

Positional determinacy over finite games

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GAMES ON GRAPHS OF INFINITE DURATION

Arena: $\mathcal{A} = (V_E, V_A, E)$

with V_E is the set of **positions** belonging to **Eva**

V_A is the set of **positions** belonging to **Adam**

V_E and V_A are disjoint

$E \subseteq V \times \Sigma \times V$ a set of **edges**

with $V = V_E \cup V_A$

Σ a **finite alphabet** fixed for the remaining of the talk

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with v_0 is an **initial position** in V

\mathcal{A} is an arena

W is a winning condition

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Winner in a confrontation of strategies:

Given a strategy s_E for Eva and a strategy s_A for Adam.

Eva wins with s_E against s_A if the only play $(v_0, a_0, v_1)(v_1, a_1, v_2) \dots$ consistent with both s_E and s_A is such that $a_0 a_1 a_2 \dots \in W$. Else **Adam wins**.

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Winner of a game/determinacy: Player Eva wins the game if there is a strategy s_E for Eva such that Eva wins with this strategy against all strategies of Adam.

The game is said **determined** if either Eva or Adam is winning it.

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Parity condition: A winning condition W is a (generalized) **parity condition** if there exists $p : \Sigma \rightarrow \mathbb{N}$ such that

$$a_0 a_1 a_2 \cdots \in W \quad \text{iff} \quad \limsup_n p(a_n) \text{ is even.}$$

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TH:(Żielonka/Walukiewicz/C,Niwiński)

A winning condition is positionally determined iff it is a parity condition.

LIMITATIONS

TH: A winning condition is positionally determined iff it is a parity condition.

The left to right implication is not true anymore if

- W is not prefix independent (e.g. $W = a(a + b)^\omega$)
- labels are on vertices instead of edges (e.g. $(a + b)^*(ab)^\omega$)
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What about determinacy over finite games ?

Can we obtain a result of the form:

“A winning condition is positionally determined over every finite games iff it is (to be completed) ?”

SOME WINNING CONDITIONS

Mean-payoff condition (Ehrenfeucht, Mycielski): W is a **mean-payoff** condition if there exists a monoid morphism $\rho : \Sigma^* \rightarrow \mathbb{R}$ such that

$$a_0 a_1 \cdots \in W \quad \text{iff} \quad \limsup_n \frac{\rho(a_0 a_1 \cdots a_n)}{n} > 0 \quad (\text{we write } W = MPC(\rho))$$

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Unboundedness condition: W is an **unboundedness** condition if there exists a monoid morphism $\rho : \Sigma^* \rightarrow \mathbb{R}$ such that

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TH: Both mean-payoff and unboundedness conditions are positionally determined over finite games.

Furthermore for every finite arena \mathcal{A} the same player wins the game $(v_0, \mathcal{A}, UC(\rho))$ and the game $(v_0, \mathcal{A}, MPC(\rho))$, and the positional winning strategies are the same in both games.

The two conditions are distinct but indistinguishable...

EQUIVALENCE BETWEEN CONDITIONS

Fact: Let W and W' be winning conditions positionally determined over finite games.

Then the games (v_0, \mathcal{A}, W) and (v_0, \mathcal{A}, W') have the same winner for every finite arena \mathcal{A} iff W and W' coincide on periodic words.

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Remark: Some equivalence classes contain an uncountable number of conditions.

Can we characterize the set of periodic words contained in a winning condition positionally determined over finite games ?

Can we characterize positionally determined winning condition up to equivalence?

PARTIAL ANSWER: PERMUTING CONDITIONS

Def: A winning condition is **permuting** if whenever $(uvw)^\omega$ is in W then $(uww)^\omega$ is also in W .

Then a word u is completely characterized by its Parikh vector $p(u)$.

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Remark: For a condition W positionally determined over finite games,

If $u^\omega \in W$ and $v^\omega \in W$ then $(uv)^\omega$. (The same holds for \overline{W} .)

Then E as well as \overline{E} are closed under addition.

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From this remark we can conclude:

TH: Every permuting winning condition positionally determined over finite games is equivalent to a generalized mean-payoff condition. (The generalized mean-payoff conditions are positionally determined over finite games.)

Def: A winning condition W is a **chain mean-payoff** condition if there exists monoid morphisms $\rho_1, \rho_2, \dots, \rho_k : \Sigma^* \rightarrow \mathbb{R}$ such that

$$a_0 a_1 a_2 a_3 \cdots \in W \quad \text{iff for some } m, \begin{cases} \text{for } l < m & \limsup_n \frac{\rho_l(a_0 a_1 \dots a_n)}{n} = 0 \\ \text{and} & \limsup_n \frac{\rho_m(a_0 a_1 \dots a_n)}{n} > 0 \end{cases}$$

PROVING POSITIONAL DETERMINACY

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We need only to consider one-player games!

STRATEGY IMPROVEMENTS

We assume games played only by Eva.

A strategy is then simply an infinite path, that we write as a word in $(V\Sigma)^*$.

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What should we assume on W for ensuring that the loop terminates?

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Let $V = V_E$ be $\{1, 2, 3, 4, 5, 6, 7\}$.

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We can construct a “parenthesized” version of the original strategy representing the successive improvement steps:

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Remark 2: Nesting depth of parentheses is **bounded**.

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We can construct a “parenthesed” version of the original strategy representing the successive improvement steps:

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Remark 1: The word appearing in each parenthesis — omitting inner parentheses — is an elementary cycle losing for Eva.

Remark 2: Nesting depth of parentheses is **bounded**.

Conclusion: Assuming that words admitting a “parenthesisation” satisfying remarks 1 and 2 are losing for Eva is sufficient for ensuring termination.

NECESSARY AND SUFFICIENT CONDITION

Weak interleaving:

W is weakly interleaving if for any $u_1u_2u_3 \cdots \in W$ and any $v^\omega \in W$ then $u_1vu_2vu_3v \cdots \in W$.

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Ultimately weakly interleaving: W is ultimately weakly interleaving if it contains the least set weakly interleaving containing ultimately periodic words. (formalizes the “parenthesisation” property)

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Lemma: Eva has positional winning strategies on every arena played only by her iff W is co-ultimately weakly interleaving.

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W is weakly interleaving if for any $u_1u_2u_3 \cdots \in W$ and any $v^\omega \in W$ then $u_1vu_2vu_3v \cdots \in W$.

Ultimately weakly interleaving: W is ultimately weakly interleaving if it contains the least set weakly interleaving containing ultimately periodic words. (formalizes the “parenthesisation” property)

Lemma: Eva has positional winning strategies on every arena played only by her iff W is co-ultimately weakly interleaving.

TH: A winning condition is positionally determined over finite games iff it is both ultimately weakly interleaving and co-ultimately weakly interleaving.

CONCLUSION

Main points

- Characterising winning condition over finite games is more difficult than over all games
- This “must” be done first up to equivalence over periodic words
- Under permuting hypothesis, chained unboundedness characterizes exactly positional determinacy over finite games up to equivalence
- We do not know of any winning condition positionally determined over finite games which is not permuting
- “Parenthesiation” allows to characterize how should be the winning condition over non periodic words

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Questions

- Half positional determinacy ? Seems much more difficult.
(partial results of Eryk Kopczynski)
- Why considering finite games or infinite games ? Are there usefull arenas of other shapes ?