Games in computer science: a survey

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“Provability” versus “proofs”

- Games to reason about programs
- Programs **as** strategies

Model-checking 1/2

Satisfiability problem for various logics (modal, temporal, μ) for automata or concurrent systems

$$\Leftrightarrow$$

Existence of winning strategies in associated games.

Model-checking 2/2

Also $\Leftrightarrow$ (non-)emptyness problem for languages recognized by various kinds of automata on (infinite) words or trees.

Also, bisimulation in concurrency theory.

Game semantics 1/2

Strategies as proofs / programs / morphims.

Composition corresponds to cut elimination / normalization. Games semantics is very active since a decade.

Game semantics 2/2

New results in the semantics of programming languages: simple and direct semantics for programming features such as control or references,

Full abstraction results connecting denotational and operational semantics tightly.

PROLOGUE 1/13

A theorem on lattices

Joyal (1997) used games to give a nice proof of the following theorem (Whitman 1947): The free lattice L over a partial order X (with $i : X \to L$) is characterized by

PROLOGUE 2/13

- L is a lattice and i is monotonous
- If $u_1 \wedge u_2 \leq v_1 \vee v_2$, then $u_1 \wedge u_2 \leq v_1$ or $u_1 \wedge u_2 \leq v_2$ or $u_1 \leq v_1 \vee v_2$ or $u_2 \leq v_1 \vee v_2$
- If $i(x) \leq v_1 \vee v_2$, then $i(x) \leq v_1$ or $i(x) \leq v_2$
- If $u_1 \wedge u_2 \leq i(x)$, then $u_1 \leq i(x)$ or $u_2 \leq i(x)$
- If $i(x) \leq i(y)$, then $x \leq y$
- L is generated by $i(X)$

PROLOGUE 3/13

Uniqueness easy. For existence, construct a suitable preorder on the following set of terms:

$$\frac{x \in X}{x \in T(X)} \qquad \frac{}{V \in T(X)} \qquad \frac{}{F \in T(X)}$$

$$\frac{A_1 \in T(X) \quad A_2 \in T(X)}{A_1 \wedge A_2 \in T(X)} \qquad \frac{A_1 \in T(X) \quad A_2 \in T(X)}{A_1 \vee A_2 \in T(X)}$$

PROLOGUE 4/13

The preorder is defined by: $A \leq B$ if and only if (A, B) is a winning position in some graph game.

The set of nodes is $T(X) \times T(X)$.

PROLOGUE 5/13

Edges:

$$(A_1 \vee A_2, B) \rightarrow (A_1, B) \quad (A_1 \vee A_2, B) \rightarrow (A_2, B)$$
$$(A, B_1 \wedge B_2) \rightarrow (A, B_1) \quad (A, B_1 \wedge B_2) \rightarrow (A, B_2)$$

$$(A_1 \wedge A_2, B_1 \vee B_2) \rightarrow (A_1, B_1 \vee B_2) \quad (A_1 \wedge A_2, B_1 \vee B_2) \rightarrow (A_2, B_1 \vee B_2)$$
$$(A_1 \wedge A_2, B_1 \vee B_2) \rightarrow (A_1 \wedge A_2, B_1) \quad (A_1 \wedge A_2, B_1 \vee B_2) \rightarrow (A_1 \wedge A_2, B_2)$$
$$(A_1 \wedge A_2, F) \rightarrow (A_1, F) \quad (A_1 \wedge A_2, F) \rightarrow (A_2, F)$$
$$(V, B_1 \vee B_2) \rightarrow (V, B_1) \quad (V, B_1 \vee B_2) \rightarrow (V, B_2)$$
$$(A_1 \wedge A_2, x) \rightarrow (A_1, x) \quad (A_1 \wedge A_2, x) \rightarrow (A_2, x)$$
$$(x, B_1 \vee B_2) \rightarrow (x, B_1) \quad (x, B_1 \vee B_2) \rightarrow (x, B_2)$$

PROLOGUE 6/13

Each node has a polarity $\in \{P, O, N\}$ (Player, Opponent, Neutral).

$$\overset{O}{(A_1 \vee A_2, B)} \quad \overset{O}{(F, B)} \quad \overset{O}{(A, B_1 \wedge B_2)} \quad \overset{O}{(A, V)}$$

$$\overset{P}{(A_1 \wedge A_2, B_1 \vee B_2)} \quad \overset{P}{(V, B_1 \vee B_2)}$$

$$\overset{P}{(V, F)} \quad \overset{P}{(A_1 \wedge A_2, F)}$$

$$\overset{P}{(x, B_1 \vee B_2)} \quad \overset{P}{(A_1 \wedge A_2, x)}$$

$$\overset{P}{(x, F)} \quad \overset{P}{(V, x)} \quad \overset{N}{(x, y)}$$

PROLOGUE 7/13

A strategy is a full subgraph S s.t.

- If $\overset{P}{(A, B)} \in S$, then S contains **at least one** edge out of (A, B).
- If $\overset{O}{(A, B)} \in S$, then S contains **all** edges of G out of (A, B).
- If $\overset{N}{(x, y)} \in S$, then $x \leq y$ in X.

PROLOGUE 8/13

We say that (A, B) is a winning position if (A, B) belongs to some strategy. We then write $A \leq B$.

PROLOGUE 9/13

A proof is a strategy which satisfies:

- In the first condition, replace "at least one" by "**exactly one**".
- There is a root (an edge from which all other edges can be reached following (oriented) paths of the strategy).

PROLOGUE 10/13

Lemma 1. (A, B) is winning iff there is proof rooted in (A, B).

Lemma 2. $A_1 \wedge A_2$ is a greatest lowert bound of A_1 and A_2, etc... .

Lemma 3. $\leq$ is transitive.

PROLOGUE 11/13

(1) Easy (induction on formulas)

(2) Use the presentation by proofs

(3) Use the presentation by strategies. The composition of two strategies S and T witnessing $A \leq B$ and $B \leq C$ is:

$$S \circ T = \{(x, z) \mid \exists\, y \ (x, y) \in S \text{ et } (y, z) \in T\}\ .$$

PROLOGUE 12/13

This example embodies ideas of using games for **both**

- model-checking (we are interested in the mere existence of strategies for inequality predicates) **and**
- game semantics: we want a compositional semantics: combine strategies to build other strategies.

PROLOGUE 13/13

The situation proofs / strategies somehow matches the operational / denotational distinction in the semantics of programming languages: Proofs compose by normalization / cut-elimination / interaction, while strategies compose as mathematical functions. (Cf. also functions as relations vs functions as algorithms).

AUTOMATA, LOGICS . . .

Büchi (1962): Two-way correspondence between automata on infinite words and monadic second order logic over infinite words α:

$$\forall \alpha \ (\alpha \models \phi \Leftrightarrow \mathcal{A} \text{ accepts } \alpha)$$

This logic is decidable.

and GAMES

second order monadic		parity
logic	$\leftrightarrow$	automata
		$\updownarrow$
		parity
		games

McNaughton, Rabin, Gurevitch-Harrington, Zielonka, Thomas,. . .

Determinacy

Parity games are determined, and who wins is **decidable**.

A nice proof of Santocanale goes along the hypothenuse of the above triangle (but the target is a logic of fixed points).

Parity automata and fixpoints 1/8

A (partial) game is

- an oriented graph $G = (G_0, G_1)$

- the nodes have a polarity ($\epsilon : G_0 \to \{P, O, N\}$, if $\epsilon(x) = N$, then x is terminal)

If $\epsilon^{-1}(N) = \emptyset$, the game is called total.

Parity automata and fixpoints 2/8

One also gives a set W_P of infinite winning paths for P (W_O is its complement).

Winning strategy for P (resp. O) = strategy all of whose infinite paths $\in W_P$ (resp. $\in W_O$). Winning position = belongs to a winning strategy.

Parity automata and fixpoints 3/8

Given $X \subseteq \epsilon^{-1}(N)$, given $S(x) \subseteq G_0$ and $OP^x \in \{\wedge, \vee\}$ for all $x \in X$, define the games

$\mu_S.G[X]$ (short for $\mu_{S,OP}.G[X]$) , $\nu_S.G[X]$:

- add $x \rightarrow g$ for all $x \in X$, $g \in S(x)$,
- change polarity of $x \in X$ to P (resp. O) if $OP^x = \vee$ (resp. $OP^x = \wedge$).

Parity automata and fixpoints 4/8

The two games differ only in the definition of winning:

- $\mu_S.G[X]$: the winning paths of P are those infinite paths in the new graph which eventually are winning for P in the old.
- $\nu_S.G[X]$: (dual) the ... of O in the new graph which eventually ... for O in the old.

Parity automata and fixpoints 5/8

$G[X \cap A]$ defined by changing the polarity of $x \in X$ to P (resp. O) if $x \notin A$ (resp. $x \in A$).

Lemma 1. If all games $G[X \cap A]$ are determined, then $\mu_S.G[X]$ (resp. $\nu_S.G[X]$) is determined and its set of winning positions is obtained as a least (resp. greatest) fixed point of a monotonous operator.

Parity automata and fixpoints 6/8

A parity game is a (total) game in which the nodes also have a colour ($p : G_0 \to \{1, \ldots, n\}$) and the colours have a parity ($\chi : \{1, \ldots, n\} \to \{P,O\}$).

W_P consists of those paths such that if m is the maximum colour visited infinitely often along the path, then $\chi(m)$=P.

Parity automata and fixpoints 7/8

Lemma 2. Each parity game G can be written as $Q_{S_n}.\cdots.Q_{S_1}.G_0[X_1]\cdots[X_n]$ where

- X_i is the set of nodes of colour i,
- $S_i(x)$ is the set of successors of x in G,
- $OP^x = \vee$ (resp. $OP^x = \wedge$) if x has polarity P (resp. O),
- $Q_{S_i} = \mu$ (resp. $Q_{S_i} = \nu$) if $\chi(i) =$P (resp. $\chi(i) =$O).

Parity automata and fixpoints 8/8

Determinacy of parity games follows from Lemmas 2 and 1.

Proof of lemmas 1 and 2 (hints) 1/3

$\mathit{WP}_P[G] =_{def} \{g \in G_0 \mid \exists$ a winning strategy for P containing $g\}$

Lemma A: $\mathit{WP}_P[G] \cap \mathit{WP}_O[G] = \emptyset$.

Lemma B: A path γ that visits X infinitely often is winning in $\mu_S.G[X]$.

Lemma C: A path that is eventually winning in $G[X]$ is winning in $\nu_S.G[X]$.

Proof of lemmas 1 and 2 (hints) 2/3

$$F_P(A) =_{def} \{g \in G_0 \mid (\epsilon g = P \Rightarrow \exists\, g' \ (g \to g' \text{ and } g' \in A)) \text{ and } (\epsilon g = O \Rightarrow \forall\, g' \ (g \to g' \Rightarrow g' \in A)$$

When a play reaches $F_P(A)$, P can force the play to go into A.

The operator of Lemma 1 is

$A \mapsto \mathit{WP}_P[G[X \cap F_P(A)]]$.

Proof of lemmas 1 and 2 (hints) 3/3

A glimpse of the proof of Lemma 1. If Z is a postfixpoint, i.e., $Z \subseteq \mathit{WP}_P[G[X \cap F_P(Z)]]$, then construct the following strategy: play according to $G[X \cap F_P(Z)]$, until eventually reaching $X \cap F_P(Z)$, then force the play to come to Z, and continue to play according to $G[X \cap F_P(Z)]$, etc...

GAME SEMANTICS 1/2

The goal is to make semantics akin to syntax and to model computation as interaction between

$$\left.\begin{array}{r}\text{a system}\\ \text{a program}\\ \text{P}\end{array}\right\} \text{ and } \left\{\begin{array}{l}\text{its environment}\\ \text{its context}\\ \text{O}\end{array}\right.$$

GAME SEMANTICS 2/2

while keeping a suitable level of mathematical abstraction (categories), and hence the possibility to use powerful reasoning tools.

Abramsky-Jagadeesan-Malacaria, Hyland-Ong (1993)

PRECURSORS

- Dialogue games of Lorenzen, Lorenz, Felscher (1960)
- Sequential algorithms of Berry and Curien (1978) (like M. Jourdain, we did not know that we were talking about games and strategies!)
- Object spaces model of Reddy (1996)