AN EXTENSION TO OVERPARTITIONS OF THE ROGERS-RAMANUJAN IDENTITIES FOR EVEN MODULI

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ABSTRACT. We define a class of well-poised basic hypergeometric series $\tilde{J}_{k,i}(a;x;q)$ and interpret these series as generating functions for overpartitions defined by multiplicity conditions. We also show how to interpret the $\tilde{J}_{k,i}(a;1;q)$ as generating functions for overpartitions whose successive ranks are bounded, for overpartitions that are invariant under a certain class of conjugations, and for special restricted lattice paths. We highlight the cases $(a,q) = (1/q,q), (1/q,q^2)$, and $(0,q)$, where the some of the functions $\tilde{J}_{k,i}(a;1;q)$ become infinite products. The latter case corresponds to Bressoud's family of Rogers-Ramanujan identities for even moduli.

1. INTRODUCTION

Over the years, a great number of combinatorial identities [1, 2, 3, 4, 8, 10, 17, 21, 23] have been extracted from Andrews' functions [7, Ch. 7] $J_{k,i}(a;x;q)$, which are defined by

$$J_{k,i}(a;x;q) = H_{k,i}(a;xq;q) + axqH_{k,i-1}(a;xq;q), \tag{1.1}$$

where

$$H_{k,i}(a;x;q) = \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+n-in} x^{kn}(1-x^i q^{2ni})(-1/a)_n(-axq^{n+1})_\infty}{(q)_n(xq^n)_\infty}. \tag{1.2}$$

Here we have employed the usual basic hypergeometric series notation [19]. Most recently [17], the first and third authors made a thorough combinatorial study of these functions, providing an interpretation of the general $J_{k,i}(a;x;q)$ in terms of overpartitions, which unified work of Andrews [4], Gordon [20], and the second author [21]. Moreover, it was shown that the $J_{k,i}(a;1;q)$ can be interpreted as generating functions for overpartitions with bounded successive ranks, for overpartitions with a specified Durfee dissection, and for certain restricted lattice paths. All of these interpretations generalized work of Andrews, Bressoud, and Burge on ordinary partitions [5, 6, 13, 14, 15].

In this paper we introduce and study a new class of functions, which we call $\tilde{J}_{k,i}(a;x;q)$ and define by

$$\tilde{J}_{k,i}(a;x;q) = \tilde{H}_{k,i}(a;xq;q) + axq\tilde{H}_{k,i-1}(a;xq;q), \tag{1.3}$$

where

$$\tilde{H}_{k,i}(a;x;q) = \sum_{n\geq 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+n-in} x^{(k-1)n}(1-x^i q^{2ni})(-x,-1/a)_n(-axq^{n+1})_\infty}{(q^2;q^2)_n(xq^n)_\infty}. \tag{1.4}$$

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Again the most natural combinatorial setting is that of overpartitions. Given an overpartition λ, let $f_\ell(\lambda)$ ($f_{\overline{\ell}}(\lambda)$) denote the number of occurrences of ℓ non-overlined (overlined) in λ. Let $V_\lambda(\ell)$ denote the number of overlined parts in λ less than or equal to ℓ. The following combinatorial interpretation of the general $\tilde{J}_{k,i}(a;x;q)$ is the principal result of the first half of this paper:

Theorem 1.1. *For $1 \leq i \leq k$ define the function $c_{k,i}(j,m,n)$ to be the number of overpartitions λ of n with m parts and j overlined parts such that (i) $f_1(\lambda)+f_{\overline{1}}(\lambda) \leq i-1$, (ii) $f_\ell(\lambda)+f_{\ell+1}(\lambda)+f_{\overline{\ell+1}}(\lambda) \leq k-1$, and (iii) if λ is saturated at ℓ, that is, if the maximum in (ii) is achieved, then $\ell f_\ell(\lambda)+(\ell+1)f_{\ell+1}(\lambda)+(\ell+1)f_{\overline{\ell+1}}(\lambda) \equiv i-1+V_\lambda(\ell) \pmod 2$. Then*

$$\tilde{J}_{k,i}(a;x;q) = \sum_{j,m,n\geq 0} c_{k,i}(j,m,n)a^j x^m q^n. \tag{1.5}$$

It turns out that the $\tilde{J}_{k,i}(a;1;q)$ are infinite products for $(a,q)=(0,q)$ and $(1/q,q^2)$, as well as for $(a,q)=(1/q,q)$ when $i=1$, and hence we can deduce partition theorems from Theorem 1.1. In the case $(a,q)=(0,q)$, the product is

$$\tilde{J}_{k,i}(0;1;q) = \frac{(q^i,q^{2k-i},q^{2k};q^{2k})_\infty}{(q)_\infty},$$

and we have a new proof of Bressoud's Rogers-Ramanujan identities for even moduli [10]:

Corollary 1.2 (Bressoud)**.** *For $k \geq 2$ and $1 \leq i \leq k-1$, let $\tilde{A}_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k$. Let $\tilde{B}_{k,i}(n)$ denote the number of partitions λ of n such that (i) $f_1(\lambda) \leq i-1$, (ii) $f_\ell(\lambda)+f_{\ell+1}(\lambda) \leq k-1$, and (iii) if $f_\ell(\lambda)+f_{\ell+1}(\lambda)=k-1$, then $\ell f_\ell(\lambda)+(\ell+1)f_{\ell+1}(\lambda) \equiv i-1 \pmod 2$. Then $\tilde{A}_{k,i}(n)=\tilde{B}_{k,i}(n)$.*

When $(a,q)=(1/q,q^2)$, the product is

$$\tilde{J}_{k,i}(1/q;1;q^2) = \frac{(-q;q^2)_\infty(q^{2i-1},q^{4k-2i-1},q^{4k-2};q^{4k-2})_\infty}{(q^2;q^2)_\infty},$$

and the result is a mod $4k-2$ companion to Andrews' generalization of the Göllnitz-Gordon identities [4]:

Corollary 1.3. *For $1 \leq i \leq k-1$, let $\tilde{A}^2_{k,i}(n)$ denote the number of partitions of n where even parts are multiples of 4 not divisible by $8k-4$ and odd parts are not congruent to $\pm(2i-1)$ modulo $4k-2$, with parts congruent to $2k-1$ modulo $4k-2$ not repeatable. Let $\tilde{B}^2_{k,i}(n)$ denote the number of partitions λ of n such that (i) $f_1(\lambda)+f_2(\lambda) \leq i-1$, (ii) $f_{2\ell}(\lambda)+f_{2\ell+1}(\lambda)+f_{2\ell+2}(\lambda) \leq k-1$, and (iii) if the maximum in (ii) is achieved at ℓ, then $\ell f_{2\ell}(\lambda)+(\ell+1)f_{2\ell+2}(\lambda)+(\ell+1)f_{2\ell+1}(\lambda) \equiv i-1+V^o_\lambda(\ell) \pmod 2$. (Here $V^o_\lambda(\ell)$ is the number of odd parts of λ less than 2ℓ). Then $\tilde{A}^2_{k,i}(n)=\tilde{B}^2_{k,i}(n)$.*

Finally, when $(a,q)=(1/q,q)$ and $i=1$, the product is

$$\tilde{J}_{k,1}(1/q;1;q) = \frac{(-q)_\infty(q,q^{2k-2},q^{2k-1};q^{2k-1})_\infty}{(q)_\infty},$$

and the result is a odd modulus companion to Theorem 1.2 of [21].

Corollary 1.4. *For $k \geq 2$, let $\tilde{A}_k^3(n)$ denote the number of overpartitions whose non-overlined parts are not congruent to $0, \pm 1$ modulo $2k-1$. Let $\tilde{B}_k^3(n)$ denote the number of overpartitions λ of n such that (i) $f_1(\lambda) = 0$, (ii) $f_\ell(\lambda) + f_{\overline{\ell}}(\lambda) + f_{\ell+1}(\lambda) \leq k-1$, and (iii) if the maximum in condition (ii) is achieved at ℓ, then $\ell f_\ell(\lambda) + \ell f_{\overline{\ell}}(\lambda) + (\ell+1) f_{\ell+1}(\lambda) \equiv V_\lambda(\ell) \pmod 2$. Then $\tilde{A}_k^3(n) = \tilde{B}_k^3(n)$.*

In the second half of the paper, we discuss three more combinatorial interpretations of the $\tilde{J}_{k,i}(a;1;q)$: one involving the theory of successive ranks for overpartitions as developed in [17], one involving a two-parameter generalization to overpartitions of Garvan's k-conjugation for partitions [18], and one involving a generalization of some lattice paths of Bressoud and Burge [13, 14, 15]. The following is the main theorem of this part, the combinatorial concepts being necessarily fully defined later in the paper. When $a = 0$ and $X = C$, D, or E, we recover some of the main results of [13, 14, 15].

Theorem 1.5.

- *Let $\tilde{B}_{k,i}(n,j)$ denote the number of overpartitions λ of n counted by $c_{k,i}(j,m,n)$ for some m.*
- *Let $\tilde{C}_{k,i}(n,j)$ denote the number of overpartitions of n with j overlined parts whose successive ranks lie in $[-i+2, 2k-i-2]$.*
- *Let $\tilde{D}_{k,i}(n,j)$ denote the number of self-(k,i)-conjugate overpartitions of n with j overlined parts.*
- *Let $\tilde{E}_{k,i}(n,j)$ denote the number of Bressoud-Burge lattice paths of major index n with j South steps which start at $k-i$, whose height is less than k and where the peaks of coordinates $(x, k-1)$ are such that $x-u$ is congruent to $i-1$ modulo 2 (u is the number of South steps to the left of the peak).*

Then for $X = B$, C, D, or E,

$$\sum_{n,j \geq 0} \tilde{X}_{k,i}(n,j) a^j q^n = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1/a)_n (-1)^n a^n q^{(2k-1)\binom{n+1}{2} - in + n}}{(-aq)_n}. \tag{1.6}$$

Again, the right-hand side of (1.6) is in many cases an infinite product, and hence there are results like Corollaries 1.2 - 1.4 involving the functions $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$. However, we shall not highlight these corollaries.

The paper is organized as follows. In the next section we study the basic properties of the $\tilde{J}_{k,i}(a;x;q)$ and give proofs of Theorem 1.1 and Corollaries 1.2 - 1.4. In Section 3, we compute the generating function of the paths counted by $\tilde{E}_{k,i}(n,j)$ to show that they are in bijection with the overpartitions counted by $\tilde{B}_{k,i}(n,j)$. In Section 4, we present a direct bijection between the paths counted by $\tilde{E}_{k,i}(n,j)$ and the overpartitions counted by $\tilde{C}_{k,i}(n,j)$. In Section 5, we compute the generating function of the overpartitions counted by $\tilde{D}_{k,i}(n,j)$ to show that they are in bijection with the paths counted by $\tilde{E}_{k,i}(n,j)$. The techniques used in Sections 3,4, and 5 are very similar to [17]. We conclude in Section 6 with some suggestions for future research.

2. The $\tilde{J}_{k,i}(a;x;q)$

We begin by proving some facts about the functions $\tilde{H}_{k,i}(a;x;q)$ and $\tilde{J}_{k,i}(a;x;q)$ defined in the introduction.

Lemma 2.1.

$$\tilde{H}_{k,0}(a;x;q) = 0 \tag{2.1}$$

$$\tilde{H}_{k,-i}(a;x;q) = -x^{-i}\tilde{H}_{k,i}(a;x;q) \tag{2.2}$$

$$\tilde{H}_{k,i}(a;x;q) - \tilde{H}_{k,i-2}(a;x;q) = x^{i-2}(1+x)\tilde{J}_{k,k-i+1}(a;x;q). \tag{2.3}$$

Proof. The first part is trivial and the second part follows from the fact that

$$-x^{-i}q^{-in}(1-x^iq^{2ni}) = q^{-n(-i)}(1-x^{-i}q^{2n(-i)}).$$

For the third part, we have

$$\begin{aligned}
&\tilde{H}_{k,i}(a;x;q) - \tilde{H}_{k,i-2}(a;x;q) \\
&= \sum_{n\geq 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+n} x^{(k-1)n}(-x,-1/a)_n(-axq^{n+1})_\infty}{(q^2;q^2)_n(xq^n)_\infty}(q^{-in}-x^iq^{in}-q^{(2-i)n}+(xq^n)^{i-2}) \\
&= \sum_{n\geq 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+n} x^{(k-1)n}(-x,-1/a)_n(-axq^{n+1})_\infty q^{-in}(1-q^{2n})}{(q^2;q^2)_n(xq^n)_\infty} \\
&+ \sum_{n\geq 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+n} x^{(k-1)n}(-x,-1/a)_n(-axq^{n+1})_\infty x^{i-2}q^{n(i-2)}(1-x^2q^{2n})}{(q^2;q^2)_n(xq^n)_\infty} \\
&= \sum_{n\geq 1} \frac{(-a)^n q^{kn^2-\binom{n}{2}+n} x^{(k-1)n}(-x,-1/a)_n(-axq^{n+1})_\infty q^{-in}}{(q^2;q^2)_{n-1}(xq^n)_\infty} \\
&+ \sum_{n\geq 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+n} x^{(k-1)n}(-x)_{n+1}(-1/a)_n(-axq^{n+1})_\infty x^{i-2}q^{n(i-2)}}{(q^2;q^2)_n(xq^{n+1})_\infty} \\
&= \sum_{n\geq 0} \frac{(-a)^{n+1} q^{kn^2+2kn+k-\binom{n+1}{2}+n+1} x^{kn+k-n-1}(-x,-1/a)_{n+1}(-axq^{n+2})_\infty q^{-in-i}}{(q^2;q^2)_n(xq^{n+1})_\infty} \\
&+ \sum_{n\geq 0} \frac{(-a)^n q^{kn^2-\binom{n}{2}+n} x^{(k-1)n}(-x)_{n+1}(-1/a)_n(-axq^{n+1})_\infty x^{i-2}q^{n(i-2)}}{(q^2;q^2)_n(xq^{n+1})_\infty}
\end{aligned}$$

$$
\begin{aligned}
&= x^{i-2}\sum_{n\geq 0}\frac{(-a)^n q^{kn^2-\binom{n}{2}+ni-n}x^{(k-1)n}(-x)_{n+1}(-1/a)_n(-axq^{n+2})_\infty}{(q^2;q^2)_n(xq^{n+1})_\infty}\\
&\times \left((1+axq^{n+1}) - ax^{k-i+1}q^{2kn-2ni+n+k-i+1}(1+q^n/a)\right)\\
&= x^{i-2}\sum_{n\geq 0}\frac{(-a)^n q^{kn^2-\binom{n}{2}+ni-n}x^{(k-1)n}(-x)_{n+1}(-1/a)_n(-axq^{n+2})_\infty}{(q^2;q^2)_n(xq^{n+1})_\infty}(1-x^{k-i+1}q^{(k-i+1)(2n+1)})\\
&+ x^{i-2}\sum_{n\geq 0}\frac{(-a)^n q^{kn^2-\binom{n}{2}+ni-n}x^{(k-1)n}(-x)_{n+1}(-1/a)_n(-axq^{n+2})_\infty}{(q^2;q^2)_n(xq^{n+1})_\infty}axq^{n+1}(1-x^{k-i}q^{(k-i)(2n+1)})\\
&= x^{i-2}(1+x)\sum_{n\geq 0}\frac{(-a)^n q^{kn^2-\binom{n}{2}+ni-n}x^{(k-1)n}(-xq)_n(-1/a)_n(-axq^{n+2})_\infty}{(q^2;q^2)_n(xq^{n+1})_\infty}(1-x^{k-i+1}q^{(k-i+1)(2n+1)})\\
&+ x^{i-2}(1+x)axq\sum_{n\geq 0}\frac{(-a)^n q^{kn^2-\binom{n}{2}+ni}x^{(k-1)n}(-xq)_n(-1/a)_n(-axq^{n+2})_\infty}{(q^2;q^2)_n(xq^{n+1})_\infty}(1-x^{k-i}q^{(k-i)(2n+1)})\\
&= x^{i-2}(1+x)\left(\tilde{H}_{k,k-i+1}(a;xq;q) + axq\tilde{H}_{k,k-i}(a;xq;q)\right)\\
&= x^{i-2}(1+x)\tilde{J}_{k,k-i+1}(a;xq;q).
\end{aligned}
$$

□

Now assume that $1\leq i\leq k$. The following recurrences for the $\tilde{J}_{k,i}(a;x;q)$ are fundamental.

Theorem 2.2.

$$
\begin{aligned}
\tilde{J}_{k,1}(a;x;q) &= \tilde{J}_{k,k}(a;xq;q) && (2.4)\\
\tilde{J}_{k,2}(a;x;q) &= (1+xq)\tilde{J}_{k,k-1}(a;xq;q) + axq\tilde{J}_{k,k}(a;xq;q) && (2.5)\\
\tilde{J}_{k,i}(a;x;q) - \tilde{J}_{k,i-2}(a;x;q) &= (xq)^{i-2}(1+xq)\tilde{J}_{k,k-i+1}(a;xq;q) && (2.6)\\
&+ a(xq)^{i-2}(1+xq)\tilde{J}_{k,k-i+2}(a;xq;q) \quad (3\leq i\leq k)
\end{aligned}
$$

Proof. Using (2.3) followed by (2.2) and then (2.1), we have

$$
\begin{aligned}
\tilde{J}_{k,k}(a;xq;q) &= \frac{\tilde{H}_{k,1}(a;xq;q) - \tilde{H}_{k,-1}(a;xq;q)}{(xq)^{-1}(1+xq)}\\
&= \frac{\tilde{H}_{k,1}(a;xq;q) + (xq)^{-1}\tilde{H}_{k,1}(a;xq;q)}{(xq)^{-1}(1+xq)}\\
&= \tilde{H}_{k,1}(a;xq;q)\\
&= \tilde{H}_{k,1}(a;xq;q) + axqH_{k,0}(a;xq;q)\\
&= \tilde{J}_{k,1}(a;x;q),
\end{aligned}
$$

which is (2.4). For (2.5), we have

$$\begin{aligned}\tilde{J}_{k,2}(a;xq;q) &= \tilde{H}_{k,2}(a;xq;q) + axq\tilde{H}_{k,1}(a;xq;q)\\ &= \tilde{H}_{k,2}(a;xq;q) - \tilde{H}_{k,0}(a;xq;q) + axq\tilde{H}_{k,1}(a;xq;q)\\ &= (1+x)\tilde{J}_{k,k-1}(a;xq;q) + axq\tilde{J}_{k,1}(a;x;q).\end{aligned}$$

Finally, using (2.3) we have

$$\begin{aligned}\tilde{J}_{k,i}(a;x;q) - \tilde{J}_{k,i-2}(a;x;q) &= \tilde{H}_{k,i}(a;xq;q) + axq\tilde{H}_{k,i-1}(a;xq;q)\\ &- \tilde{H}_{k,i-2}(a;xq;q) - axq\tilde{H}_{k,i}(a;xq;q)\\ &= (xq)^{i-2}(1+xq)\tilde{J}_{k,k-i+1}(a;xq;q)\\ &+ axq(xq)^{i-3}(1+xq)\tilde{J}_{k,k-i+2}(a;xq;q),\end{aligned}$$

which is (2.6) and which completes the proof of the Theorem. □

We now turn to the proof of Theorem 1.1. If we write

$$\tilde{J}_{k,i}(a;x;q) = \sum_{j,m,n\geq 0} b_{k,i}(j,m,n)a^j x^m q^n,$$

then the recurrences in Theorem 2.2 imply that

$$b_{k,1}(j,m,n) = b_{k,k}(j,m,n-m), \tag{2.7}$$

$$\begin{aligned}b_{k,2}(j,m,n) &= b_{k,k-1}(j,m,n-m)\\ &+ b_{k,k-1}(j,m-1,n-m)\\ &+ b_{k,k}(j-1,m-1,n-m),\end{aligned} \tag{2.8}$$

and for $3 \leq i \leq k$,

$$\begin{aligned}b_{k,i}(j,m,n) - b_{k,i-2}(j,m,n) &= b_{k,k-i+1}(j,m-i+2,n-m)\\ &+ b_{k,k-i+1}(j,m-i+1,n-m)\\ &+ b_{k,k-i+2}(j-1,m-i+2,n-m)\\ &+ b_{k,k-i+2}(j-1,m-i+1,n-m).\end{aligned} \tag{2.9}$$

We shall demonstrate that the $c_{k,i}(j,m,n)$ also satisfy these recurrences. In what follows we shall repeatedly employ a mapping $\lambda \to \widehat{\lambda}$, where $\widehat{\lambda}$ is obtained by removing all of the ones from λ and then subtracting one from each remaining part. Before continuing, we make a couple of observations regarding this mapping. First, if λ satisfies condition (ii) in the statement of the theorem, so does $\widehat{\lambda}$. Second, if λ is an overpartition counted by $c_{k,i}(j,m,n)$ and $\widehat{\lambda}$ is saturated at ℓ, then λ was saturated at $\ell+1$, so we have

$$\begin{aligned}
\ell f_\ell(\widehat{\lambda}) + (\ell+1)f_{\ell+1}(\widehat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\widehat{\lambda}) &= \ell f_{\ell+1}(\lambda) + (\ell+1)f_{\ell+2}(\lambda) + (\ell+1)f_{\overline{\ell+2}}(\lambda) \\
&= (\ell+1)f_{\ell+1}(\lambda) + (\ell+2)f_{\ell+2}(\lambda) + (\ell+2)f_{\overline{\ell+2}}(\lambda) \\
&- (f_\ell(\widehat{\lambda}) + f_{\ell+1}(\widehat{\lambda}) + f_{\overline{\ell+1}}(\widehat{\lambda})) \qquad (2.10) \\
&\equiv i - 1 + V_\lambda(\ell+1) \\
&- (f_\ell(\widehat{\lambda}) + f_{\ell+1}(\widehat{\lambda}) + f_{\overline{\ell+1}}(\widehat{\lambda})) \pmod 2 \\
&\equiv V_\lambda(\ell+1) + k - i \pmod 2
\end{aligned}$$

Finally, it is clear that

$$V_{\widehat{\lambda}}(\ell) \equiv \begin{cases} V_\lambda(\ell+1) \pmod 2, & \text{if } \overline{1} \notin \lambda \\ V_\lambda(\ell+1) + 1 \pmod 2, & \text{if } \overline{1} \in \lambda \end{cases} \qquad (2.11)$$

We begin with (2.7). Given an overpartition λ counted by $c_{k,1}(j,m,n)$, $\widehat{\lambda}$ is an overpartition of $n-m$ with m parts, j of which are overlined. Since λ could have had at most $k-1$ twos, $\widehat{\lambda}$ has at most $k-1$ ones. If $\widehat{\lambda}$ is saturated at ℓ, then from (2.10) and (2.11) we have $\ell f_\ell(\widehat{\lambda}) + (\ell+1)f_{\ell+1}(\widehat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k-1+V_{\widehat{\lambda}}(\ell) \pmod 2$. Thus $\widehat{\lambda}$ is an overpartition counted by $c_{k,k}(j,m,n-m)$. Since the mapping from λ to $\widehat{\lambda}$ is reversible, we have the recurrence (2.7) for the functions $c_{k,i}(j,m,n)$.

We turn to (2.8). Suppose now that λ is an overpartition counted by $c_{k,2}(j,m,n)$. Then λ has at most one 1. We consider three cases.

First, if λ has no ones, then it can have at most $k-2$ twos. For if λ had $k-1$ twos, then $1f_1(\lambda) + 2f_2(\lambda) + 2f_{\overline{2}}(\lambda) \equiv 0 \pmod 2$ violates condition (iii) in the definition of the $c_{k,2}(j,m,n)$. Hence $\widehat{\lambda}$ is an overpartition of $n-m$ into m parts, ℓ of which are overlined, and having at most $k-2$ ones. If $\widehat{\lambda}$ is saturated at ℓ, then from (2.10) and (2.11) we have $\ell f_\ell(\widehat{\lambda}) + (\ell+1)f_{\ell+1}(\widehat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k-2+V_{\widehat{\lambda}}(\ell) \pmod 2$. Hence $\widehat{\lambda}$ is an overpartition counted by $c_{k,k-1}(j,m,n-m)$.

Second, if 1 occurs (non-overlined) in λ, then there can be up to $k-2$ twos, so $\widehat{\lambda}$ has at most $k-2$ ones. If $\widehat{\lambda}$ is saturated at ℓ, then from (2.10) and (2.11) we have $\ell f_\ell(\widehat{\lambda}) + (\ell+1)f_{\ell+1}(\widehat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k-2+V_{\widehat{\lambda}}(\ell) \pmod 2$. Hence $\widehat{\lambda}$ is an overpartition counted by $c_{k,k-1}(j,m-1,n-m)$.

Third and finally, if $\overline{1}$ occurs in λ, then there can be at most $k-1$ twos, so $\widehat{\lambda}$ has at most $k-1$ ones. If $\widehat{\lambda}$ is saturated at ℓ, then from (2.10) and (2.11) we have $\ell f_\ell(\widehat{\lambda}) + (\ell+1)f_{\ell+1}(\widehat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k-1+V_{\widehat{\lambda}}(\ell) \pmod 2$. Hence $\widehat{\lambda}$ is an overpartition counted by $c_{k,k}(j-1,m-1,n-m)$.

Since the mappings are reversible, we have the recurrence (2.8) for the functions $c_{k,i}(j,m,n)$.

For the recurrence (2.9), everything continues to work out nicely as above. Note that for $3 \le i \le k$, $c_{k,i}(j,m,n) - c_{k,i-2}(j,m,n)$ counts those overpartitions λ counted by $c_{k,i}(j,m,n)$ having exactly $i-1$ or $i-2$ ones. We consider two cases. First, if $\overline{1}$ does not occur, then if λ has $i-1$ ones then there can be at most $k-i$ twos in λ and therefore at most $k-i$ ones in $\widehat{\lambda}$. If λ has $i-2$ ones there can still be at most $k-i$ twos, or else the defining condition (iii) would be violated. So in either case, there are at most $k-i$ ones in $\widehat{\lambda}$. And, in either case, if $\widehat{\lambda}$ is saturated at ℓ, using

(2.10) and (2.11) as usual shows that $\ell f_\ell(\widehat{\lambda}) + (\ell+1)f_{\ell+1}(\widehat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k - i + V_{\widehat{\lambda}}(\ell)$ (mod 2). So $\widehat{\lambda}$ is an overpartition counted by $c_{k,k-i+1}(j, m-i+1, n-m)$ in the first case, and $c_{k,k-i+1}(j, m-i+2, n-m)$ in the second case.

Now if $\overline{1}$ does occur in λ, then whether there are $i-1$ or $i-2$ ones there can be up to $k-i+1$ twos, and so $\widehat{\lambda}$ has at most $k-i+1$ ones. Finally, if $\widehat{\lambda}$ is saturated at ℓ, then $\ell f_\ell(\widehat{\lambda}) + (\ell+1)f_{\ell+1}(\widehat{\lambda}) + (\ell+1)f_{\overline{\ell+1}}(\widehat{\lambda}) \equiv k-i+1+V_{\widehat{\lambda}}(\ell)$ (mod 2). Therefore $\widehat{\lambda}$ is an overpartition counted by $c_{k,k-i+2}(j-1, m-i+1, n-m)$ if λ has $i-1$ ones and $c_{k,k-i+2}(j-1, m-i+2, n-m)$ if λ has $i-2$ ones. Again the mappings here are reversible, so we have the recurrence (2.9) for the functions $c_{k,i}(j,m,n)$.

To finalize the claim that the two families of functions are equal, we note that

$$b_{k,i}(j,m,n) = \begin{cases} 0, & \text{if } j \leq 0,\ m \leq 0 \text{ or } n \leq 0, \text{ and } (j,m,n) \neq (0,0,0) \\ 1, & \text{if } (j,m,n) = (0,0,0), \end{cases} \tag{2.12}$$

which is indeed also true for the $c_{k,i}(j,m,n)$. □

We now deduce Corollaries 1.2 - 1.4. First, we'll prove a proposition which is a piece of Theorem 1.5 and from which it follows that several instances of the $\tilde{J}_{k,i}(a;1;q)$ are infinite products.

Proposition 2.3. *We have*

$$\tilde{J}_{k,i}(a;1;q) = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1/a)_n(-1)^n a^n q^{(2k-1)\binom{n+1}{2} - in + n}}{(-aq)_n}. \tag{2.13}$$

Proof. Using the definition, we have

$$\begin{aligned}
\tilde{J}_{k,i}(a;1;q) &= \tilde{H}_{k,i}(a;q;q) + aq\tilde{H}_{k,i}(a;q;q) \\
&= \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-in}(1-q^{i(2n+1)})(-1/a)_n}{(-aq)_{n+1}} \\
&+ aq\frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-(i-1)n}(1-q^{(i-1)(2n+1)})(-1/a)_n}{(-aq)_{n+1}} \\
&= \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-in}(-1/a)_n}{(-aq)_{n+1}} \\
&\quad \left(1 - q^{(2n+1)i} + aq^{n+1} - aq^{n+1+(i-1)(2n+1)}\right) \\
&= \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-in}(-1/a)_n}{(-aq)_n} \\
&- \frac{(-aq)_\infty}{(q)_\infty} \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+kn-\binom{n}{2}-in+i(2n+1)}(-1/a)_n}{(-aq)_{n+1}}(1+aq^{-n}).
\end{aligned}$$

In this last sum, we replace n by $-n-1$ and simplify using the fact that

$$(x)_{-n} = \frac{(-1)^n q^{\binom{n+1}{2}}}{x^n (q/x)_n}.$$

The result is precisely (2.13). □

Corollary 2.4. *We have*

$$\tilde{J}_{k,i}(0;1;q) = \frac{(q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty}{(q)_\infty}, \tag{2.14}$$

$$\tilde{J}_{k,i}(1/q;1;q^2) = \frac{(-q;q^2)_\infty (q^{2i-1}, q^{4k-2i-1}, q^{4k-2}; q^{4k-2})_\infty}{(q^2;q^2)_\infty}, \tag{2.15}$$

and

$$\tilde{J}_{k,1}(1/q,1;q) = \frac{(-q)_\infty (q, q^{2k-2}, q^{2k-1}; q^{2k-1})_\infty}{(q)_\infty}. \tag{2.16}$$

Proof. These are immediate upon invoking Proposition 2.3 and the Jacobi triple product identity,

$$\sum_{n\in\mathbb{Z}} z^n q^{\binom{n+1}{2}} = (-zq, -1/z, q)_\infty. \tag{2.17}$$

□

We are now ready to prove the corollaries. In the following, we consider that λ is an overpartition of n with j overlined parts, hence it is counted in the coefficient of $q^n a^j$ of $\tilde{J}_{k,i}(a,1;q)$. This overpartition is such that (i) $f_1(\lambda) + f_{\overline{1}}(\lambda) \leq i-1$, (ii) $f_\ell(\lambda) + f_{\ell+1}(\lambda) + f_{\overline{\ell+1}}(\lambda) \leq k-1$, and (iii) if λ is saturated at ℓ, that is, if the maximum in (ii) is achieved, then $\ell f_\ell(\lambda) + (\ell+1) f_{\ell+1}(\lambda) + (\ell+1) f_{\overline{\ell+1}}(\lambda) \equiv i-1+V_\lambda(\ell) \pmod 2$.

For Corollary 1.2, we consider the functions $\tilde{J}_{k,i}(0;1;q)$. From Theorem 1.1 we easily see that the coefficient of q^n in $J_{k,i}(0;1;q)$ is $\tilde{B}_{k,i}(n)$. Indeed when $(a,q)=(0,q)$, this implies that λ has no overlined parts, that is $f_{\overline{\ell}} = V_\lambda(\ell) = 0$ for all ℓ. Therefore the conditions (i), (ii) and (iii) are now (i) $f_1(\lambda) \leq i-1$, (ii) $f_\ell(\lambda) + f_{\ell+1}(\lambda) \leq k-1$, and (iii) if the maximum in (ii) is achieved at ℓ, then $\ell f_\ell(\lambda) + (\ell+1) f_{\ell+1}(\lambda) \equiv i-1 \pmod 2$. On the other hand, from (2.14) of Corollary 2.4, the coefficient of q^n in $\tilde{J}_{k,i}(0;1;q)$ is also $\tilde{A}_{k,i}(n)$. □

For Corollary 1.3, we use the functions $\tilde{J}_{k,i}(1/q;1;q^2)$. A little thought reveals that the coefficient of q^n in $\tilde{J}_{k,i}(1/q;1;q^2)$ is $\tilde{B}^2_{k,i}(n)$. When $(a,q) = (1/q, q^2)$, for any ℓ all the parts equal to $\overline{\ell}$ in λ are changed to $2\ell-1$ and all the parts equal to ℓ in λ are changed to 2ℓ. This implies that (i) $f_1(\lambda)+f_2(\lambda) \leq i-1$, (ii) $f_{2\ell}(\lambda) + f_{2\ell+1}(\lambda) + f_{2\ell+2}(\lambda) \leq k-1$, and (iii) if the maximum in (ii) is achieved at ℓ, then $\ell f_{2\ell}(\lambda) + (\ell+1) f_{2\ell+2}(\lambda) + (\ell+1) f_{2\ell+1}(\lambda) \equiv i-1+V^o_\lambda(\ell) \pmod 2$.

Rewriting of the product in (2.15) as

$$(q^2;q^4)_\infty (q^{8k-4};q^{8k-4})_\infty (q^{2i-1}, q^{4k-2i-1}; q^{4k-2})_\infty (-q^{2k-1}; q^{4k-2})_\infty \prod_{n \not\equiv 2k-1 \pmod{4k-2}} \frac{1}{(1-q^n)}$$

shows that the coefficient of q^n in $\tilde{J}_{k,i}(1/q;1;q^2)$ is also $\tilde{A}^2_{k,i}(n)$. □

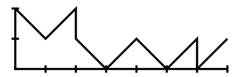

FIGURE 1. This path has four peaks : two NES peaks (located at $(2,2)$ and $(6,1)$) and two NESE peaks (located at $(4,1)$ and $(7,1)$). Its major index is $2+4+6+7=19$.

Finally, for Corollary 1.4, we use the functions $\tilde{J}_{k,1}(1/q;1;q)$. Again it may readily be seen that the coefficient of q^n therein is $\tilde{B}^3_k(n)$. Indeed when $i=1$, $f_1(\lambda)=f_{\overline{1}}(\lambda)=0$, and when $(a,q)=(1/q,q)$ all the overlined parts of λ are decreased by one. That implies that (i) $f_1(\lambda)=0$, (ii) $f_\ell(\lambda)+f_{\overline{\ell}}(\lambda)+f_{\ell+1}(\lambda)\le k-1$, and (iii) if the maximum in condition (ii) is achieved at ℓ, then $\ell f_\ell(\lambda)+(\ell+1)f_{\overline{\ell}}(\lambda)+(\ell+1)f_{\ell+1}(\lambda)\equiv V_\lambda(\ell-1) \pmod 2$. As $V_\lambda(\ell-1)+f_{\overline{\ell}}(\lambda)=V_\lambda(\ell)$, this is equivalent to $\ell f_\ell(\lambda)+\ell f_{\overline{\ell}}(\lambda)+(\ell+1)f_{\ell+1}(\lambda)\equiv V_\lambda(\ell) \pmod 2$. On the other hand, from (2.16) of Corollary 2.4, the coefficient q^n in $\tilde{J}_{k,1}(1/q;1;q)$ is also $\tilde{A}^3_k(n)$. □

3. Lattice Paths

We study paths in the first quadrant that use four kinds of unitary steps:

- North-East $NE:\ (x,y)\to(x+1,y+1)$,
- South-East $SE:\ (x,y)\to(x+1,y-1)$,
- South $S:\ (x,y)\to(x,y-1)$,
- East $E:\ (x,0)\to(x+1,0)$.

The *height* corresponds to the y-coordinate. A South step can only appear after a North-East step and an East step can only appear at height 0. The paths must end with a North-East or South step. A *peak* is a vertex preceeded by a North-East step and followed by a South step (in which case it will be called a *NES peak*) or by a South-East step (in which case it will be called a *NESE peak*). If the path ends with a North-East step, its last vertex is also a NESE peak. The *major index* of a path is the sum of the x-coordinates of its peaks (see Figure 1 for an example). When the paths have no South steps, this is the definition of the paths in [13].

Let k and i be a positive integers with $i\le k$. Let $\tilde{E}_{k,i}(n,j)$ be the number of paths of major index n with j South steps which satisfy the following *special* (k,i)*-conditions*: (i) the paths start at height $k-i$, (ii) their height is less than k, (iii) every peak of coordinates $(x,k-1)$ satisfies $x-u\equiv i-1 \pmod 2$ where u is the number of South steps to the left of the peak.

Let $\tilde{\mathcal{E}}_{k,i}(a,q)$ be the generating function of those paths, that is $\tilde{\mathcal{E}}_{k,i}(a,q)=\sum_{n,j}\tilde{E}_{k,i}(n,j)a^jq^n$. Let $\tilde{\mathcal{E}}_{k,i}(N)$ be the generating function of paths counted by $\tilde{\mathcal{E}}_{k,i}(a,q)$ which have N peaks. Moreover, for $0\le i<k$, let $\tilde{\Gamma}_{k,i}(N)$ be the generating function of paths obtained by deleting the first NE step of a path which is counted in $\tilde{\mathcal{E}}_{k,i+1}(N)$ and begins with a NE step. Then

Proposition 3.1.

$$\tilde{\mathcal{E}}_{k,i}(N) = q^N\tilde{\mathcal{E}}_{k,i+1}(N)+q^N\tilde{\Gamma}_{k,i-1}(N);\qquad 0<i<k \tag{3.1}$$

$$\tilde{\Gamma}_{k,i}(N) = q^N\tilde{\Gamma}_{k,i-1}(N)+(a+q^{N-1})\tilde{\mathcal{E}}_{k,i+1}(N-1);\quad 0<i<k \tag{3.2}$$

$$\tilde{\mathcal{E}}_{k,k}(N) = q^N\tilde{\mathcal{E}}_{k,k-1}(N)+q^N\tilde{\Gamma}_{k,k-1}(N) \tag{3.3}$$

$$\tilde{\mathcal{E}}_{k,i}(0)=1 \qquad \tilde{\Gamma}_{k,0}(N)=0. \tag{3.4}$$

Proof. We prove that by induction on the length of the path. If the path is empty, it is the path counted in $\tilde{\mathcal{E}}_{k,i}(0)$. For $1 < i < k$, if the path is not empty, then we take off its first step. When we do this, we increase or decrease i by 1 and thus change the parity of $i-1$; moreover, all the peaks are shifted by 1, so the parity of $x-u-i$ is not changed (if the step we remove is a South step, the peaks are not shifted but u decreases by 1 for all peaks, so the result is the same). The case $i=0$ is straightforward as a path that starts at height $k-1$ can not start with a North-East step. The case $i=k$ needs further explanation. For these paths the fact that every peak of coordinates $(x, k-1)$ satisfies $x-u \equiv k-1 \pmod 2$ is equivalent to the fact that every peak of coordinates $(x, k-1)$ has an even number of East steps to its left. Therefore the paths counted in $\tilde{\mathcal{E}}_{k,k}(N)$ that start with an East step where this step is deleted are in bijection with the paths counted in $\tilde{\mathcal{E}}_{k,k-1}(N)$. This bijection is easy to describe. If the path does not have any other East step, then the path is shifted up, i.e each vertex of the path (x,y) is changed to $(x, y+1)$. This creates a path in $\tilde{\mathcal{E}}_{k,k-1}(N)$ that does not have any vertex of the form $(x,0)$. If the path does contain an East step, then the path before the first East step is shifted up, the East step is changed to a South-East step and the rest of the path is not changed. This creates a path in $\tilde{\mathcal{E}}_{k,k-1}(N)$ that has at least one vertex of the form $(x,0)$. Moreover it is easy to see that the paths counted in $\tilde{\mathcal{E}}_{k,k}(N)$ that start with a South-East step where this step is deleted are the paths counted in $\tilde{\Gamma}_{k,k-1}(N)$. □

These recurrences uniquely define the series $\tilde{\mathcal{E}}_{k,i}(N)$ and $\tilde{\Gamma}_{k,i}(N)$. We get that

Theorem 3.2.

$$\tilde{\mathcal{E}}_{k,i}(N) = a^N q^{\binom{N+1}{2}}(-1/a)_N \sum_{n=-N}^{N} (-1)^n \frac{q^{(k-1)n^2+(k-i)n}}{(q)_{N-n}(q)_{N+n}} \tag{3.5}$$

$$\tilde{\Gamma}_{k,i}(N) = a^N q^{\binom{N}{2}}(-1/a)_N \sum_{n=-N}^{N-1} (-1)^n \frac{q^{(k-1)n^2+(k-i-1)n}}{(q)_{N-n-1}(q)_{N+n}} \tag{3.6}$$

The proof is omitted. It uses simple algebraic manipulation to prove that these generating functions satisfy the recurrence relations of Proposition 3.1.

We recall a proposition proved in [17] that will enable us to compute $\tilde{\mathcal{E}}_{k,i}(a,q)$ from the recurrences:

Proposition 3.3. [17] *For any* $n \in \mathbb{Z}$

$$\sum_{N \geq |n|} \frac{(-azq)_n(-q^n/a)_{N-n} q^{\binom{N+1}{2}-\binom{n+1}{2}} z^{N-n} a^{N-n}}{(zq)_{N+n}(q)_{N-n}} = \frac{(-azq)_\infty}{(zq)_\infty}.$$

From (3.5), summing on N using Proposition 3.3, we get

$$\sum_{n,j \geq 0} \tilde{E}_{k,i}(n,j) a^j q^n = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1/a)_n (-1)^n a^n q^{(2k-1)\binom{n+1}{2} - in + n}}{(-aq)_n}. \tag{3.7}$$

This is Equation (1.6).

We now need to recall the definition of the relative height of a peak. This notion was defined by Bressoud in [13]. The definition we use is a simpler version taken from [9].

Definition 3.4. [9] *The relative height of a peak* (x, y) *is the largest integer* h *for which we can find two vertices on the path,* $(x', y-h)$ *and* $(x'', y-h)$*, such that* $x' < x < x''$ *and such that between these two vertices there are no peaks of height* y *and every peak of height* y *has weight* $\geq x$*.*

We now state a result of Bressoud [13] which will be used to prove our next proposition.

Lemma 3.5. [13]

$$\frac{q^{n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}}{(q)_{n_2-n_3}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}$$

is the generating function of the paths with no South steps which start at height $k-i$*, whose height is less than* $k-1$*, with* n_j *peaks of relative height* $\geq j-1$ *for* $2 \leq j \leq k-1$ *and where the peaks of coordinates* $(x, k-2)$ *are such that* x *is congruent to* $i-2$ *modulo* 2*.*

Proposition 3.6.

$$\frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}$$

is the generating function of the paths (counted by major index and number of south steps) satisfying the special (k,i)*-conditions and having* n_j *peaks of relative height* $\geq j$ *for* $1 \leq j \leq k-1$*.*

Proof. It is similar to that of Proposition 6.1 of [17] except that its starting point is Lemma 3.5. We first insert a NES peak at each peak. This "volcanic uplift" operation increases the relative height of each peak by one. Moreover, it transforms a peak of height $k-2$ into a peak of height $k-1$ and changes the parity of $x-u$ for all peaks. Indeed, if we consider the jth peak from the left, its x-coordinate increases by j and it has $j-1$ South steps to its left after the uplift. Therefore, for this peak, $x-u$ has increased by $j-(j-1)=1$. We then insert n_1-n_2 NES peaks of relative height one at the begining of the path, transform some of the NES peaks into NESE peaks and move some of the peaks of relative height one we inserted. These operations do not modify the parity of $x-u$ for any peak. Besides, it is important to note that when we move the peaks, the distribution of relative heights is not modified. □

4. Successive Ranks

The Frobenius representation of an overpartition [16, 22] of n is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_N \\ b_1 & b_2 & \ldots & b_N \end{pmatrix}$$

where $(a_1, \ldots, a_N)$ is a partition into distinct nonnegative parts and $(b_1, \ldots, b_N)$ is an overpartition into nonnegative parts where the first occurrence of a part can be overlined and $N + \sum(a_i + b_i) = n$.

We call that the Frobenius representation of an overpartition because it is in bijection with overpartitions. We say that the *generalized Durfee square* of an overpartition λ has side N if N is the largest integer such that the number of overlined parts plus the number of non-overlined parts greater or equal to N is greater than or equal to N.

Proposition 4.1. [17] *There exists a bijection between overpartitions whose Frobenius representation has* N *columns and whose bottom line has* j *non-overlined parts and overpartitions with generalized Durfee square of size* N *and* j *overlined parts.*

We now define the successive ranks.

Definition 4.2. [17] *The* successive ranks *of an overpartition can be defined from its Frobenius representation. If an overpartition has Frobenius representation*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ b_1 & b_2 & \cdots & b_N \end{pmatrix}$$

then its ith successive rank r_i is $a_i - b_i$ minus the number of non-overlined parts in $\{b_{i+1}, \dots, b_N\}$.

For example, the successive ranks of $\begin{pmatrix} 7 & 4 & 2 & 0 \\ \overline{3} & 3 & 1 & \overline{0} \end{pmatrix}$ are $(2, 0, 1, 0)$.

The purpose of this Section is to prove the following Proposition.

Proposition 4.3. *There exists a one-to-one correspondence between the paths of major index n with j South steps, counted by $\tilde{E}_{k,i}(n, j)$ and the overpartitions of n with j non-overlined parts in the bottom line of their Frobenius representation and whose successive ranks lie in $[-i + 2, 2k - i - 2]$, counted by $\tilde{C}_{k,i}(n, j)$. This correspondence is such that the paths have N peaks if and only if the Frobenius representation of the overpartition has N columns.*

Proof. Let $\overline{E}_{k,i}(n, j)$ be the number of paths counted by $\tilde{E}_{k,i}(n, j)$ where the last condition ($x - u \equiv i - 1 \pmod 2$ for the peaks of height $k - 1$) is dropped. In [17], we proposed a bijection between paths counted by $\overline{E}_{k,i}(n, j)$ and overpartitions of n with j non-overlined parts in the bottom line of their Frobenius representation and whose successive ranks lie in $[-i+2, 2k-i-1]$.

We recall now this map. Given a lattice path which starts at $(0, k - i)$ and a peak (x, y), let the parameter u be the number of South steps to the left of the peak. We map this peak to the pair (s, t) where

$$s = (x + k - i - y + u)/2$$
$$t = (x - k + i + y - 2 - u)/2$$

if there are an even number of East steps to the left of the peak, and

$$s = (x + k - i + y - 1 + u)/2$$
$$t = (x - k + i - y - 1 - u)/2$$

if there are an odd number of East steps to the left of the peak. Moreover, we overline t if the peak is a NESE peak. The map is easily reversible. In both cases, s and t are integers and we have $s + t + 1 = x$. The successive rank of that pair (s, t) is $r = s - t - u$ and the conditions on the paths imply that $-i + 2 \le r \le 2k - i - 1$.

Let N be the number of peaks in the path and j the number of South steps. Let (x_i, y_i) be the coordinates of the ith peak from the right and (s_i, t_i) be the corresponding pair. Then we proved in [17] that

$$\begin{pmatrix} s_1 & s_2 & \cdots & s_N \\ t_1 & t_2 & \cdots & t_N \end{pmatrix}$$

is the Frobenius representation of an overpartition whose weight is

$$\sum_{i=1}^{N} (s_i + t_i + 1) = \sum_{i=1}^{N} x_i$$

(i.e. the major index of the corresponding path), with j non-overlined parts in the bottom line (i.e. the number of South steps of the corresponding path) and whose successive ranks lie in $[-i+2, 2k-i-1]$.

If we apply this map to a path counted by $\tilde{E}_{k,i}(n,j)$ then we can show that no successive rank can be equal to $2k-i-1$. Indeed, if it was the case, we would have $s-t-u = 2k-i-1$ and from the map we know that $s-t-u = k-i-y+1$ or $k-i+y$. The first case is therefore impossible. The second case implies that $y = k-1$ and $s = \frac{1}{2}(x+u+2k-i-2)$. As s is an integer, we have $x - u \equiv i \pmod 2$. This is forbidden by the last condition of the definition of $\tilde{E}_{k,i}(n,j)$. □

5. Generalized self-conjugate overpartitions

We define an operation for overpartitions called k-conjugation, where $k \geq 2$ is an integer. From the Frobenius representation of an overpartition π, we use Algorithm III of [22] to get three partitions λ_1, λ_2 and μ as described in the following paragraph.

Let N be the number of columns of the Frobenius representation. We get λ_1, which is a partition into N nonnegative parts, by removing a staircase from the top row (i.e. we remove 0 from the smallest part, 1 from the next smallest, and so on). We get λ_2 (which is a partition into N nonnegative parts) and μ (which is a partition into distinct nonnegative parts less than N) as follows. First, we initialize λ_2 to the bottom row. Then, if the mth part of the bottom row is overlined, we remove the overlining of the mth part of λ_2, we decrease the $m-1$ first parts of λ_2 by one and we add a part $m-1$ to μ. For example, the overpartition whose Frobenius representation is

$$\begin{pmatrix} 7 & \overline{5} & 4 & 2 & \overline{0} \\ 6 & \overline{4} & 4 & 3 & \overline{1} \end{pmatrix}$$

gives $\lambda_1 = (3,2,2,1,0)$, $\lambda_2 = (4,3,3,2,1)$ and $\mu = (4,1)$.

Let λ_1' (resp. λ_2') be the conjugate of λ_1 (resp. of λ_2). λ_1' and λ_2' are thus partitions into parts less than or equal to N. Recall that the Durfee square of a partition is the largest square contained in its diagram [7] and that the i^{th} Durfee square is the Durfee square of the partition that is under the $(i-1)^{st}$ Durfee square.

We now consider two regions. The first region is the portion of λ_2' below its $(k-2)$-th Durfee square (for $k=2$, this region is λ_2'). The second region consists of the parts of λ_1' which are less than or equal to the size of the $(k-2)$-th Durfee square of λ_2' (for $k=2$, this region is λ_1').

Definition 5.1. *The* k-conjugation *consists in interchanging these two regions (if λ_2' has less than $k-2$ Durfee squares, the k-conjugation is the identity).*

Example 1. *We consider the overpartition π whose Frobenius representation is*

$$\begin{pmatrix} 14 & \overline{13} & 10 & \overline{8} & 7 & \overline{5} & 3 & 0 \\ 14 & \overline{12} & 10 & \overline{8} & 7 & \overline{5} & 3 & 2 \end{pmatrix}.$$

The above algorithm gives us $\lambda_1 = (7,7,5,4,4,3,2,0)$, $\lambda_2 = (11,10,8,7,6,5,3,2)$ and $\mu = (5,3,1)$. We thus have $\lambda_1' = (7,7,6,5,3,2,2)$ and $\lambda_2' = (8,8,7,6,6,5,4,3,2,2,1)$. λ_1' and λ_2' are represented in Figure 2, where the two regions defined above (for $k=4$) are highlighted. If we swap these two regions, we get $\lambda_1' = (7,7,6,5,2,2,1)$ and $\lambda_2' = (8,8,7,6,6,5,4,3,3,2,2)$. Conjugating thse two partitions, we have $\lambda_1 = (7,6,4,4,4,3,2,0)$ and $\lambda_2 = (11,11,9,7,6,5,3,2)$.

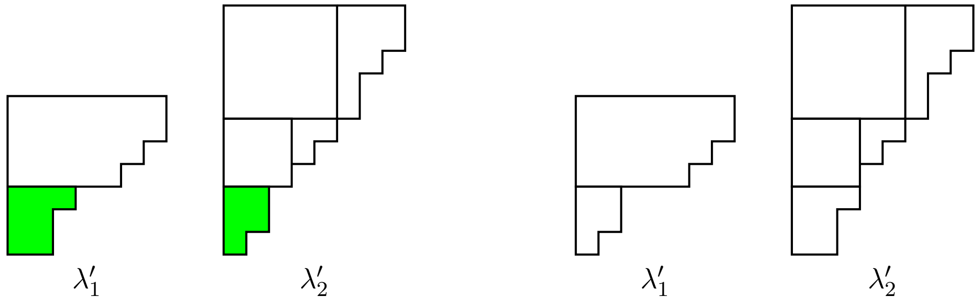

Figure 2. Illustration of the 4-conjugation (see Example 1). For the initial overpartition π (on the left), we have $\lambda'_1 = (7,7,6,5,3,2,2)$ and $\lambda'_2 = (8,8,7,6,6,5,4,3,2,2,1)$. The regions highlighted are interchanged by 4-conjugation, which gives $\lambda'_1 = (7,7,6,5,2,2,1)$ and $\lambda'_2 = (8,8,7,6,6,5,4,3,3,2,2)$ for $\pi^{(4)}$, the 4-conjugate of π (on the right).

Applying Algorithm III of [22] *in reverse (remember that $\mu = (5,3,1)$, we get that the 4-conjugate of π is*

$$\pi^{(4)} = \begin{pmatrix} 14 & \overline{12} & 9 & \overline{8} & 7 & \overline{5} & 3 & 0 \\ 14 & \overline{13} & 11 & \overline{8} & 7 & \overline{5} & 3 & 2 \end{pmatrix}$$

Remark 5.2. *For $k = 2$, we just swap λ'_1 and λ'_2 (which boils down to swapping λ_1 and λ_2) and we get the F-conjugation defined by Lovejoy* [22].

Remark 5.3. *If there are no overlined parts, we get the k-conjugation for partitions defined by Garvan* [18]. *Indeed, for partitions, the $(k-2)$-th Durfee square of λ'_2 is in fact the $(k-1)$-th Durfee square of the partition π. Consequently, the parts of λ'_2 below this Durfee square (first region) are the parts of π below its $(k-1)$-th Durfee square. Moreover, the parts of λ'_1 which are less than or equal to the size of the $(k-2)$-th Durfee square of λ'_2 (second region) are the columns of π to the right of its first Durfee square whose length is less than or equal to the size of the $(k-1)$-th Durfee square of π. We thus see that the regions we interchange in the k-conjugation are the same as in* [18].

Definition 5.4. *We say that an overpartition is* self-k-conjugate *if it is fixed by k-conjugation.*

Proposition 5.5. *The generating function of self-k-conjugate overpartitions is*

$$\sum_{n_1 \geq n_2 \geq \dots \geq n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2}(-1/a)_{n_1} a^{n_1}}{(q)_{n_1-n_2}\dots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}$$

where n_1 is the number of columns of the Frobenius symbol and $n_2, \dots, n_{k-1}$ are the sizes of the $k-2$ first successive Durfee squares of λ'_2.

Proof. We decompose a self-k-conjugate overpartition in the following way :

- μ (region IV in Figure 3), which is counted by

$$a^{n_1}(-1/a)_{n_1}$$

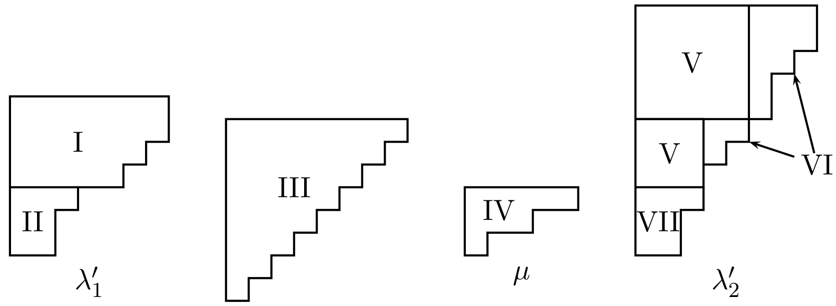

FIGURE 3. Decomposition of a self-k-conjugate overpartition (in this example, $k = 4$).

- the staircase of the top row and the part n_1 (region III), which are counted by
$$q^{\binom{n_1+1}{2}}$$
- the $k-2$ Durfee squares of λ_2' (region V), which are counted by
$$q^{n_2^2+\cdots+n_{k-1}^2}$$
- the regions between the Durfee squares of λ_2' (region VI), which are counted by
$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q$$
- the parts in λ_1' which are $> n_{k-1}$ and of course $\leq n_1$ (region I) : they are counted by
$$\frac{1}{(1-q^{n_{k-1}+1})\cdots(1-q^{n_1})} = \frac{(q)_{n_{k-1}}}{(q)_{n_1}}$$
- the two identical regions (regions II and VII), which are counted by
$$\frac{1}{(q^2;q^2)_{n_{k-1}}}.$$

Summing on $n_1, n_2, \ldots, n_{k-1}$, we get the generating function :
$$\sum_{n_1 \geq n_2 \geq \ldots \geq n_{k-1} \geq 0} (-1/a)_{n_1} a^{n_1} q^{\binom{n_1+1}{2}} q^{n_2^2+\cdots+n_{k-1}^2} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \cdots \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \frac{(q)_{n_{k-1}}}{(q)_{n_1}} \frac{1}{(q^2;q^2)_{n_{k-1}}}$$
$$= \sum_{n_1 \geq n_2 \geq \ldots \geq n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2}(-1/a)_{n_1} a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}$$

□

Corollary 5.6. *When there are no overlined parts, $a \to 0$ and we get the generating function of self-k-conjugate partitions* [18].

Definition 5.7. *Let i and k be integers with $1 \leq i \leq k$. We say that an overpartition is* self-(k,i)-conjugate *if it is obtained by taking a self-k-conjugate overpartition and adding a part n_j (n_j is the size of the $(j-1)$-th successive Durfee square of λ_2') to λ_2' for $i \leq j \leq k-1$ (if $i = k$, no parts are added).*

Remember that we denote by $\tilde{D}_{k,i}(n,j)$ the number of self-(k,i)-conjugate overpartitions with j overlined parts (or, equivalently, the number of self-(k,i)-conjugate overpartitions whose Frobenius representation has j non-overlined parts in its bottom row: see Proposition 4.1).

Proposition 5.8.

$$\tilde{\mathcal{E}}_{k,i}(a,q) = \sum_{n,j} \tilde{D}_{k,i}(n,j)a^j q^n.$$

Proof. It is obvious from Proposition 5.5 that

$$\sum_{n,j} \tilde{D}_{k,i}(n,j)a^j q^n = \sum_{n_1 \geq n_2 \geq \dots \geq n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_N a^N}{(q)_{n_1-n_2}\dots(q)_{n_{k-2}-n_{k-1}}(q^2;q^2)_{n_{k-1}}}$$

which is $\tilde{\mathcal{E}}_{k,i}(a,q)$ by Proposition 3.6. □

6. Concluding Remarks

We would like to mention that the $J_{k,i}(a;x;q)$ and $\tilde{J}_{k,i}(a;x;q)$ can be embedded in a family of functions that satisfy recurrences like those in Lemma 2.1 and are sometimes infinite products when $x=1$. For $m \geq 1$ we define

$$J_{k,i,m}(a;x;q) = H_{k,i,m}(a;xq;q) + axqH_{k,i-1,m}(a;xq;q), \tag{6.1}$$

where

$$H_{k,i,m}(a;x;q) = \sum_{n\geq 0} \frac{(-a)^n q^{kn^2+n-in-(m-1)\binom{n}{2}} x^{n(k-m-1)}(1-x^i q^{2ni})(-1/a)_n(-axq^{n+1})_\infty (x^m;q^m)_n}{(q^m;q^m)_n(x)_\infty}. \tag{6.2}$$

The case $m=1$ gives the $J_{k,i}(a;x;q)$ and $m=2$ corresponds to the $\tilde{J}_{k,i}(a;x;q)$. Equations (2.1) and (2.2) of Lemma 2.1 are true for the $H_{k,i,m}(a;x;q)$, and following the proof of (2.3), one may show that

$$H_{k,i,m}(a;x;q) - H_{k,i-m,m}(a;x;q) = x^{i-m}(1+x+x^2+\cdots+x^{m-1})J_{k,k-i+1,m}(a;x;q).$$

It would certainly be worth investigating what kinds of combinatorial identities are stored in these general series.

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