 Σ_2 Σ_3 Σ_4 $B(\Sigma_1)$ $B(\Sigma_2)$ etc…

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[C.&Löding 08] The star-height is decidable over trees.

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We have $L \subseteq \mathcal{L}(L_{E,n})$.

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[Regular cost functions] immediately yes.

Regular cost functions as languages

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[Toruńczyk 11] One can identify regular cost functions with a language of profinite words, in a natural way.

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- idealization: (simplified) there exists a non-standard integer.

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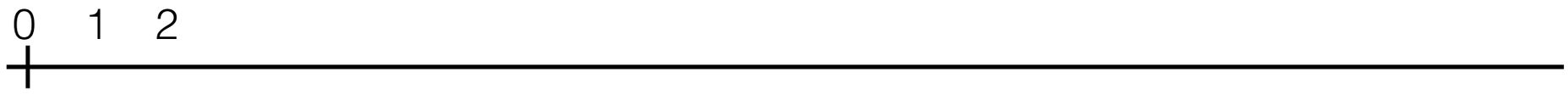

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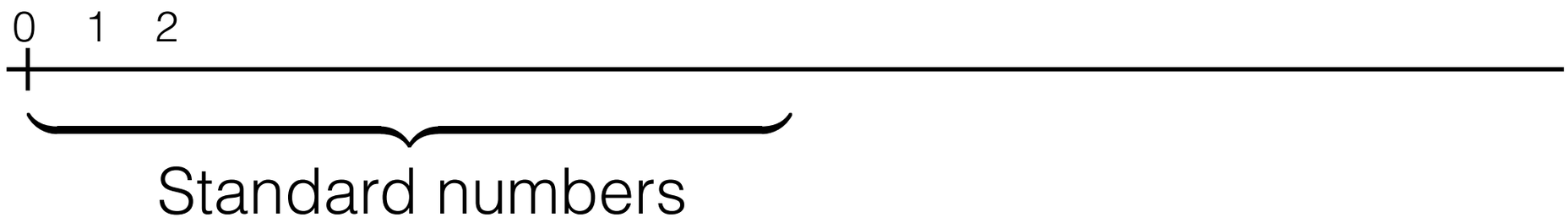

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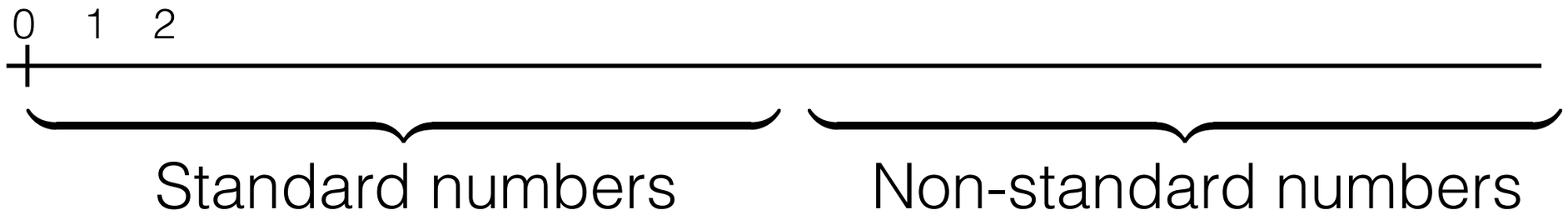

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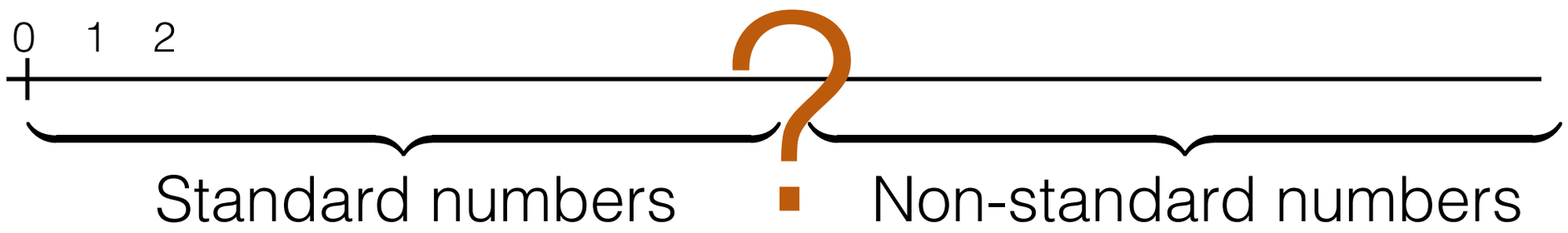

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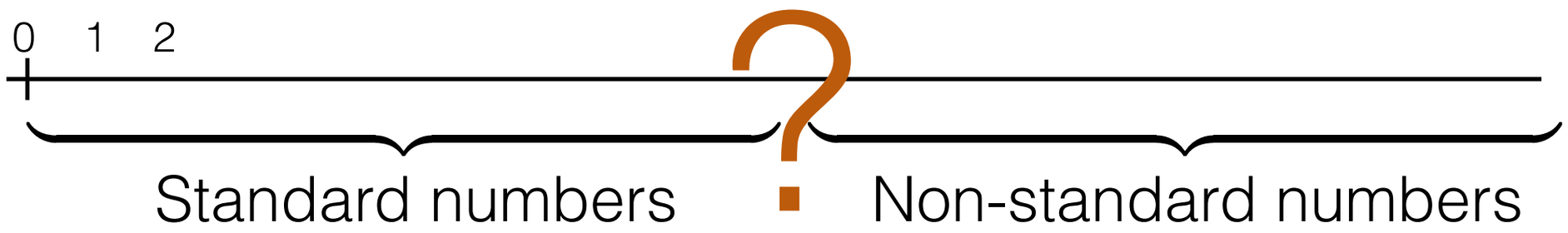

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The comprehension schema

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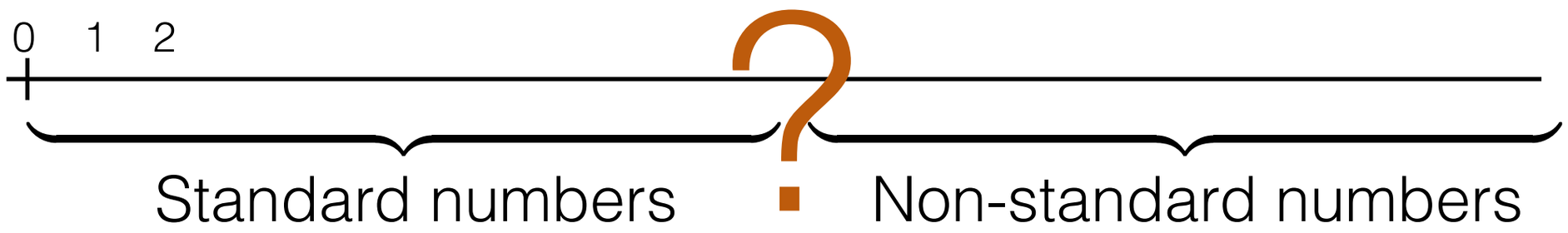

The set of standard integers does not exist!

The comprehension schema

$$\{x \in A \mid \varphi\}$$

is only valid for internal formulae!!!!!

Regular cost functions in IST

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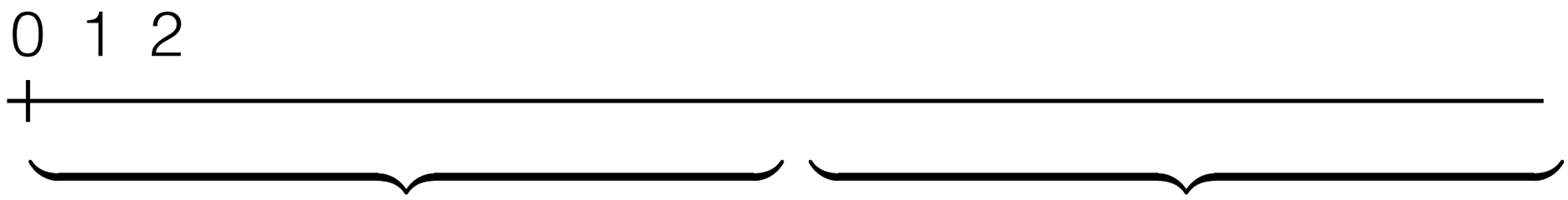

Given a standard function $f : A^* \to \mathbb{N} \cup \{\infty\}$, define:

$$L_f = \{u \in A^* \mid \underbrace{\texttt{St}(f(u)) \wedge f(u) \neq \infty}_{f(u) \text{ is small}}\}^{external}$$

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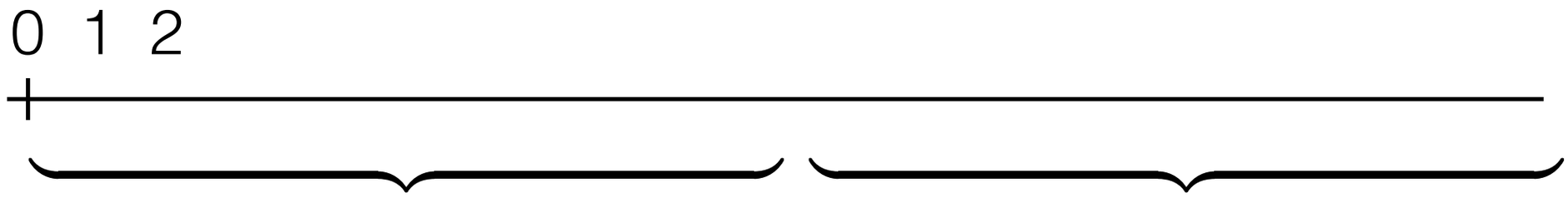

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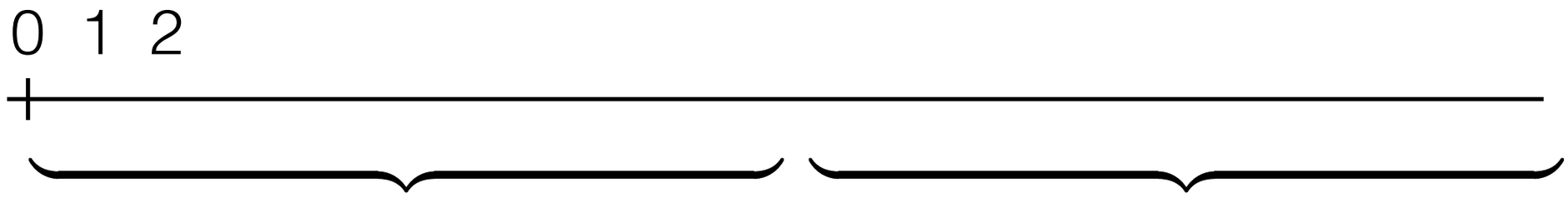

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Under this view, the theory of regular cost functions can be recast into a theory of regular external languages.

Regular external languages

Regular external languages

B-rational expression

rational expression + $L^{\mathtt{St}}$

$(a^{\mathtt{St}}b)^*a^{\mathtt{St}}$

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rational expression + $L^{\neg\mathtt{St}}$

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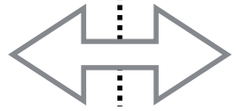

complement

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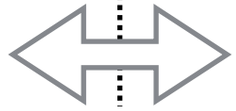
B-automata

ND automata + counters
with standardness constraints



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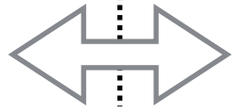

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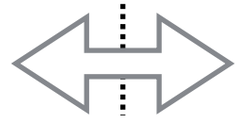
cost MSO

MSO + $\mathtt{St}(|X|)$ (positively)

‘there is a maximal set of consecutive a’s of standard size’



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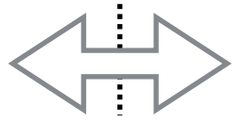

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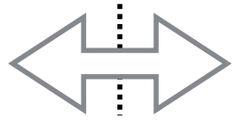

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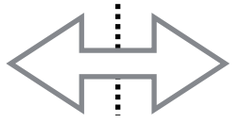

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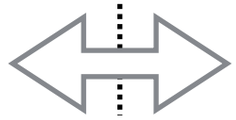

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Emptiness of Boolean combinations is decidable.

Conclusion

Regular cost functions are functions computed by variants of tropical automata modulo a ‘boundedness equivalence relation’.

The most important results that hold for regular languages are still valid.

It can be used to decide several problems involving the existence of bounds, be it explicitly or implicitly.

Thanks to non-standard analysis, the results can be seen as a language theory over less usual words, in the same spirit as infinite words/words of countable length.