Omega Regularity with Bounds

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**Omega Regular Languages**

An $\omega$-word is an infinite word indexed by $\omega$: $a_1a_2a_3 \ldots$
An \( \omega \)-\textit{word} is an infinite word indexed by \( \omega: a_1 a_2 a_3 \ldots \)

\textbf{THM(Buchi,\ldots)} The following property of a language of \( \omega \)-\textit{words} \( L \) are equivalent:

- \( L \) is the evaluation of an \( \omega \)-\textit{regular expression} 
  \[ (a^*b)^\omega \]
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\( \omega \)-regularity seems to be the notion corresponding to regularity on words.

**GOAL:** Consider another notion — more expressive — of regularity
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- \( L \) is the set of \( \omega \)-words models of an MSOL-formula
  \[ \forall x. \exists y. x < y \land b(y) \]
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- $L$ is recognized by an $\omega$-semigroup morphism
- $L$ is accepted by a Rabin/Street/Muller/Parity automaton (deterministic or not).
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Syntax of $\omega$-regular expressions:

\[ R = \emptyset \mid \varepsilon \mid a \mid R + R \mid R.R \mid R^* \]

\[ O = R^\omega \mid R.O \mid O + O \]
Syntax of $\omega BS$-regular expressions:

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R = \emptyset \mid \varepsilon \mid a \mid R + R \mid R.R \mid R^* \mid R^B \mid R^S \quad O = R^\omega \mid R.O \mid O + O
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- $B$ constrains the number of iteration to be bounded ($B$ stands for ‘bounded’)

$$(a^B b)^\omega$$

there is an infinite number of $b$’s and, the number of $a$’s between consecutive $b$’s is bounded

$$ababab \ldots \in (a^B b)^\omega$$

$$aba^2baba^3baba^4b \ldots \notin (a^B b)^\omega$$

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- $S$ constrains the number of iterations to tend toward the infinite ($S$ stands for ‘strictly unbounded’)

$(a^S b)^\omega$ there is an infinite number of $b$’s and,

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$$ababab \ldots \notin (a^S b)^\omega \quad aba^2baba^3baba^4b \ldots \notin (a^S b)^\omega$$

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\textbf{SEMANTIC OF $\omega BS$-REGULAR LANGUAGES}

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SEMsATIC OF $\omega BS$-REGULAR LANGUAGES

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The languages with an infinite number of $b$'s such that the size of segments of $a$ is not bounded.
E.g. $aba^2baba^3baba^4b\cdots \in ((a^*b)a^Sb)\omega$
Other Examples

\[((a^*b)^*a^Sb)^\omega\]

The languages with an infinite number of \(b\)'s such that the size of segments of \(a\) is not bounded.
E.g. \(aba^2baba^3baba^4b \cdots \in ((a^*b)a^Sb)^\omega\)

\[((a^Sb)^Sa^Sc)^\omega\]

There is an infinite number of \(c\)'s. The number of \(b\)'s between two \(c\)'s tends toward the infinite. The number of \(a\)'s between two \(b\) or \(c\)'s tends toward the infinite.
The Diamond

\(\omega BS\)-regular expressions
e.g. \((a^B b^S c)^\omega\)

\(\omega S\)-regular expressions
e.g. \((a^S b)^\omega\)

\(\omega B\)-regular expressions
e.g. \((a^B b)^\omega\)

\(\omega\)-regular expressions
e.g. \((a^* b)^\omega\)

Proposition:
- Emptyness of \(\omega BS\)-regular languages is decidable.

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The inclusions in the diamond are strict.
**The Diamond**

\[ \omega BS \text{-regular expressions} \]
\[ \text{e.g. } (a^B b^S c)^\omega \]

\[ \omega S \text{-regular expressions} \]
\[ \text{e.g. } (a^S b)^\omega \]

\[ \omega B \text{-regular expressions} \]
\[ \text{e.g. } (a^B b)^\omega \]

\[ \omega \text{-regular expressions} \]
\[ \text{e.g. } (a^* b)^\omega \]

**PROP:** Emptyness of \( \omega BS \)-regular languages is decidable
**THE DIAMOND**

ωBS-regular expressions

\[ (a^B b^S c)^\omega \]

ωS-regular expressions

\[ (a^S b)^\omega \]

ωB-regular expressions

\[ (a^B b)^\omega \]

ω-regular expressions

\[ (a^* b)^\omega \]

**PROP:** Emptyness of ωBS-regular languages is decidable

**PROP:** The inclusions in the diamond are strict.
DEF: One can define $\omega B$, $\omega S$, $\omega BS$-automata. Essentially: finite state automata with modified accepting condition (more expressive than Buchi). They come in two variants hierarchical or not.
**DEF:** One can define \( \omega B, \omega S, \omega BS \)-automata. Essentially: finite state automata with modified accepting condition (more expressive than Buchi). They come in two variants hierarchical or not.

**THM:** The following are equivalent:

- \( L \) is evaluation of an \( \omega BS \)-regular expression
- \( L \) is accepted by an \( \omega BS \)-automaton
- \( L \) is accepted by a hierarchical \( \omega BS \)-automaton

And the same holds for \( \omega B \) and \( \omega S \) regular languages.
THE DIAMOND (2: AUTOMATA)

\[ \omega_{BS}\text{-regular expressions} \]
\[ \text{hierarchical } \omega_{BS}\text{-automata} \]
\[ \omega_{BS}\text{-automata} \]
\[ \omega_{S}\text{-regular expressions} \]
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\[ \omega_{B}\text{-automata} \]
\[ \omega\text{-regular expressions} \]
\[ \text{Büchi automata} \]
**THM:** The $\omega BS$-regular languages are closed under union, intersection, projection.

**PROP:** $BS$-regular languages are not closed under complementation.

**MP:** The complement of $L = (\bigcup a \text{B} \cup a \text{S})(\bigcup b)$ is not $BS$-regular.

The language $L$ contains $a f(1)$ $ba f(2)$ $b \cdots \iff$ there exists infinitely many values appearing infinitely often in $f$. E.g.: $a_1 b a_1 b a_2 b a_2 b a_3 b \cdots \in L$.

**THM:** The complement of an $BS$-regular language is $S$-regular.

The complement of an $S$-regular language is $BS$-regular.

Example: $(a \text{B} b) \notin (a \text{S} b \cup a \text{B} b \cup a \text{B} b)$.
**THM:** The $\omega BS$-regular languages are closed under union, intersection, projection.

**Proof:** Union and projection are syntactic on $\omega BS$-regular expressions. Intersection is obtained by product of $\omega BS$-automata.

**Example:**

\[
\begin{array}{c}
(a \cup b) \\
(a \cup b) \\
(a \cup b)
\end{array}
\]

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\begin{array}{c}
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**Example of intersections:**

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**Example of intersections:**

$$(b^{*}ab^{*})^{B \#})^{\omega} \cap ((a^{*}ba^{*})^{B \#})^{\omega} = ((a + b)^{B \#})^{\omega}$$
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((b^*ab^*)^S \#)^\omega \cap ((a^*ba^*)^B \#)^\omega = ((a^*ba^*)^B a^S (a^*ba^*)^B \#)^\omega
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((b^* ab^*)^S \#)^\omega \cap ((a^* ba^*)^S \#)^\omega = \left(\begin{array}{c}
(a + b)^* a^S (a + b)^* b^S (a + b)^* \\
+ (a + b)^* b^S (a + b)^* a^S (a + b)^* \\
+ b^* (a + b^+) S a^*
\end{array}\right)^\omega
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**PROP:** $\omega BS$-regular languages are not closed under complementation.
**Closure**

**THM:** The $\omega BS$-regular languages are closed under union, intersection, projection.

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**Mp:** The complement of $L = ((a^B + a^S)\#)^\omega$ is not $\omega BS$-regular. The language $\overline{L}$ contains $a^{f(1)}b a^{f(2)}b \cdots$ iff there exists infinitely many values appearing infinitely often in $f$.

**E.g:** $a^1b \ a^1ba^2b \ a^1ba^2ba^3b \cdots \in \overline{L}$
**Closure**

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The \( \omega B S \)-regular languages are closed under union, intersection, projection.

\textbf{PROP:} \( \omega B S \)-regular languages are not closed under complementation.

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\textbf{Example:}

\[
(a^B b)^\omega = (a + b)^* a^\omega + ((a^* b)^* a^S b)^\omega
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**Closure**

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**Example:**

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\]

\[
\overline{(a^S b)^\omega} = (a + b)^* a^\omega + (((a^* b)^* a^B b)^\omega
\]
THE DIAMOND (3: CLOSURE)

union, intersection, projection

\( \omega BS \)-regular

complementation

\( \omega S \)-regular

\( \omega B \)-regular

union, intersection, projection

complementation, projection

union, intersection, projection, complementation, projection

\( \omega \)-regular
The logic MSOLB

The logic MSOL:

$$\phi = \phi \lor \phi \mid \phi \land \phi \mid \neg \phi \mid \exists X.\phi \mid \exists x.\phi \mid x \in X \mid x = S(y) \mid a(x)$$
The logic **MSOLB** (Bojanczyk05):

\[ \phi = \phi \lor \phi \mid \phi \land \phi \mid \neg \phi \mid \exists X. \phi \mid \exists x. \phi \mid x \in X \mid x = S(y) \mid a(x) \mid \bigcup X. \phi \]

With \( \bigcup X. \phi \equiv \forall n. \exists X. (|X| > n) \land \phi \)

“there exists a set as big as I want”

**THM** (Buchi):

SAT of **MSOL** is decidable over infinite words.

**Rq:** Adding equality/comparison of cardinality to **MSOL** leads to undecidability of SAT (already for finite words).

**QESTION:** Is SAT of **MSOLB** decidable over \(-\)words?

The question is open. But \(-\)regularity provides a partial answer.
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And also \( \Box X. \phi \equiv \neg \bigcup X. \neg \phi \equiv \exists n. \forall X. (|X| > n) \rightarrow \phi \)

“for all sets above a certain size”
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"there exists a set as big as I want"

And also \(\bigtriangleup X. \phi \equiv \neg \bigcup X. \neg \phi \equiv \exists n. \forall X. (|X| > n) \to \phi\)

"for all sets above a certain size"

And also \(\bigtriangledown X. \phi \equiv \neg \bigcup X. \phi \equiv \exists n. \forall X. \phi \to (|X| < n)\)

"there is a bound on the size of sets satisfying"
The logic **MSOLB** (Bojanczyk05):

\[ \phi = \phi \lor \phi \mid \phi \land \phi \mid \neg \phi \mid \exists X. \phi \mid \exists x. \phi \mid x \in X \mid x = S(y) \mid a(x) \mid \mathbb{U} X. \phi \]

With \( \mathbb{U} X. \phi \equiv \forall n. \exists X. (|X| > n) \land \phi \)

“there exists a set as big as I want”

And also \( \mathbb{A} X. \phi \equiv \neg \mathbb{U} X. \neg \phi \equiv \exists n. \forall X. (|X| > n) \rightarrow \phi \)

“for all sets above a certain size”

And also \( \mathbb{B} X. \phi \equiv \neg \mathbb{U} X. \phi \equiv \exists n. \forall X. \phi \rightarrow (|X| < n) \)

“there is a bound on the size of sets satisfying”

**THM(Buchi):** SAT of MSOL is decidable over infinite words.
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\[ \phi = \phi \lor \phi \mid \phi \land \phi \mid \neg \phi \mid \exists X. \phi \mid \exists x. \phi \mid x \in X \mid x = S(y) \mid a(x) \mid \exists X. \phi \]

With \( \exists X. \phi \equiv \forall n. \exists X. (|X| > n) \land \phi \)

"there exists a set as big as I want"

And also \( \forall X. \phi \equiv \neg \exists X. \neg \phi \equiv \exists n. \forall X. (|X| > n) \rightarrow \phi \)

"for all sets above a certain size"

And also \( \lnot X. \phi \equiv \neg \exists X. \phi \equiv \exists n. \forall X. \phi \rightarrow (|X| < n) \)

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**THM(Buchi):** SAT of MSOL is decidable over infinite words.

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With \( \forall X. \phi \equiv \forall n. \exists X. (|X| > n) \land \phi \)

“there exists a set as big as I want”

And also \( \forall X. \phi \equiv \neg \forall X. \neg \phi \equiv \exists n. \forall X. (|X| > n) \rightarrow \phi \)

“for all sets above a certain size”

And also \( \exists X. \phi \equiv \neg \exists X. \phi \equiv \exists n. \forall X. \phi \rightarrow (|X| < n) \)

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**QUESTION:** Is SAT of MSOLB decidable over \( \omega \)-words?
The logic **MSOLB** (Bojanczyk05):

\[ \phi = \phi \lor \phi \mid \phi \land \phi \mid \neg \phi \mid \exists X. \phi \mid \exists x. \phi \mid x \in X \mid x = S(y) \mid a(x) \mid \bigcup X. \phi \]

With \( \bigcup X. \phi \equiv \forall n. \exists X. (|X| > n) \land \phi \)

“there exists a set as big as I want”

And also \( \Delta X. \phi \equiv \neg \bigcup X. \neg \phi \equiv \exists n. \forall X. (|X| > n) \rightarrow \phi \)

“for all sets above a certain size”

And also \( \boxdot X. \phi \equiv \neg \bigcup X. \phi \equiv \exists n. \forall X. \phi \rightarrow (|X| < n) \)

“there is a bound on the size of sets satisfying”

**THM(Buchi):** SAT of MSOL is decidable over infinite words.

**Rq:** Adding equality/comparison of cardinality to MSOL leads to undecidability of SAT (already for finite words).

**QESTION:** Is SAT of MSOLB decidable over \( \omega \)-words? The question is open. But \( \omega BS \)-regularity provides a partial answer.
THE DIAMOND (3:logic)

\[ \begin{align*}
\forall, \land, \exists & \quad \rightarrow \quad \omega BS\text{-regular} \\
\forall, \land, \exists & \quad \rightarrow \quad \omega S\text{-regular} \\
\forall, \land, \exists & \quad \leftarrow \quad \omega B\text{-regular} \\
\forall, \land, \exists & \quad \rightarrow \quad \omega \text{-regular} \\
\forall, \land, \neg, \exists & \quad \leftarrow \quad \omega \text{-regular}
\end{align*} \]

\( \text{PROP: } \omega S\text{-regular} \text{ and } \omega BS\text{-regular languages are closed under } \forall. \)
PROP: $\omega B$-regular languages are closed under $\cup$.
THE DIAMOND (3:LOGIC)

\[
\begin{align*}
&\forall, \land, \exists \\
&\omega BS\text{-regular} \\
&\forall, \land, \exists, \forall \\
&\omega S\text{-regular} \\
&\exists, \forall \Rightarrow \omega B\text{-regular} \\
&\forall, \land, \exists, \forall, \neg \\
&\omega\text{-regular} \\
&\forall, \land, \neg, \exists, \forall \\
\end{align*}
\]
PROP: $\omega BS$-regular languages are closed under $\cup$. 

THE DIAMOND (3:LOGIC)
PROP: $\omega S$ and $\omega BS$-regular languages are closed under $\bigcup$.
PROP: \( \omega S \) and \( \omega BS \)-regular languages are closed under \( \cup \).
SAT OF MSOLB

**DEF:**

$\text{MSOLB}^+$ = MSOLB where $\cup$ appears only **positively**

$\text{MSOLB}^-$ = MSOLB where $\cup$ appears only **negatively**
SAT of MSOLB

DEF: MSOLB+ = MSOLB where \( U \) appears only positively
MSOLB- = MSOLB where \( U \) appears only negatively

COR: MSOL is equivalent to \( \omega \)-regular languages
MSOLB+ is equivalent to \( \omega S \)-regular languages
MSOLB- is equivalent to \( \omega B \)-regular languages
Boolean comb. of MSOLB+ are contained in \( \omega BS \)-regular languages

\( \Rightarrow \) SAT is decidable for those fragments of MSOLB
CONCLUSION

We have:

- Introduced an extension of $\omega$-regular expressions.
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- Introduced an extension of $\omega$-regular expressions.
- Introduced corresponding class of automata.

Two main open questions.

- Solve the full logic MSOLB over $\omega$-words.
- Find equivalent class of languages over trees.
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- Introduced an extension of $\omega$-regular expressions.
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- Shown decidability and closure properties.
CONCLUSION

We have:

- Introduced an extension of $\omega$-regular expressions.
- Introduced corresponding class of automata.
- Shown decidability and closure properties.
- Used it for solving a fragment of the logic MSOLB over $\omega$-words.

Two main open questions.

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CONCLUSION

We have:

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- Solve the full logic MSOLB over $\omega$-words.
CONCLUSION

We have:

- Introduced an extension of \( \omega \)-regular expressions.
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- Shown decidability and closure properties.
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Two main open questions.

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