Omega Regularity with Bounds

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Omega Regular languages

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GOAL: Consider another notion — more expressive — of regularity

EXTENDING ω-REGULAR EXPRESSIONS

Syntax of ω-regular expressions:

$$R = \emptyset \mid \varepsilon \mid a \mid R + R \mid R.R \mid R^* \qquad\qquad O = R^\omega \mid R.O \mid O + O$$

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- $U^\omega \mapsto \{u_1u_2 \dots \ : \ \vec{u} \in U\}$

OTHER EXAMPLES

$$((a^*b)^*a^Sb)^\omega$$

The languages with an infintie number of b's such that the size of segments of a is not bounded.
E.g. $aba^2baba^3baba^4b\cdots \in ((a^*b)a^Sb)^\omega$

Other examples

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E.g. $aba^2baba^3baba^4b\cdots \in ((a^*b)a^Sb)^\omega$

$$((a^Sb)^Sa^Sc)^\omega$$

There is an infinite number of c's. The number of $b's$ between two c's tends toward the infinite. The number of a's between two b or c's tends toward the infinite.

THE DIAMOND

ωBS-regular expressions
e.g. $(a^B b^S c)^\omega$

$\subset$ $\supset$

ωS-regular expressions
e.g. $(a^S b)^\omega$

ωB-regular expressions
e.g. $(a^B b)^\omega$

$\supset$ $\subset$

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e.g. $(a^* b)^\omega$

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e.g. $(a^B b^S c)^\omega$

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PROP: Emptyness of ωBS-regular languages is decidable

The diamond

ωBS-regular expressions
e.g. $(a^B b^S c)^\omega$

ωS-regular expressions $\subset$ ωBS-regular expressions $\supset$ ωB-regular expressions

ωS-regular expressions
e.g. $(a^S b)^\omega$

ωB-regular expressions
e.g. $(a^B b)^\omega$

ω-regular expressions $\subset$ ωS-regular expressions, ω-regular expressions $\subset$ ωB-regular expressions

ω-regular expressions
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PROP: The inclusions in the diamond are strict.

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DEF: One can define ωB, ωS, ωBS-automata.
Essentially: finite state automata with modified accepting condition (more expressive than Buchi).
They come in two variants hierarchical or not.

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THM: The following are equivalent:

- L is evaluation of an ωBS-regular expression
- L is accepted by an ωBS-automaton
- L is accepted by a hierarchical ωBS-automaton

And the same holds for ωB and ωS regular languages.

THE DIAMOND (2:AUTOMATA)

ωBS-regular expressions
hierarchical ωBS-automata
ωBS-automata

ωS-regular expressions
hierarchical ωS-automata
ωS-automata

ωB-regular expressions
hierarchical ωB-automata
ωB-automata

ω-regular expressions
Büchi automata

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Mp: The complement of $L = ((a^B + a^S)\sharp)^\omega$ is not ωBS-regular.

The language $\overline{L}$ contains $a^{f(1)}ba^{f(2)}b\cdots$ iff there exists infinitely many values appearing infinitely often in f.

E.g: $a^1b \;\; a^1ba^2b \;\; a^1ba^2ba^3b \;\; \cdots \in \overline{L}$

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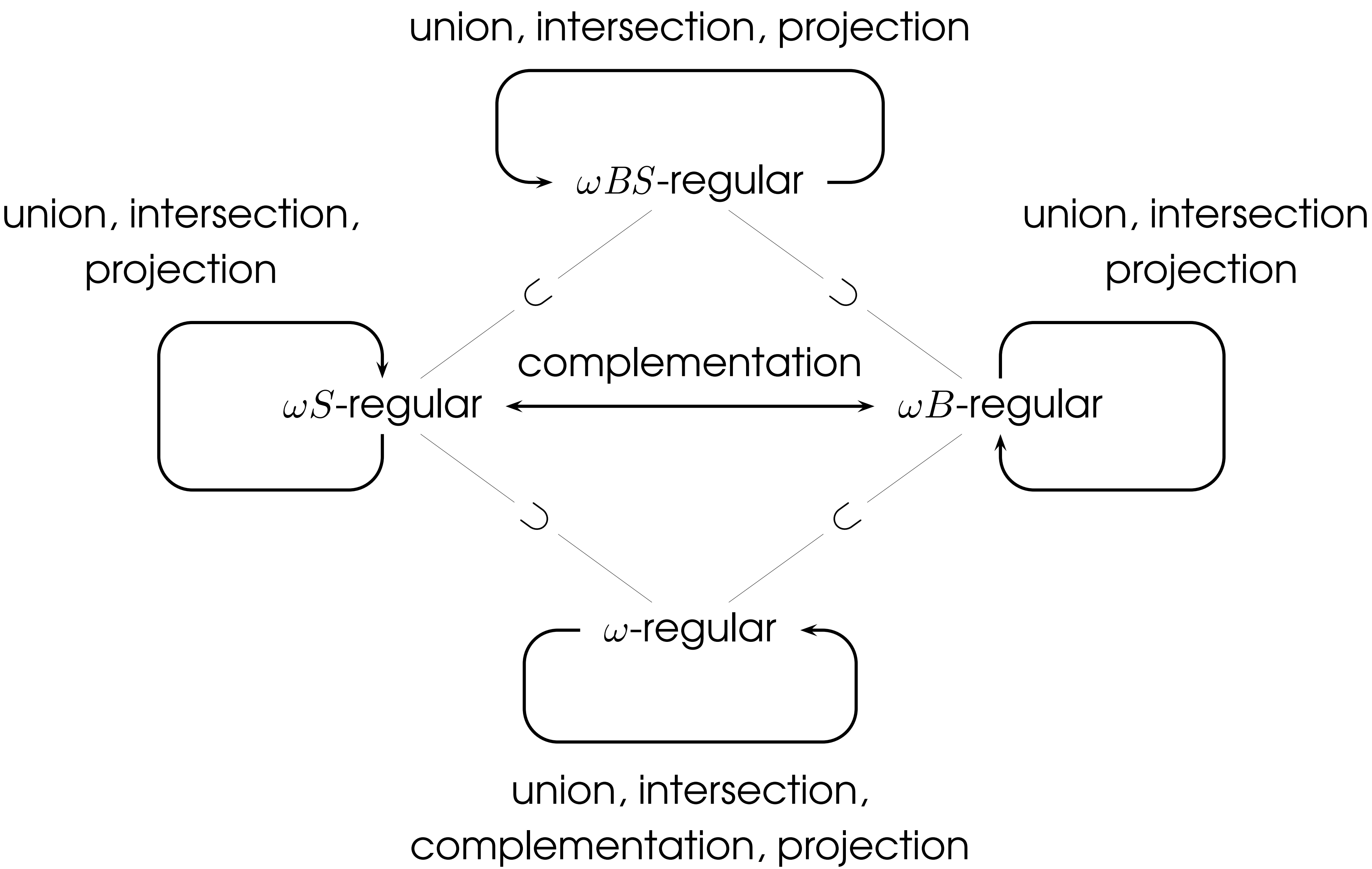
THE DIAMOND (3:CLOSURE)



The logic MSOLB

The logic MSOL :

$$\phi = \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \exists X.\phi \mid \exists x.\phi \mid x \in X \mid x = S(y) \mid a(x)$$

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Rq: Adding equality/comparison of cardinality to MSOL leads to undecidability of SAT (already for finite words).

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Rq: Adding equality/comparison of cardinality to MSOL leads to undecidability of SAT (already for finite words).

QESTION: Is SAT of MSOLB decidable over ω-words?

THE LOGIC MSOLB

The logic **MSOLB** (Bojanczyk05) :

$$\phi = \ \phi \vee \phi \mid \phi \wedge \phi \mid \neg \phi \mid \exists X.\phi \mid \exists x.\phi \mid x \in X \mid x = S(y) \mid a(x) \quad \mid \mathbb{U}X.\phi$$

With $\mathbb{U}X.\phi \quad \equiv \quad \forall n.\, \exists X.\, (|X| > n) \wedge \phi$

"there exists a set as big as I want"

And also $\mathbb{A}X.\phi \quad \equiv \ \neg \mathbb{U}X.\neg\phi \quad \equiv \ \exists n.\, \forall X.\, (|X| > n) \rightarrow \phi$

"for all sets above a certain size"

And also $\mathbb{B}X.\phi \quad \equiv \ \neg \mathbb{U}X.\phi \quad \equiv \ \exists n.\, \forall X.\, \phi \rightarrow (|X| < n)$

"there is a bound on the size of sets satisfying"

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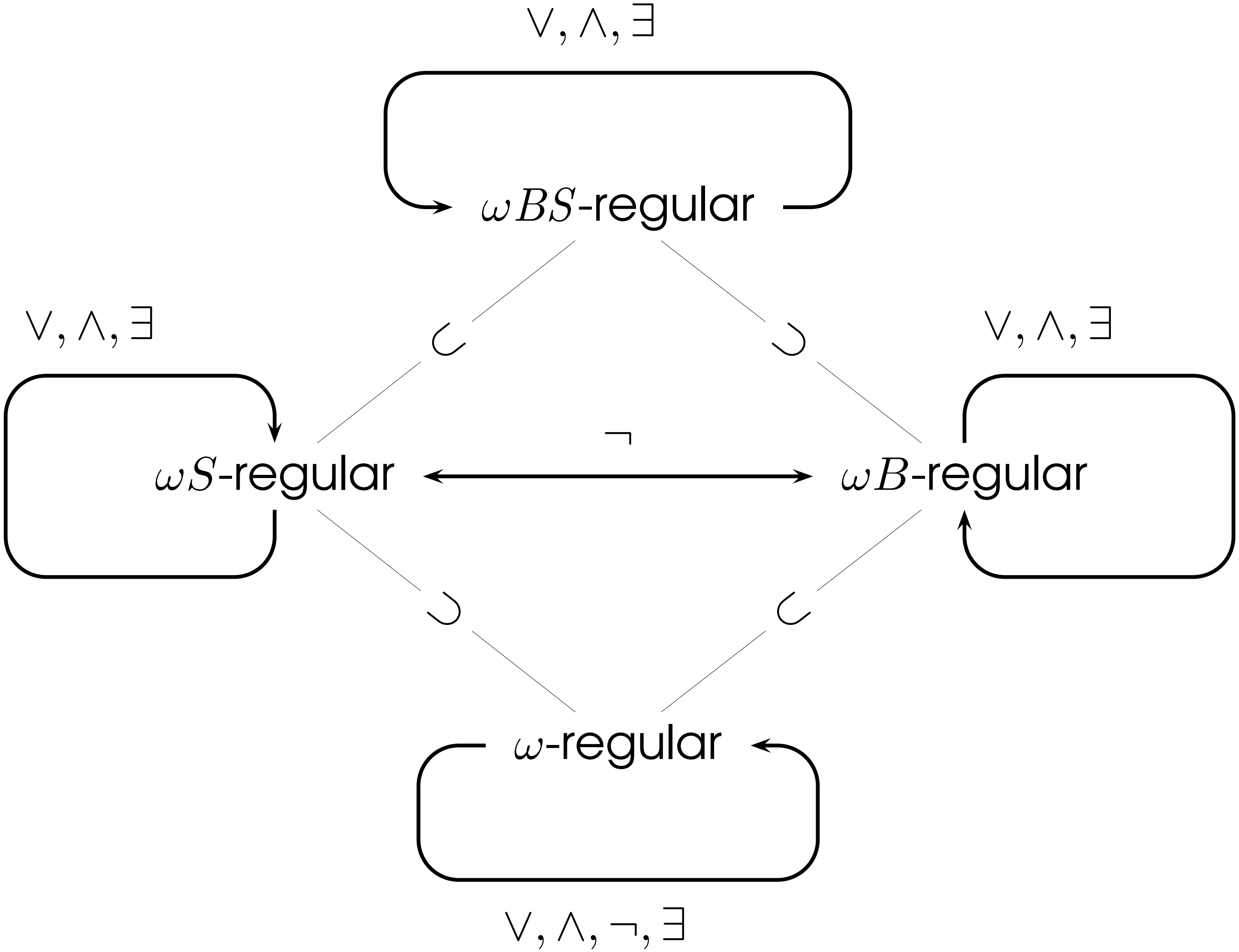
QESTION: Is SAT of MSOLB decidable over ω-words? The question is open.
But ωBS-regularity provides a partial answer.

THE DIAMOND (3:LOGIC)



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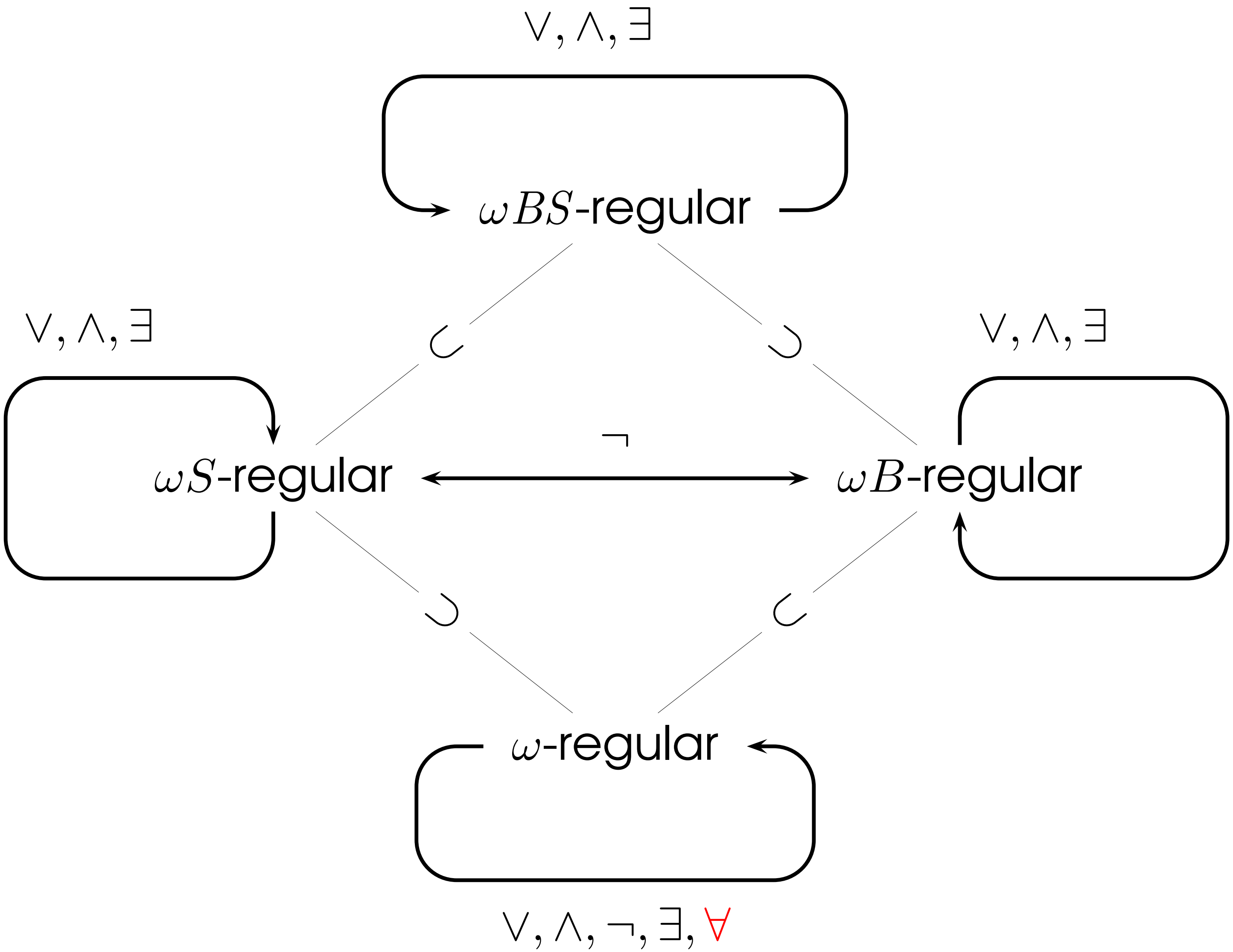

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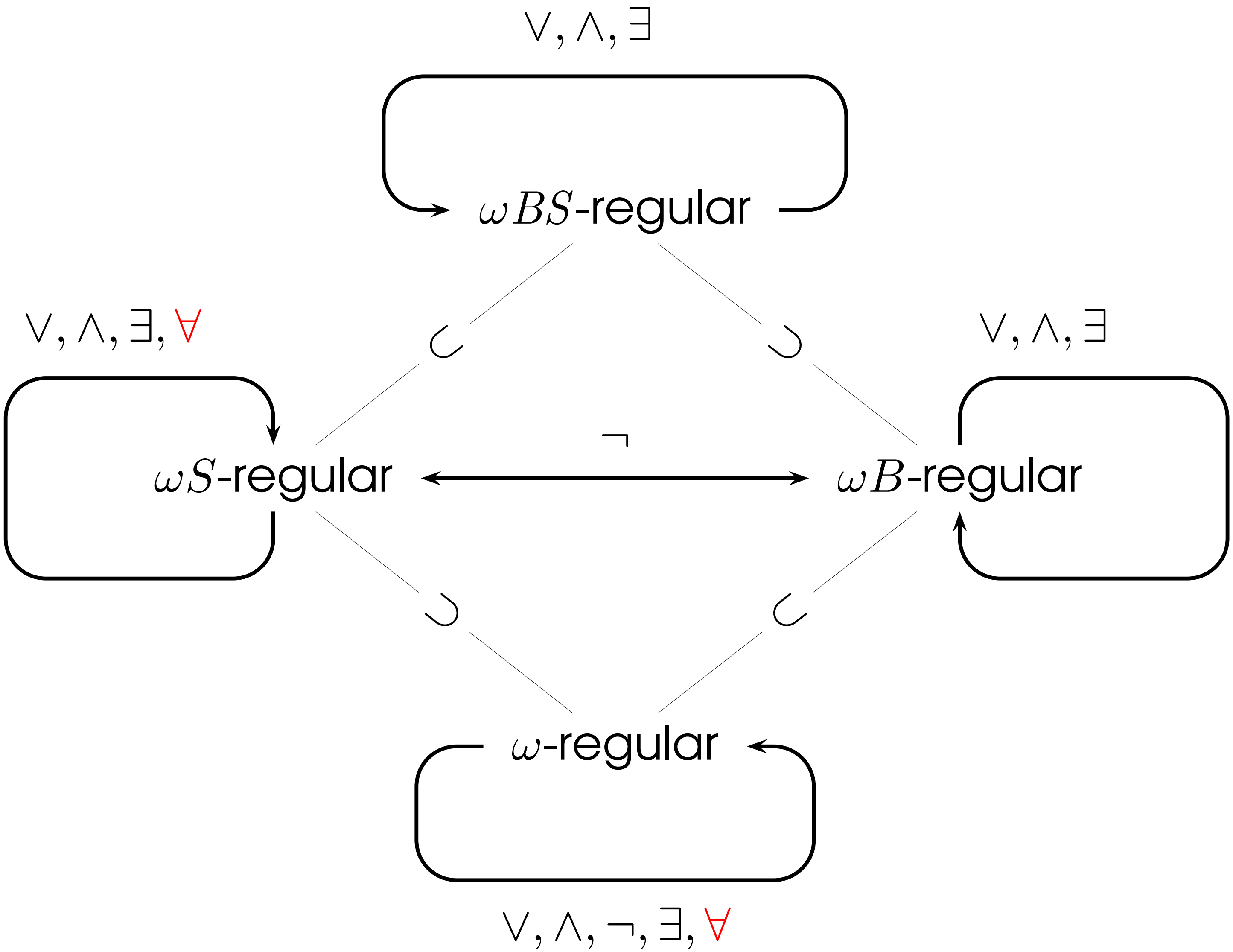

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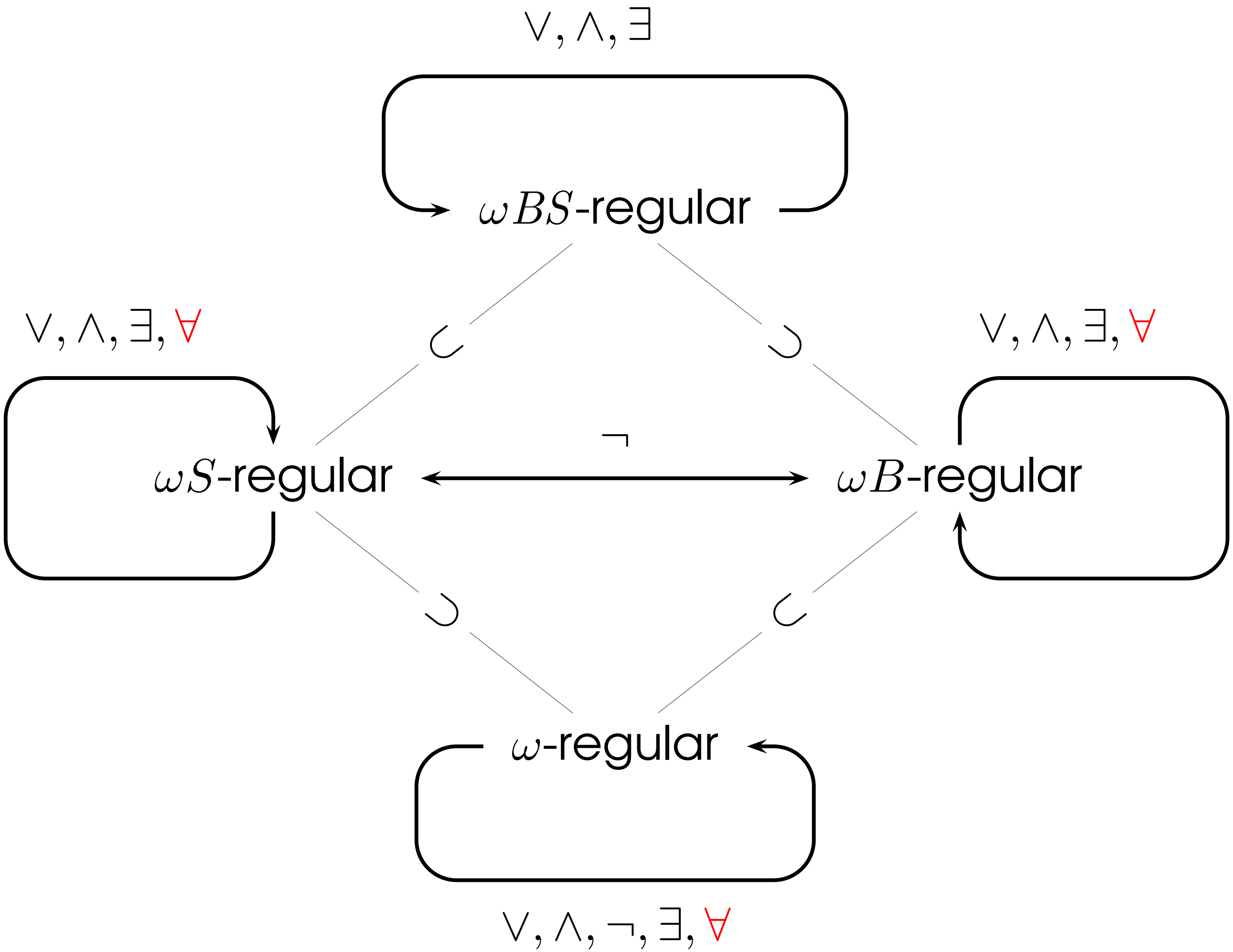

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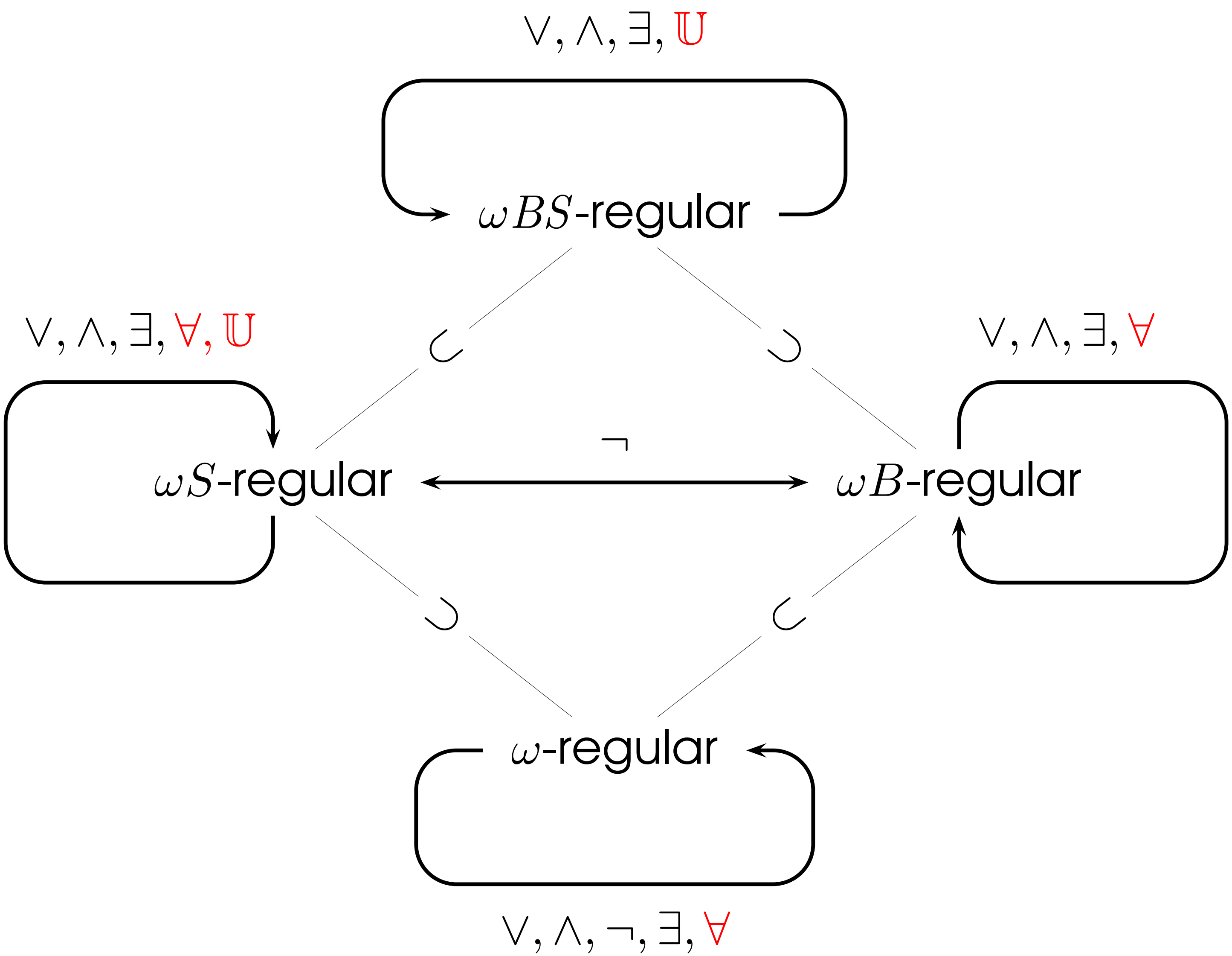

PROP: ωS and ωBS-regular languges are closed under $\mathbb{U}$.

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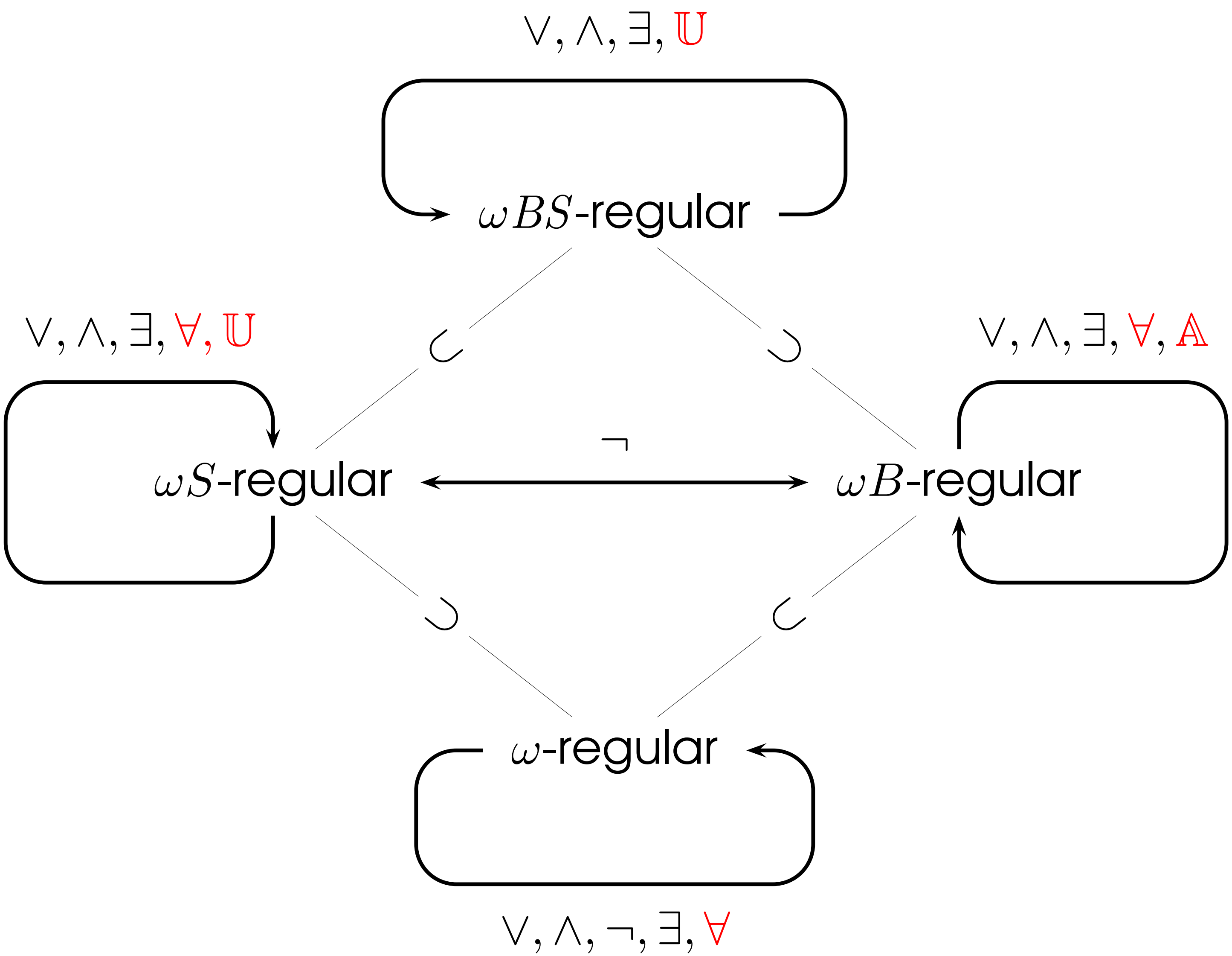

PROP: ωS and ωBS-regular langugues are closed under $\mathbb{U}$.

SAT of MSOLB

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MSOLB- = MSOLB where $\mathbb{U}$ appears only negatively

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COR: MSOL is equivalent to ω-regular languages
MSOLB+ is equivalent to ωS-regular languages
MSOLB- is equivalent to ωB-regular languages
Boolean comb. of MSOLB+ are contained in ωBS-regular language
$\Rightarrow$ **SAT is decidable for those fragments of MSOLB**

Conclusion

We have:

- Introduced an extension of ω-regular expressions.

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Two main open questions.

- Solve the full logic MSOLB over ω-words.
- Find equivalent class of languages over trees.