Finite set interpretations

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INTERPRETATIONS

Interpretations: logically defined mappings from structures to structures.

Property: interpretations are closed under composition.

Property: $\mathcal{S} \models \tilde{I}(\phi)$ iff $I(\mathcal{S}) \models \phi$.

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Example:

Theorem (Rabin): The full binary tree has a decidable MSO-theory.

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Expressive power? Completeness?
Question: are all structures of decidable MSO-theory prefix recognizable?

Seese conjecture: are all structures of decidable MSO-theory MSO-interpretations of a tree? **Partially solved (Courcelle & Oum).**

OVERVIEW

- Finite set interpretations & Elementary results
- Finite set interpretations of trees
- Focus on car parking
- Conclusion

Finite set interpretations

Structures and logics

Def: A relational structure $\mathcal{S} = (\mathcal{U}, R_1, \ldots, R_k)$

- $\mathcal{U}$ is a set called the universe of the structure
- each R_i is a relation of arity r_i over $\mathcal{U}$

Isomorphism: $\mathcal{S} \simeq \mathcal{S}'$ if $R(u_1, \ldots, u_n) \leftrightarrow R'(f(u_1), \ldots, f(u_n))$ (for f bijective)

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First-order logic (FO)

FO variables: $x, y, x_1, \ldots$ ranging over $\mathcal{U}$

FO logic: $x = y, R(x_1, \ldots, x_k), \wedge, \vee, \neg, \exists x, \forall x$

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Model: $\mathcal{S} \models \phi$, $\mathcal{S} \models \phi(a, b, c)$, $\mathcal{S} \models \phi(A, B, C)$

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Def: Finite set interpretation $(\Delta(X), \Phi_1(X_1, \ldots, X_{r_1}), \ldots, \Phi_k(X_1, \ldots, X_{r_k}))$

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Semantic of interpretations

Given an interpretation I and a structure $\mathcal{S}$.

FO interpretation: $I_{\mathrm{FO}}(\mathcal{S})$ is defined by:
Universe: $\{a \in \mathcal{U}_{\mathcal{S}} \ : \ \mathcal{S} \models \delta(a)\}$
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Motto: **Elements** of $I_{\mathrm{FS}}(\mathcal{S})$ are **finite sets of elements** of $\mathcal{S}$.

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Isomorphism: $f(E) = \sum_{n \in E} 2^n$

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Prop: $(\mathbb{N}, \mathrm{Succ}) \models \mathrm{Plus}(A, B, C)$ iff $f(A) + f(B) = f(C)$

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Conclusion: $I_{\mathrm{FS}}(\mathbb{N}, \mathrm{Succ}) \simeq (\mathbb{N}, +)$

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Prop: $I_{\mathrm{FS}}(\mathcal{S}) \models \phi$ iff $\mathcal{S} \models \widetilde{I_{\mathrm{FS}}}(\phi)$.
Where ϕ is a first order formula, $\widetilde{I_{\mathrm{FS}}}(\phi)$ is a WMSO-formula.

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Proof:

Transform in ϕ every FO-variable into a WMSO-variable.

Relativise the quantification to Δ.

Substitute the corresponding formula in I_{FS} for every relational symbol.

E.g. $\exists x. \forall y. \exists z.\ x + y = z$ becomes $\exists X. \forall Y. \exists Z.\ \mathrm{Plus}(X, Y, Z)$

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Prop: If I_{FO} is first-order, I_{FS} is finite set and I_{WMSO} is weak monadic then

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Corollary: The set $\mathrm{FS}(\mathcal{S})$ of structures obtainable by finite set interpretations from $\mathcal{S}$ is closed under FO-interpretations.

Universal interpretation/Generator

Def: The universal FS-interpretation P_{F} is defined by:
$P_{\mathrm{F}} = (\Delta(X), \Phi_{\subseteq}(X,Y), \Phi_1(X_1, \ldots, X_{r_1}), \ldots, \Phi_k(X_k, \ldots, X_{r_k})$ with:

$$\Delta(X) = \mathit{True}$$

$$\Phi_{\subseteq}(X,Y) = \forall x.\ \ x \in X \rightarrow x \in Y$$

$$\Phi_i(X_1, \ldots, X_{r_i}) = \exists x_1 \ldots \exists x_{r_i}. \bigwedge_j X_j = \{x_j\} \wedge R_i(x_1, \ldots, x_{r_i})$$

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Description: $P_{\mathrm{F}}(\mathcal{S})$ is the structure such that:

- The universe is the set of finite sets of elements of $\mathcal{S}$.
- The relation $R_{\subseteq}$ coincide with the inclusion.
- The structure $\mathcal{S}$ is replicated over singletons.

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Proof: Eliminate FO-variables using $\subseteq$.
Replace syntactically WMSO-variables by FO-variables.

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Corollary: $\mathrm{FS}(\mathcal{S})$ is the set of structures FO-interpretable in $P_{\mathrm{F}}(\mathcal{S})$ (called the generator of $\mathrm{FS}(\mathcal{S})$)

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Prop: A structure admits an automatic presentation iff it is isomorphic to $I_{\mathrm{FS}}(\mathbb{N}, +)$ for some finite set interpretation I_{FS}.

A structure admits a tree-automatic presentation iff it is isomorphic to $I_{\mathrm{FS}}(\Delta_2)$ for some finite set interpretation I_{FS}.

Finite set interpretations of trees

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Def: By tree we intend (possibly infinite) binary rooted labeled trees. In the signature it is possible to distinguish left-child and right child.

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Conclusion: Structures obtainable by finite set interpretations of trees are an important class to study.

Goal: Study $\mathrm{FS}(t)$ for t a deterministic tree.

Quotient of structures

Def: An equivalence relation $\sim$ is a congruence over a structure $\mathcal{S}$ if for every n and $u_1 \sim v_1, \ldots, u_{r_n} \sim v_{r_n}$:

$$R_n(u_1, \ldots, u_{r_n}) \quad \text{iff} \quad R_n(v_1, \ldots, v_{r_n})$$

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Prop: If $\mathcal{S} = (\mathcal{U}, R_1, \ldots, R_k, \sim)$ has a decidable FO-theory, $\mathcal{S}/_\sim$ also has a decidable FO-theory.

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Corollary: The tree-automatic structures are closed under quotient.

MAIN REPRESENTATION THEOREM

Corollary of Main Theorem: Fix I_{FS}. There exists I_{WMSO} such that for every $\mathcal{S}$ and t,

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$\mathrm{FS}(\mathcal{S}) \subseteq \mathrm{FS}(t)$ iff $\mathcal{S} = I_{\mathrm{WMSO}}(t)$ for some WMSO-interpretation I_{WMSO}

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Focus on car parking

OVERVIEW OF THE PROOF

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$$\begin{array}{rlcll} f: & \mathrm{Atoms} \subseteq \mathcal{P}(dom(t)) & \rightarrow & dom(t) & \quad\quad\quad\quad \text{is an injection.} \\ & A & \mapsto & u & \text{with } t \models \Psi(A, u) \end{array}$$

Overview of the proof

Corollary of Main Theorem: Fix I_{FS}. There exists I_{WMSO} such that for every $\mathcal{S}$ and t,

$$P_{\mathrm{F}}(\mathcal{S}) \simeq I_{\mathrm{FS}}(t) \quad \text{iff} \quad \mathcal{S} \simeq I_{\mathrm{WMSO}}(t)\ .$$

Remark: $R_{\subseteq}$ is an order on $I_{\mathrm{FS}}(t)$ which has all the properties of $\subseteq$ over finite sets. In particular it a minimal element $\perp$.

Def: Call Atoms the set of $R_{\subseteq}$-minimal non-$\perp$ elements (FO-definable).

Fact: $I_{\mathrm{FS}}(t)|_{\mathrm{Atoms}} \simeq \mathcal{S}$.

Objective: Construct $\Psi(X,x)$ WMSO st:

$$\begin{array}{rlcl} f: & \mathrm{Atoms} \subseteq \mathcal{P}(dom(t)) & \rightarrow & dom(t) \\ & A & \mapsto & u \quad \text{with } t \models \Psi(A,u) \end{array} \qquad \text{is an injection.}$$

Conclusion: By sending $I_{\mathrm{FS}}(t)|_{\mathrm{Atoms}} \simeq \mathcal{S}$ via f, one obtains $I_{\mathrm{WMSO}}(t) \simeq \mathcal{S}$.

SPARSITY

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Lemma: Every WMSO-definable mapping g such that for every zone Z,

$$|g^{-1}(Z)| \leq |Z| + K.|F| \quad (K\text{-sparsity})$$

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Idea: Each $f(X)$ is a car. Compute the flow of cars, and an itinerary for each car.
At the end, one must have at most one car per node.

Conclusion

CONCLUSION

New object: finite set interpretations

Elementary properties:

- Composition: $I_{\mathrm{FO}} \circ I_{\mathrm{FS}}$ and $I_{\mathrm{FS}} \circ I_{\mathrm{WMSO}}$ are FS-interpretation.
- Transfer: $I_{\mathrm{FS}}(\mathcal{S}) \models \phi$ iff $\mathcal{S} \models \widetilde{I_{\mathrm{FS}}}(\phi)$.
- Existence of a universal FS-interpretation P_{F}.

Advanced results:

- Closure of $\mathrm{FS}(t)$ under quotient.
- Main result: $P_{\mathrm{F}}(\mathcal{S}) \simeq I_{\mathrm{FS}}(t)$ iff $\mathcal{S} \simeq I_{\mathrm{WMSO}}(t)$.

Applications:

- The free monoid (two elements) non FS-interpretable in a tree.
- Random graph non FS-interpretable in a tree.
- Strictness of the tree-automatic hierarchy (variant by Blumensath).
- Intrinsic regularity in $P_{\mathrm{F}}(\mathbb{N}, \mathrm{Succ})$ (extension of (Barany)).