Omega Regularity with Bounds

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Omega Regular languages

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E.g:

$$x \leq y \quad := \quad \forall X.(x \in X \wedge (\forall z.z \in X \rightarrow S(z) \in X)) \rightarrow y \in X$$

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Idea: Transform each monadic formula into an ‘equivalent’ automaton.

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Proof idea: By induction on the structure of the formula.

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$$u, \vec{X} = \vec{A} \models \exists Y.\phi \qquad \text{iff } \langle u, \vec{A} \rangle \in \pi(L(\mathcal{A}_\phi)) \quad (\pi \text{ projection cancelling } Y)$$

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In general, to each connective/quantifier and atomic predicate corresponds a language theoretic operation.

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Only complementation is non-trivial
(Based on Ramsey theorem).

A MORE GENERAL VIEW OF THE PICTURE

THM(Buchi,...)The following properties are equivalent:

- L is the set of ω-words models of a monadic formula

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- L is modelling a μ-calculus formula.

EXTENSIONS

Extension of the equivalence logic/automata

- to countable ordinals (Buchi)
- to trees (Rabin)
- to iteration structures (Muchnik,Walukiewicz)
- to countable scattered linear orderings (Carton& Rispal)

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Extension of models of decidable monadic theory

- pushdown graphs (Muller& Shupp)
- HR-equational graphs/structures (Courcelle)
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Extension of the expressivity of the logic: None going beyong ω-regularity

Extending ω-regularity with bounds

EXTENDING ω-REGULAR EXPRESSIONS

Syntax of ω-regular expressions:

$$R = \emptyset \ \mid \varepsilon \ \mid a \ \mid R + R \mid R.R \ \mid R^* \qquad\qquad O = R^\omega \ \mid R.O \ \mid O + O$$

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- B constrains the number of iteration to be bounded
  (B stands for 'bounded')

  $(a^B b)^\omega$ there is an infinite number of b's and,
  the number of a's between consecutive b's is bounded

$$ababab\cdots \in (a^B b)^\omega$$

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  (S stands for 'strictly unbounded')

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$$ababab\cdots \notin (a^S b)^\omega \qquad aba^2baba^3baba^4b\cdots \notin (a^S b)^\omega$$

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- $U^\omega \mapsto \{u_1u_2\ldots \;:\; \vec{u}\in U\}$

Other examples

$$((a^*b)^*a^Sb)^\omega$$

OTHER EXAMPLES

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The languages with an infintie number of b's such that the size of segments of a is not bounded.
E.g. $aba^2baba^3baba^4b\cdots \in ((a^*b)a^Sb)^\omega$

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There is an infinite number of c's. The number of $b's$ between two c's tends toward the infinite. The number of a's between two b or c's tends toward the infinite.

THE DIAMOND

ωBS-regular expressions
e.g. $(a^B b^S c)^\omega$

ωS-regular expressions $\subset$ ωBS-regular expressions

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PROP: Emptyness of ωBS-regular languages is decidable

The diamond

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$\subset$ $\supset$

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ωB-regular expressions
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$\supset$ $\subset$

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PROP: The inclusions in the diamond are strict.

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They come in two variants hierarchical or not.

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ωBS-automaton =

finite automaton

\+ finite set of counters, of kind B or S

\+ each counter can be left unchanged or reset or incremented

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Essentially: finite state automata with modified accepting condition (more expressive than Buchi).
They come in two variants hierarchical or not.

ωBS-automaton =

finite automaton

\+ finite set of counters, of kind B or S

\+ each counter can be left unchanged or reset or incremented

A run is accepting if

- every counter is reset infinitely many times
- every B-counter is incremented a bounded number of times between resets
- the number of increments of an S-counter between reset tends toward the infinite

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THM: The following are equivalent:

- L is evaluation of an ωBS-regular expression
- L is accepted by an ωBS-automaton
- L is accepted by a hierarchical ωBS-automaton

And the same holds for ωB and ωS regular languages.

THE DIAMOND (2:AUTOMATA)

ωBS-regular expressions
hierarchical ωBS-automata
ωBS-automata

$\cup$ $\cup$

ωS-regular expressions
hierarchical ωS-automata
ωS-automata

ωB-regular expressions
hierarchical ωB-automata
ωB-automata

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ω-regular expressions
Büchi automata

CLOSURE

THM: The ωBS-regular languages are closed under union, intersection, projection.

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PROP: ωBS-regular languages are not closed under complementation.

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PROP: ωBS-regular languages are not closed under complementation.

Mp: The complement of $L = ((a^B + a^S)\sharp)^\omega$ is not ωBS-regular.

The language $\complement L$ contains $a^{f(1)}ba^{f(2)}b\cdots$ iff there exists infinitely many values appearing infinitely often in f.

E.g: $a^1b \;\; a^1ba^2b \;\; a^1ba^2ba^3b \;\; \cdots \in \complement L$

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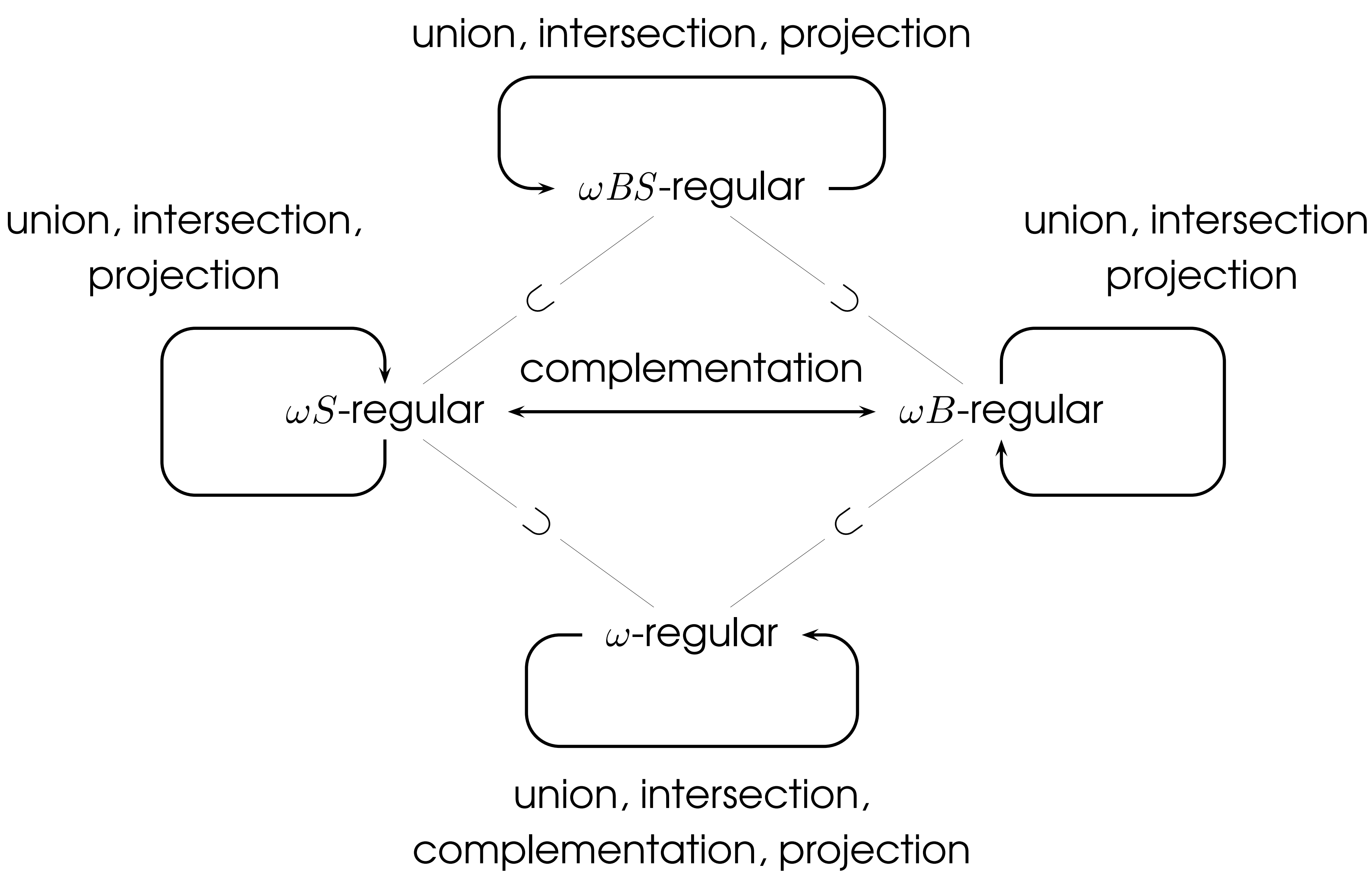
THE DIAMOND (3:CLOSURE)



Monadic logic with bounds (MSOLB)

The logic MSOL :

$$\phi = \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \exists X.\phi \mid \exists x.\phi \mid x \in X \mid x = S(y) \mid a(x)$$

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With $\mathbb{U} X . \phi \quad \equiv \quad \forall n . \ \exists X . \ (|X| > n) \wedge \phi$

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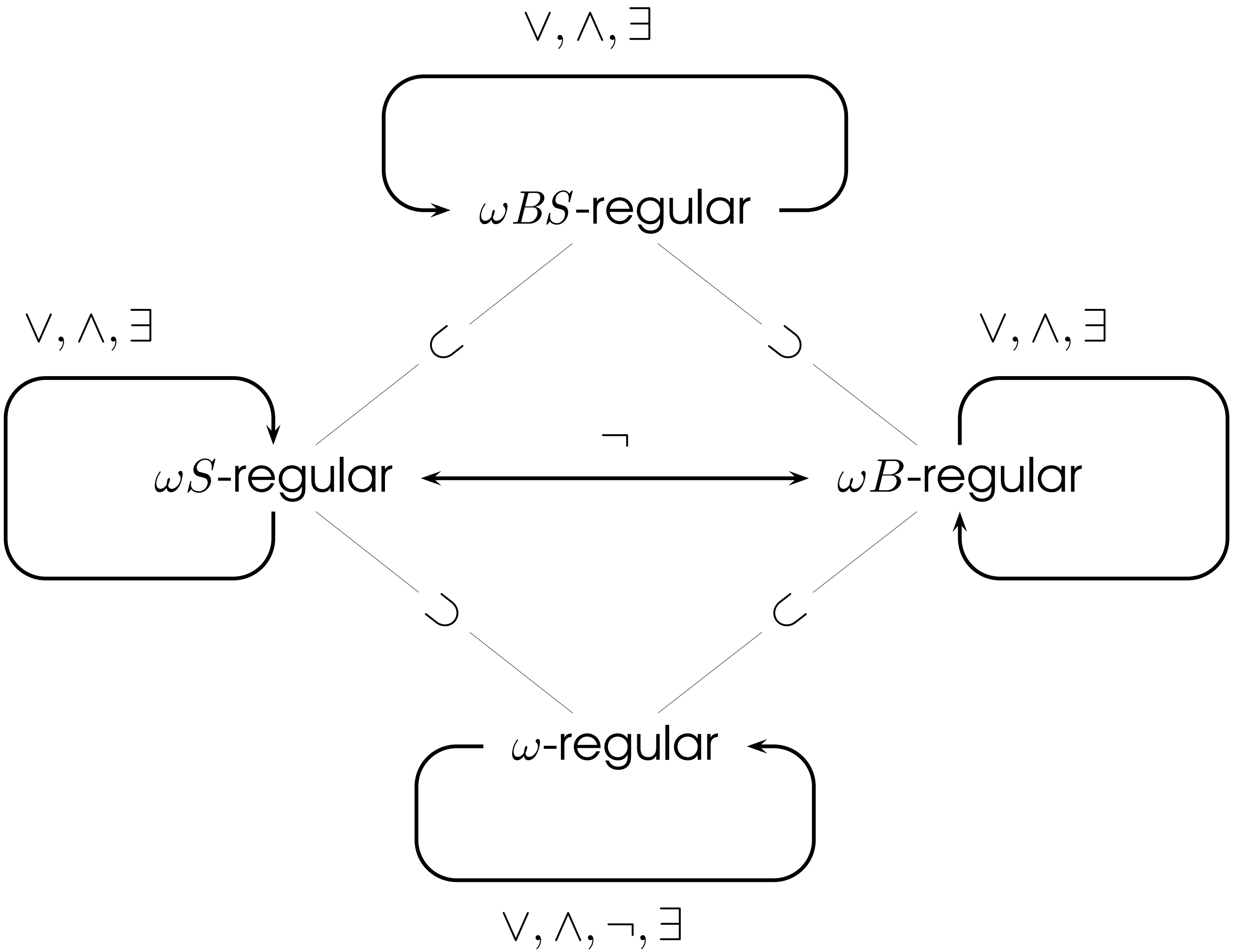
But ωBS-regularity provides a partial answer.

THE DIAMOND (3:LOGIC)



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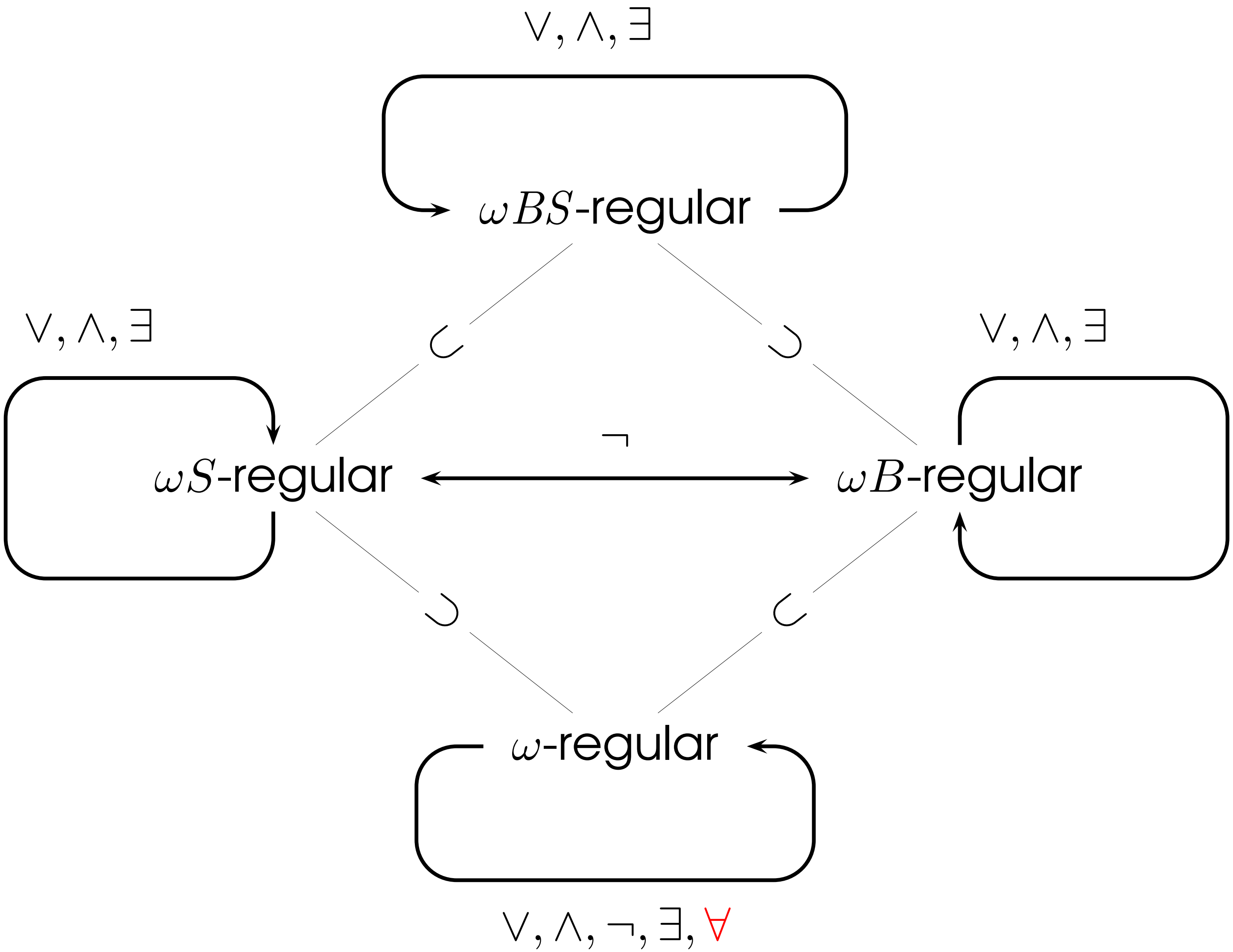

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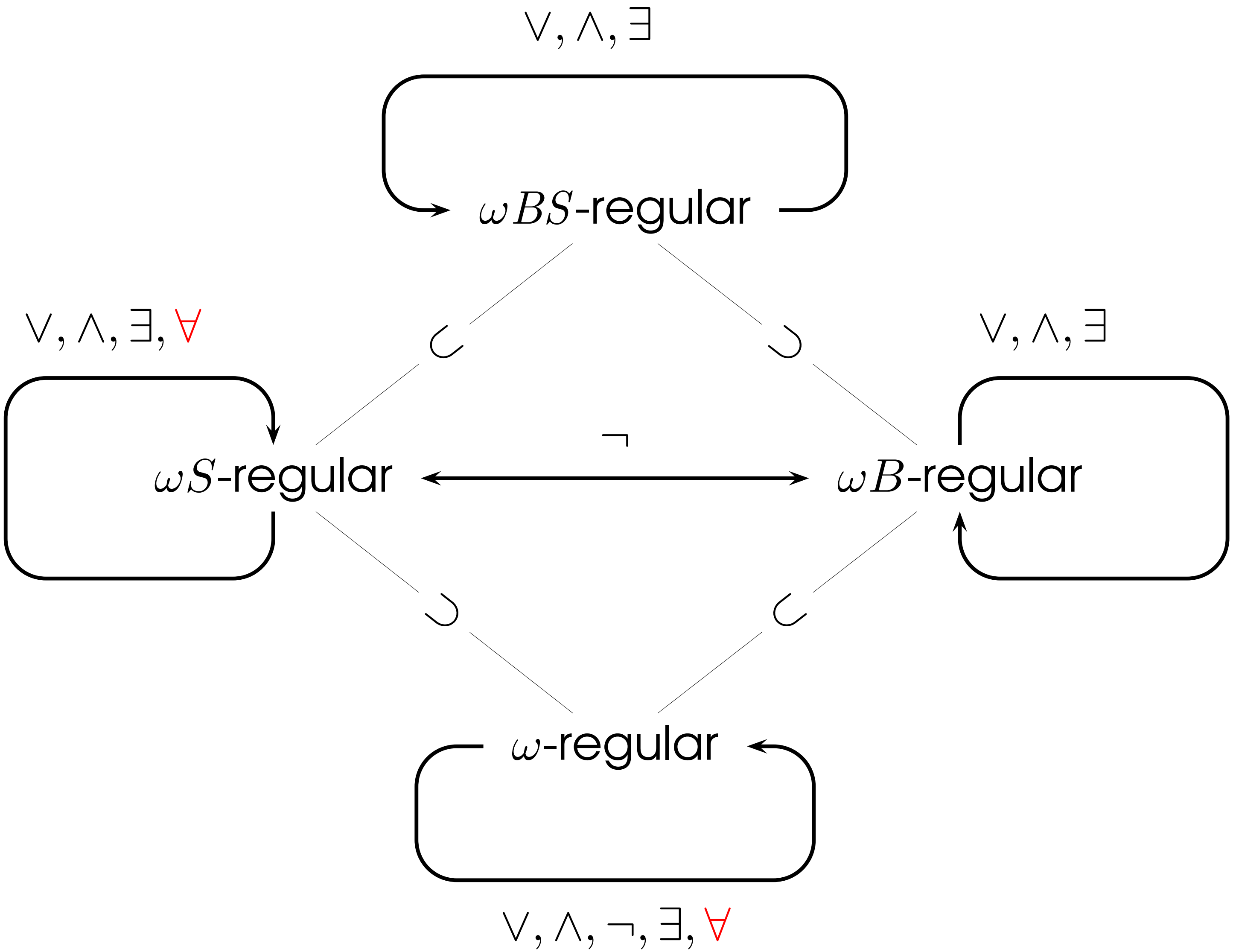

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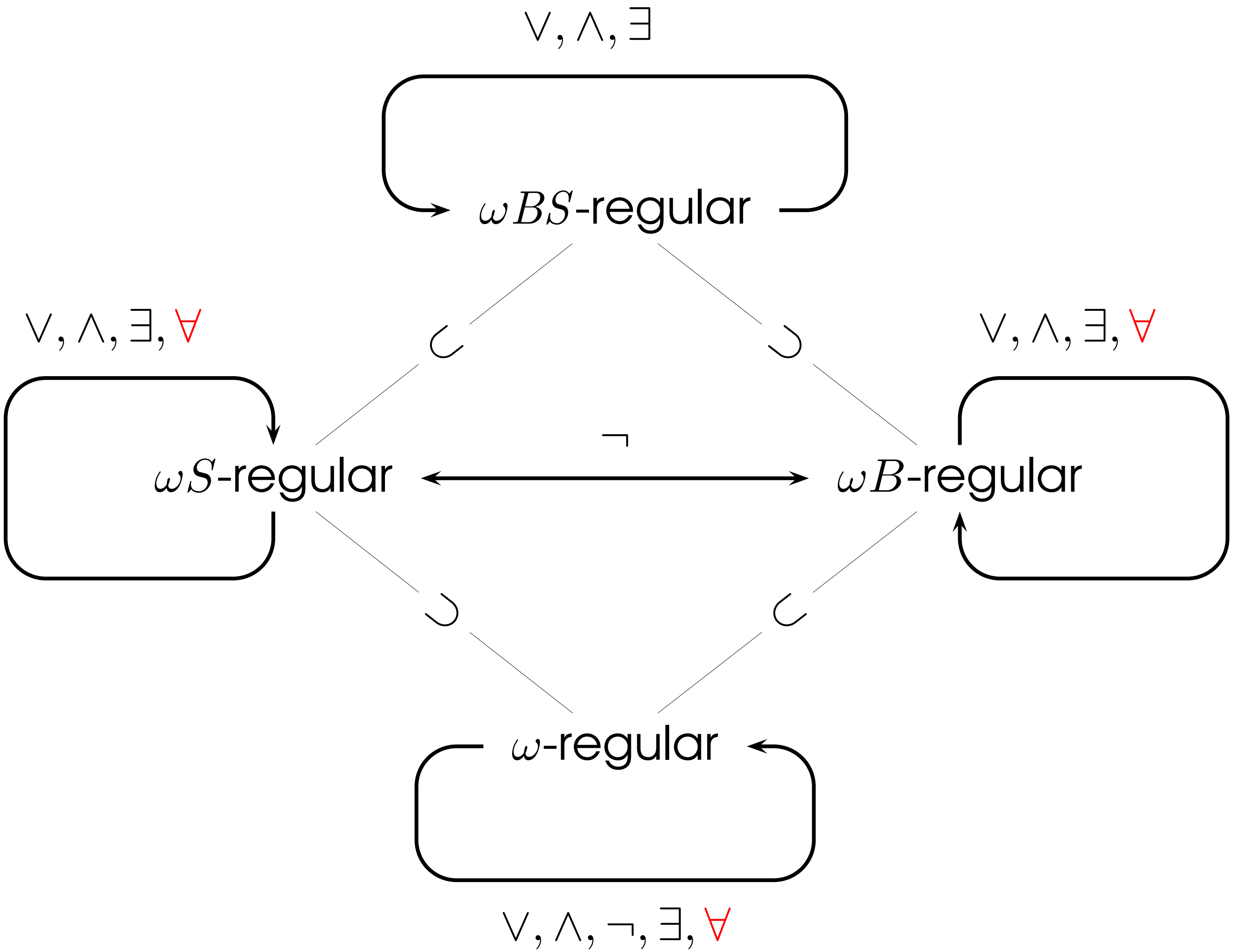

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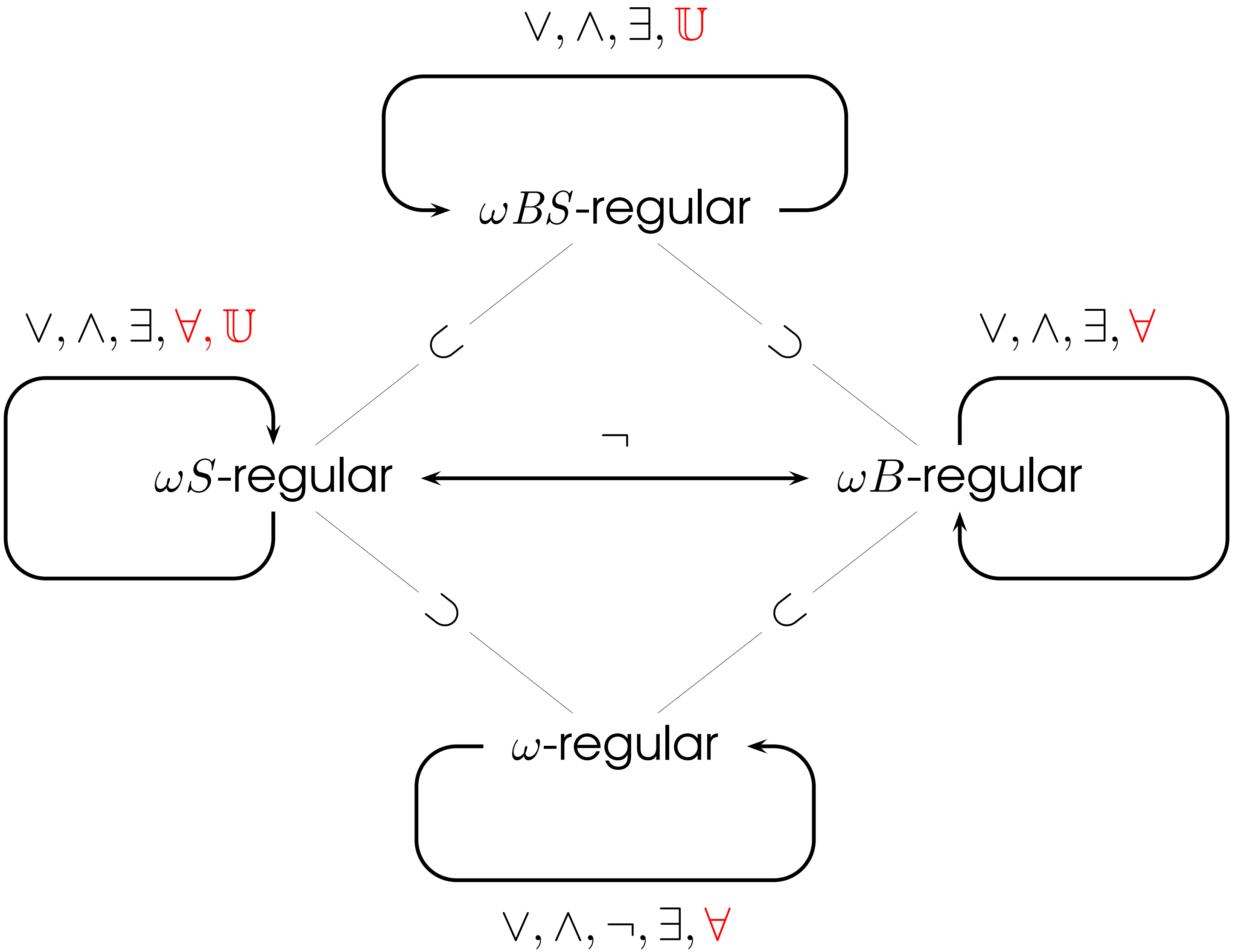

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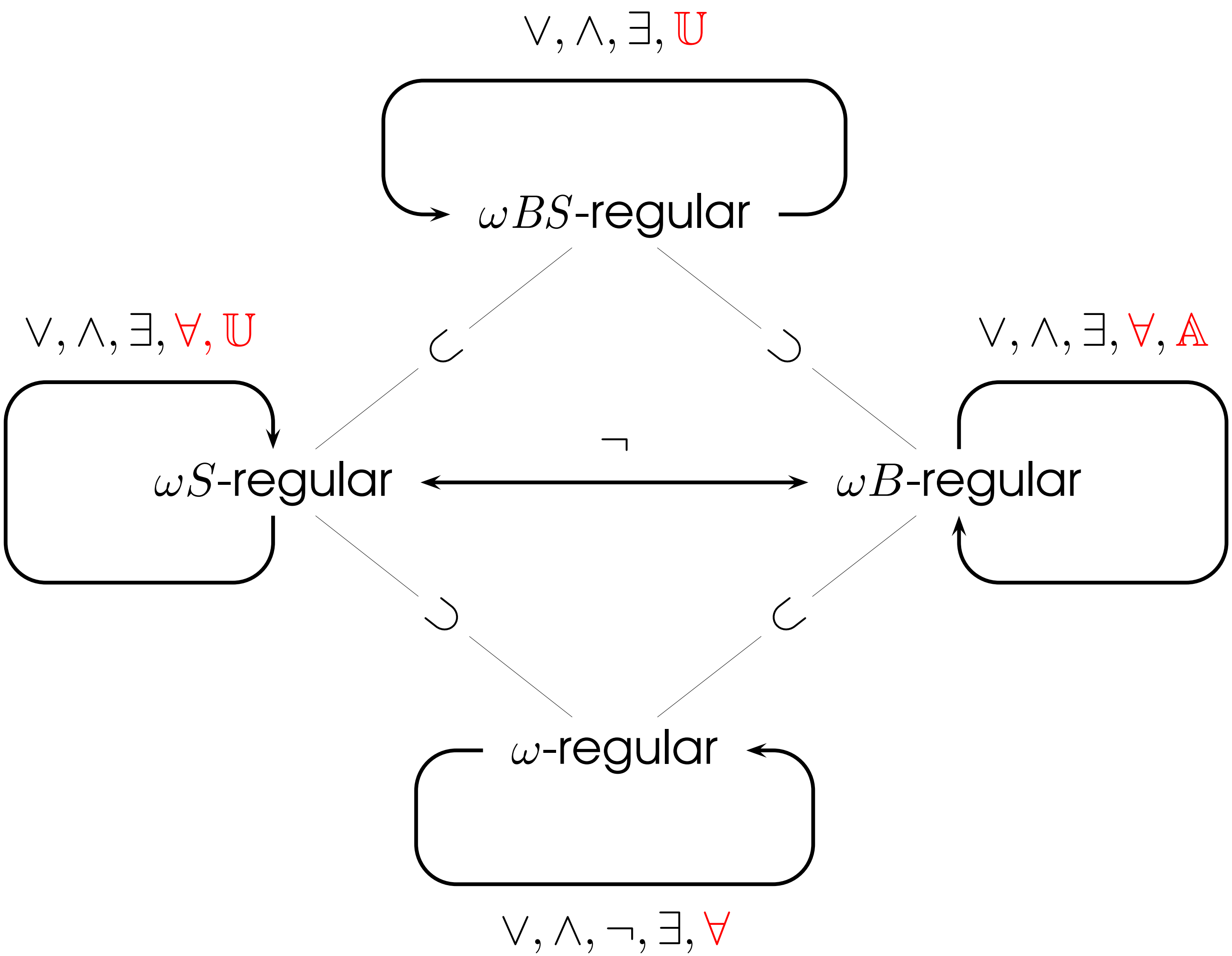

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SAT OF MSOLB

DEF: MSOLB+ = MSOLB where $\mathbb{U}$ appears only positively
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COR: MSOL is equivalent to ω-regular languages
MSOLB+ is equivalent to ωS-regular languages
MSOLB- is equivalent to ωB-regular languages
Boolean comb. of MSOLB+ are contained in ωBS-regular language
$\Rightarrow$ **SAT is decidable for those fragments of MSOLB**

CONCLUSION

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- Introduced an extension of ω-regular expressions.

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- Find equivalent class of languages over trees.
- What about the logic **GMSOLB**?:

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Epilogue: Restricted star-height problem

Q?: Given a regular language of finite words, how many nesting of kleene star are required in a regular expression for describing it?

This number is called the (restricted) star height of the language.

Among the most important decision problems in language theory.

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Limitedness problem: $L(\mathcal{A}) = L(\mathcal{A}[*/B])$?

This is decidable by our results.

Thank you.