An approach to

innocent strategies as graphs

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Denotational semantics

\[ \Downarrow \]

\[ \approx \ (\text{game semantics} \ / \ \text{ludics}) \]

\[ \Uparrow \]

Syntax

“the best of both worlds”
proofs $\rightarrow$ (tree) strategies

↓

proof nets $\rightarrow$ graph strategies
Ludics 1/6

ludics = focalized, untyped version of MALL sequent calculus.

(polarized, focalized) MALL sequent calculus proofs → designs

(MALL = Multiplicative-Additive Linear Logic)
\[
\begin{align*}
\Gamma \vdash P_1, \Gamma_1 \quad & \quad \Gamma \vdash P_2, \Gamma_1 \\
\Gamma \vdash P_1 \& P_2, \Gamma_1 \quad & \quad \Gamma \vdash N, \Gamma_2 \\
\Gamma \vdash (P_1 \& P_2) \otimes N, \Gamma_1, \Gamma_2
\end{align*}
\]
Ludics 3/6

- Formulas organized by alternating clusters of positive ($\otimes$, $\oplus$) (resp. negative ($\otimes$, $\&$)) formulas
- Each cluster becomes an address (cf. type $\text{bool}_2 \to \text{bool}_1 \to \text{bool}_\varepsilon$ in game semantics)
• Logical rules expressed in terms of actions $= (\xi, I)$ ($I$ finite set of relative immediate subaddresses of $\xi$)
  - We say that $(\xi, I)$ generates $\xi_i$ ($i \in I$)
  - A negative rule involving $\&$ gives rise to actions $(\xi, I_1), \ldots, (\xi, I_n)$ on the same address
Looks exotic? Just (sorts of) Böhm trees:

- negative $\xi \rightarrow \lambda x_1 \ldots x_i \ldots x_n.P$
- positive $\xi i \rightarrow x_i M_1 \ldots M_p$ where $x_i$ is bound higher up at a negative node of address $\xi$
Full syntax for Girard’s designs (Curien 2001):

\[ M ::= \{ J = \lambda \{ x_j : j \in J \} . P_J : J \in \mathcal{P}_f(\omega) \} \]

\[ P ::= (x \cdot I)\{ M_i : i \in I \} | \Omega | \Xi \]
An *L-net* ([Faggian/Maurel]) $\mathcal{D}$ is given by:

- An interface $\vdash \Lambda$ (positive) or $\xi \vdash \Lambda$ (negative).
- A set $A$ of nodes (or events) which are *labelled by polarized actions* (notation $k = (\xi, I)$)
- A structure on $A$ of *directed acyclic bipartite graph* (if $k \leftarrow k'$, the two nodes have opposite polarity) which satisfies (for all $k$):
L-nets 2/6

• **Views.** All the addresses used in $k^\downarrow = \{k', k' \leftarrow k\}$ are distinct.

• **Parents.** If $k = (\sigma, I)$, then either $\sigma \in \text{interface}$ (with same polarity), or it has been generated by (the action of) a $c \leftarrow k$ (of opposite polarity). Moreover, if $k$ is negative, and $b \leftarrow k$, then $b = c$ (*innocence!*)

• **Positivity.** $(k \text{ maximal w.r.t. } \leftarrow) \Rightarrow (k \text{ positive})$
L-nets 3/6

- **Sibling.** Two nodes in an additive pair have distinct labels (in the example above, \(\{1\} \neq \{2\}\)).

- **Additives.** If \(k_1 = (\xi, K_1)\), \(k_2 = (\xi, K_2)\), \(\exists w_1, w_2\) in the same additive rule such that \(w_1 \leftarrow k_1\), and \(w_2 \leftrightarrow k_2\).

(“two events on the same address are in conflict”)

(So far = L-nets, one more condition for \(L_s\)-nets)
L-nets 4/6

**Fact.** For each pair of distinct nodes $k, k'$ of an L-net $\mathcal{D}$, the sets of actions of $k\downarrow$ and $k'\downarrow$ are different.

→ L-nets as sets of (positive) views (≡ L-nets with a maximal element, and whose nodes are actions). Very useful for *superpositions as mere unions*.

(cf. event structures presented as configuration structures)
L-nets 5/6

A *switching edge* of a negative rule $R$ has its target is in $R$.
A *switching path* uses at most one switching edge for each negative rule.

- **Cycles.** For all non-empty union $C$ of switching cycles, there is an additive rule $W$ not intersecting $C$, and a pair $w_1, w_2 \in W$ such that for some nodes $c_1, c_2 \in C$, $w_1 \leftrightarrow c_1$, and $w_2 \leftrightarrow c_2$. 
L-nets 6/6

The condition *Cycles* is an anologue of Hughes and Van Glabbeek’s *toggling* condition.

It is the key to sequentialization:

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every \( L_s \)-net has a splitting conclusion
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A gradient of sequentiality 1/5

- **L-forests.** Maximally sequential L-nets are forests (Girard’s designs with \textit{mix}).

- **parallel L-nets.** Minimally sequential L-nets = our notion of multiplicative-additive (untyped, focalized) proof-nets
A gradient of sequentiality 2/5

Algebraic presentation of parallel L-nets:

\[ \mathcal{D} := \mathcal{D}^+ | \mathcal{D}^- \]
\[ \mathcal{D}^+ := \biguplus \mathcal{E}^+ \]
\[ \mathcal{E}^+ := k^+ \mid \bigcup (\xi, I)^+ \circ \mathcal{D}^\xi_i \]
\[ \mathcal{D}^- := \bigcup_{add} (\sigma, J)^- \circ \mathcal{D}^+ \]
A gradient of sequentiality 3/5

- **Rooting.** $x \circ \mathcal{D}^+$: the node $x$ is added, and only edges enforced by condition $Parents$ are added.
- **Boxing.** $x \cdot \mathcal{D}^+$: the node $x$ is added below all the conclusions of $\mathcal{D}$.
- **Additive union.** $\bigcup_{\text{add}} \mathcal{D}_I$: selective union (only the views which are common to all $\mathcal{D}_I$’s are shared)

(and associated destructors)
A gradient of sequentiality 4/5

Algebraic presentation of L-forests:

\[
\begin{align*}
\mathcal{D} & := \mathcal{D}^+ \mid \mathcal{D}^- \\
\mathcal{D}^+ & := \cup \mathcal{E}^+ \\
\mathcal{E}^+ & := k^+ \mid \cup (\xi, I)^+ \circ \mathcal{D}_{\xi_i}^- \\
\mathcal{D}^- & := \cup (\sigma, J)^- . \mathcal{D}^+ 
\end{align*}
\]
• Every $L_S$-net can be (non-deterministically) **sequentialized** to an $L$-forest.
• Every $L$-forest (more generally, every $L_S$-net) with leaves decorated by sets of actions ("axioms") can be **desequentialized** to a parallel $L$-net.

The two procedures can be applied so as to be **inverse** to each other.
Further work

- Characterization of minimal sequentiality and of the induced equational theory on L-forests
- Sequentialization/desequentialization à la carte: Di Giambernardino-Faggian (multiplicative)
- What kind of proof nets do we get when restoring types?
A wider picture

Aim: to link proof theory, game semantics, and concurrency theory.

L-nets $\rightarrow$ (typed) event structures

L-net normalisation (Faggian-Maurel) $\rightarrow$ parallel composition (+ synchronization) of (typed) event structures (Faggian-Piccolo) (cf. Varacca-Yoshida)

Operations on $L$-nets $\leftrightarrow$ operations on event structures.