The duality of computation under focus

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TYPES 2010, Warszawa, October 2010

What the talk is about

- Part I :
  - We define a term language for **a** presentation of classical (propositional) sequent calculus (with connectives $\wedge$, $\vee$, and $\neg$), based on Curien-Herbelin's $\mu\tilde{\mu}$-kit.
  - We discuss focalisation, which allows us to recover confluence. This leads us to **focalising system** L.
- Part II : We sketch the role of focalising system L as an intermediate language (close to abstract machines).
- Part III : We want to abstract over the order of left decompositions, and to account for "maximal" focalisation. This leads us to **synthetic system** L, a syntax closed to (Terui's computational) ludics.

PART I

(positive fragment of)

Focalising system L

(focalising system L was introduced by the second author, CSL 2009)

Classical sequents

$$A ::= X \parallel A \wedge A \parallel A \vee A \parallel \neg A$$

A (bilateral) sequent is a pair of two (finite) multi-sets of formulas, written

$$\Gamma \vdash \Delta$$

A presentation of classical sequent calculus LK

Axiom and cut :

$$\frac{}{\Gamma, A \vdash A, \Delta} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}$$

Right introduction rules :

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \qquad \frac{\Gamma \vdash A_1, \Delta \qquad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \wedge A_2, \Delta}$$

$$\frac{\Gamma \vdash A_1, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta} \qquad \frac{\Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta}$$

Left introduction rules

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad \frac{\Gamma, A_1 \vdash \Delta \qquad \Gamma, A_2 \vdash \Delta}{\Gamma, A_1 \vee A_2 \vdash \Delta}$$

We say that A, A_1, A_2, $\neg A$, $A_1 \wedge A_2$, $A_1 \vee A_2$ are the *active* formulas of the rules.

Implication as a derived connective

Set $A \Rightarrow B = \neg(A \wedge \neg B)$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}$$

$$\cfrac{\cfrac{\cfrac{\Gamma, A \vdash B, \Delta}{\Gamma, A, \neg B \vdash \Delta}}{\Gamma, A \wedge \neg B \vdash \Delta}}{\Gamma \vdash \neg(A \wedge \neg B), \Delta} \qquad \cfrac{\cfrac{\cfrac{}{\Gamma \vdash A, \Delta} \quad \cfrac{\Gamma, B \vdash \Delta}{\Gamma \vdash \neg B, \Delta}}{\Gamma \vdash A \wedge \neg B, \Delta}}{\Gamma, \neg(A \wedge \neg B) \vdash \Delta}$$

This encoding allows us to embed / decompose call-by-value $\lambda\mu$-calculus in our languages.

Why sequent calculus

Sequents allow formula decomposition in proof search.

The symmetry of left and right in sequent calculus allows us to account directly for *abstract machines* : *expressions* are evaluated in some *context*.

Weakening and Contraction

Weakening / Contraction

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}$$

In our presentation of LK :

- Weakening is admissible : add the weakened formulas everywhere in the sequents of the proof. In fact, our terms do not distinguish a proof of $\Gamma, A \vdash \Delta$ where A is never active from the proof of $\Gamma \vdash \Delta$ of which the former is a weakening. We say that weakening is transparent.
- Contraction is derivable :

$$\frac{\Gamma \vdash A, A, \Delta \qquad \overline{\Gamma, A \vdash A, \Delta}}{\Gamma \vdash A, \Delta}$$

Hence we call our rule the cut/contraction

Additive versus multiplicative

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}$$

is “derivable" (if weakening is viewed as transparent)

Note that multiplicative cut is just cut (and interestingly, only the cut is multiplicative in Gentzen’s original presentation)

Reversible versus irreversible

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 \vee A_2, A_2, \Delta}}{\Gamma \vdash A_1 \vee A_2, A_1 \vee A_2, \Delta}}{\Gamma \vdash A_1 \vee A_2, \Delta} \qquad \dfrac{\dfrac{\Gamma, A_1 \vdash \Delta}{\Gamma, A_1, A_2 \vdash \Delta}}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad \dfrac{\dfrac{\Gamma, A_2 \vdash \Delta}{\Gamma, A_1, A_2 \vdash \Delta}}{\Gamma, A_1 \wedge A_2 \vdash \Delta}$$

We have chosen an irreversible disjunction on the right and a reversible conjunction on the left, as an *anticipation of focalisation*

Various presentations of LK

1) Pushing weakening in the axiom makes weakening transparent, whatever style is used for all other rules. Assuming such transparent weakening, we have :

2) The cut/contraction rule is equivalent to the multiplicative cut rule + the contraction rules

3) then we have choices as to the reversibility or irreversibility for $\vee$ on the right and for $\wedge$ on the left :

1. Symmetric, both reversible : friendly for completeness
2. Symmetric, both irreversible : Gentzen's original choice (see Urban's thesis, Wadler's dual calculus for term languages)
3. Dissymmetric. There are dual choices. The one presented here ($\vee$ irreversible on the right and $\wedge$ reversible on the left) is friendly to the *call-by-value* encoding of implication $A \Rightarrow B = \neg(A \wedge \neg B)$. It is our guide in this talk.
4. Dissymmetric. The dual choice is friendly to the call-by-name encoding of implication $A \Rightarrow B = (\neg A) \vee B$

Cut elimination : logical cuts

$$\dfrac{\dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}}{\Gamma \vdash \Delta} \quad \longrightarrow \quad \dfrac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}$$

$$\dfrac{\dfrac{\Gamma, A_1 \vdash \Delta \quad \Gamma, A_1 \vdash \Delta}{\Gamma, A_1 \vee A_2 \vdash \Delta} \qquad \dfrac{\Gamma \vdash A_1, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta}}{\Gamma \vdash \Delta} \quad \longrightarrow \quad \dfrac{\Gamma, A_1 \vdash \Delta \qquad \Gamma \vdash A_1, \Delta}{\Gamma \vdash \Delta}$$

$$\dfrac{\dfrac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad \dfrac{\Gamma \vdash A_1, \Delta \quad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \wedge A_2, \Delta}}{\Gamma \vdash \Delta} \quad \longrightarrow \quad \dfrac{\dfrac{\Gamma, A_1, A_2 \vdash \Delta \quad \Gamma, A_2 \vdash A_1, \Delta}{\Gamma, A_2 \vdash \Delta} \qquad \Gamma \vdash A_2, \Delta}{\Gamma \vdash \Delta}$$

Cut elimination : commutative cuts

$$\dfrac{\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \vdash \neg B, \Delta} \qquad \Gamma \vdash A, \neg B, \Delta}{\Gamma \vdash \neg B, \Delta} \quad \longrightarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, A, B \vdash \neg B, \Delta \qquad \Gamma, B \vdash A, \neg B, \Delta}{\Gamma, B \vdash \neg B, \Delta}}{\Gamma \vdash \neg B, \neg B, \Delta}}{\Gamma \vdash \neg B, \Delta}$$

Erasing :

$$\dfrac{\dfrac{}{\Gamma, A, B \vdash B, \Delta} \qquad \Gamma, B \vdash A, B, \Delta}{\Gamma, B \vdash B, \Delta} \quad \longrightarrow \quad \dfrac{}{\Gamma, B \vdash B, \Delta}$$

Duplication :

$$\dfrac{\dfrac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta} \quad \longrightarrow \quad \dfrac{\dfrac{\Gamma, A, A \vdash \Delta \qquad \Gamma, A \vdash A, \Delta}{\Gamma, A \vdash, \Delta} \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}$$

Curien-Herbelin's syntactic kit (ICFP 2000)

Expressions	Contexts	Commands
$\Gamma \vdash v : A \mid \Delta$	$\Gamma \mid e : A \vdash \Delta$	$c : (\Gamma \vdash \Delta)$

where Γ is a set of pairs $x : N$ and Δ is a set of pairs $\alpha : P$ (ordinary variables, continuation variables)

$$\frac{\dfrac{c_2 : (\Gamma, x : A \vdash \Delta)}{\Gamma \mid \tilde{\mu}x.c_2 : A \vdash \Delta} \qquad \dfrac{c_1 : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha.c_1 : A \mid \Delta}}{\langle \mu\alpha.c_1 \mid \tilde{\mu}x.c_2 : A \rangle : (\Gamma \vdash \Delta)}$$

$$\begin{array}{l} v ::= x \parallel \mu\alpha.c \parallel \ldots \\ e ::= \alpha \parallel \tilde{\mu}x.c \parallel \ldots \\ c ::= \langle v \mid e \rangle \end{array}$$

The variable x is bound in $\tilde{\mu}x.c$ (likewise for $\mu\alpha.c$)

We use the collective name of "system L" for syntaxes based on this kit

All proofs are equal...

Operational semantics (first try) :

$$\langle \mu\alpha.c \mid e\rangle \longrightarrow c[e/\alpha] \qquad\qquad \langle v \mid \tilde{\mu}x.c\rangle \longrightarrow c[v/x]$$

Lafont's critical pair (if α is not free in c_1 and x is not free in c_2) :

$$c_1 = c_1[\tilde{\mu}x.c_2 : A/\alpha] \longleftarrow \langle \mu\alpha.c_1 \mid \tilde{\mu}x.c_2 : A\rangle \longrightarrow c_2[\mu\alpha.c_1/x] = c_2$$

To recover confluence (and consistency at the level of proofs), we shall need to reinforce the dissymmetry from the level of proof rules to the level of *computation* rules.

But first, let us define a (non confluent) term language for (this presentation of) LK.

A faithful (uninspiring) proof language for LK 1/2

Commands	$c ::= \langle x \mid \alpha \rangle \parallel \langle v \mid \alpha \rangle \parallel \langle x \mid e \rangle \parallel \langle \mu\alpha.c \mid \tilde{\mu}x.c \rangle$
Expressions	$v ::= (\tilde{\mu}x.c)^\bullet \parallel (\mu\alpha.c, \mu\alpha.c) \parallel inl(\mu\alpha.c) \parallel inr(\mu\alpha.c)$
Contexts	$e ::= \tilde{\mu}\alpha^\bullet.c \parallel \tilde{\mu}(x_1, x_2).c \parallel \tilde{\mu}[inl(x_1).c_1 \mid inr(x_2).c_2]$

(In $\langle v \mid \alpha \rangle$ (resp. $\langle x \mid e \rangle$), we suppose α (resp. x) fresh for v (resp. e).)

$$\frac{}{\langle x \mid \alpha \rangle : (\Gamma, x : A \vdash \alpha : A, \Delta)} \qquad \frac{c : (\Gamma \vdash \alpha : A, \Delta) \qquad d : (\Gamma, x : A \vdash \Delta)}{\langle \mu\alpha.c \mid \tilde{\mu}x.d \rangle : (\Gamma \vdash \Delta)}$$

$$\frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \vdash (\tilde{\mu}x.c)^\bullet : \neg A \mid \Delta} \qquad \frac{c_1 : (\Gamma \vdash \alpha_1 : A_1, \Delta) \qquad c_2 : (\Gamma \vdash \alpha_2 : A_2, \Delta)}{\Gamma \vdash (\mu\alpha_1.c_1, \mu\alpha_2.c_2) : A_1 \wedge A_2 \mid \Delta} \qquad \frac{c_1 : (\Gamma \vdash \alpha_1 : A_1, \Delta)}{\Gamma \vdash inl(\mu\alpha_1.c_1) : A_1 \vee A_2 \mid \Delta}$$

$$\frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \mid \tilde{\mu}\alpha^\bullet.c : \neg A \vdash \Delta} \qquad \frac{c : (\Gamma, x_1 : A_1, x_2 : A_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x_1, x_2).c : A_1 \wedge A_2 \vdash \Delta} \qquad \frac{c_1 : (\Gamma, x_1 : A_1 \vdash \Delta) \qquad c_2 : (\Gamma, x_2 : A_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}[inl(x_1).c_1 \mid inr(x_2).c_2] : A_1 \vee A_2 \vdash \Delta}$$

Deactivation :

$$\frac{\Gamma \vdash v : A \mid \Delta}{\langle v \mid \alpha \rangle : (\Gamma \vdash \alpha : A, \Delta)} \qquad \frac{\Gamma \mid e : A \vdash \Delta}{\langle x \mid e \rangle : (\Gamma, x : A \vdash \Delta)}$$

A faithful (uninspiring) proof language for LK 2/2

Logical rules (redexes of the form $\langle \mu\alpha.\langle v \mid \alpha\rangle \mid \tilde{\mu}x.\langle x \mid e\rangle\rangle$) :

$\langle \mu\alpha.\langle (\tilde{\mu}x.c)^\bullet \mid \alpha\rangle \mid \tilde{\mu}y.\langle y \mid \tilde{\mu}\alpha^\bullet.d\rangle\rangle \longrightarrow \langle \mu\alpha.d \mid \tilde{\mu}x.c\rangle$ (similar rules for conjunction and disjunction)

Commutative rules (going "up left", redexes of the form $\langle \mu\alpha.\langle v \mid \beta\rangle \mid \tilde{\mu}x.c\rangle$) :

$\langle \mu\alpha.\langle (\tilde{\mu}y.c)^\bullet \mid \beta\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle \mu\beta'.\langle (\tilde{\mu}y.\langle \mu\alpha.c \mid \tilde{\mu}x.d\rangle)^\bullet \mid \beta'\rangle \mid \tilde{\mu}y.\langle y \mid \beta\rangle\rangle$ ($\neg$ right)
(similar rules of commutation with the other right introduction rules and with the left introduction rules)
$\langle \mu\alpha.\langle \mu\beta.\langle y \mid \beta\rangle \mid \tilde{\mu}y'.c\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle \mu\beta.\langle y \mid \beta\rangle \mid \tilde{\mu}y'.\langle \mu\alpha.c \mid \tilde{\mu}x.d\rangle\rangle$ (contraction right)
$\langle \mu\alpha.\langle \mu\beta'.c \mid \tilde{\mu}y.\langle y \mid \beta\rangle\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle \mu\beta'.\langle \mu\alpha.c \mid \tilde{\mu}x.d\rangle \mid \tilde{\mu}y.\langle y \mid \beta\rangle\rangle$ (contraction left)
$\langle \mu\alpha.\langle \mu\alpha'.c \mid \tilde{\mu}x'.\langle x' \mid \alpha\rangle\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle \mu\alpha.\langle \mu\alpha'.c \mid \tilde{\mu}x.d\rangle \mid \tilde{\mu}x.d\rangle$ (duplication)
$\langle \mu\alpha.\langle y \mid \beta\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle y \mid \beta\rangle$ (erasing)

Commutative rules (going "up right", redexes of the form $\langle \mu\alpha.c \mid \tilde{\mu}x.\langle y \mid e\rangle\rangle$) : similar rules.

A simple twist makes it more inspiring !

Making activations $\mu\alpha$ and $\tilde{\mu}x$ “first class” (system L !)

Commands $c ::= \langle v \mid e\rangle \parallel c[\sigma]$
Expressions $v ::= x \parallel \mu\alpha.c \parallel e^\bullet \parallel (v,v) \parallel inl(v) \parallel inr(v) \parallel v[\sigma]$
Contexts $e ::= \alpha \parallel \tilde{\mu}x.c \parallel \tilde{\mu}\alpha^\bullet.c \parallel \tilde{\mu}(x_1,x_2).c \parallel \tilde{\mu}[inl(x_1).c_1 | inr(x_2).c_2] \parallel e[\sigma]$

where σ is a list $v_1/x_1,\ldots,v_m/x_m,e_1/\alpha_1,\ldots,e_n/\alpha_n$

$$\frac{}{\Gamma\,,\,x:A\vdash x:A\,|\,\Delta} \qquad \frac{}{\Gamma\,|\,\alpha:A\vdash\alpha:A\,,\,\Delta} \qquad \frac{\Gamma\vdash v:A\,|\,\Delta \qquad \Gamma\,|\,e:A\vdash\Delta}{\langle v\mid e\rangle:(\Gamma\vdash\Delta)}$$

$$\frac{c:(\Gamma\,,\,x:A\vdash\Delta)}{\Gamma\,|\,\tilde{\mu}x.c:A\vdash\Delta} \qquad \frac{c:(\Gamma\vdash\alpha:A\,,\,\Delta)}{\Gamma\vdash\mu\alpha.c:A\,|\,\Delta}$$

$$\frac{\Gamma\,|\,e:A\vdash\Delta}{\Gamma\vdash e^\bullet:\neg A\,|\,\Delta} \qquad \frac{\Gamma\vdash v_1:A_1\,|\,\Delta \qquad \Gamma\vdash v_2:A_2\,|\,\Delta}{\Gamma\vdash(v_1,v_2):A_1\wedge A_2\,|\,\Delta} \qquad \frac{\Gamma\vdash v_1:A_1\,|\,\Delta}{\Gamma\vdash inl(v_1):A_1\vee A_2\,|\,\Delta}$$

$$\frac{c:(\Gamma,x_1:A_1,\ldots,x_m:A_m\vdash\alpha_1:B_1,\ldots,\alpha_n:B_n)\ \ldots\ \Gamma\vdash v_i:A_i\,|\,\Delta\ \ldots\ \Gamma\,|\,e_j:B_j\vdash\Delta\ \ldots}{c[v_1/x_1,\ldots,v_m/x_m,e_1/\alpha_1,\ldots,e_n/\alpha_n]:(\Gamma\vdash\Delta)}$$

(idem $v[\sigma], e[\sigma]$)

(rules unchanged for the $\tilde{\mu}$’s)

Commutative cuts as explicit substitutions

Logical cut-elimination rules as pattern-matching

(control)	$\langle \mu\alpha.c \mid e \rangle \longrightarrow c[e/\alpha]$
	$\langle v \mid \tilde{\mu}x.c \rangle \longrightarrow c[v/x]$
(logical)	$\langle e^{\bullet} \mid \tilde{\mu}\alpha^{\bullet}.c \rangle \longrightarrow c[e/\alpha]$
	$\langle (v_1, v_2) \mid \tilde{\mu}(x_1, x_2).c \rangle \longrightarrow c[v_1/x_1, v_2/x_2]$
	$\langle inl(v_1) \mid \tilde{\mu}[inl(x_1).c_1 \mid inr(x_2).c_2] \rangle \longrightarrow c_1[v_1/x_1]$
(commutation)	$\langle v \mid e \rangle[\sigma] \longrightarrow \langle v[\sigma] \mid e[\sigma] \rangle$
	$x[\sigma] \longrightarrow x$ (x not declared in σ)
	$x[v/x, \sigma] \longrightarrow v$ (idem $\alpha[\sigma]$)
	$(\mu\alpha.c)[\sigma] \longrightarrow \mu\alpha.(c[\sigma])$ (capture avoiding)
	⋮

Remarks

The second system looks nicer, but what is the relation with the faithful one, i.e. with LK ?

There are now more terms than proofs (e.g. activation/deactivation stutterings), but both systems are related by a retraction.

- The faithful language is a sublanguage of the system L one (cf. Clément's Houtmann's talk)
- By erasing decorations in sequents, one always gets an underlying proof of LK.

  Now, we are ready to address confluence.

Focalisation

A focalised proof search alternates between right and left phases, as follows :

- *Left phase* : Decompose (copies of) formulas on the left, in any order. Every decomposition of a negation on the left feeds the right part of the sequent. At any moment, one can change the phase from left to right.

- *Right phase* : Choose a formula A on the right, and *hereditarily* decompose a copy of it in all branches of the proof search. This *focusing* in any branch can only end with an axiom (which ends the proof search in that branch), or with a decomposition of a negation, which prompts a phase change back to the left. Etc. . .

Polarisation

To account for right focalisation, we introduce a fourth kind of judgement : the *values*, typed as $(\Gamma \vdash V : A \,;\, \Delta)$

Focusing (Andreoli) $\leftrightarrow$ **Stoup (Girard)** $\leftrightarrow$ **Values (Plotkin)**

We also make official the existence of two disjunctions (since the behaviours of the conjunction on the left and of the disjunction on the right are different) and two conjunctions, by renaming $\wedge, \vee, \neg$ as $\otimes, \oplus, \neg^{+}$, respectively (**positive** formulas) :

$$P ::= X \parallel P \otimes P \parallel P \oplus P \parallel \neg^{+} P$$

We can define their De Morgan duals (**negative** formulas) :

$$N ::= \overline{X} \parallel N \parr N \parallel N \& N \parallel \neg^{-} N$$

They restore the duality of connectives (think of P on the left as being a $\overline{P}$ in a unilateral sequent $\vdash \overline{\Gamma}, \Delta$).

Syntax of focalising system L

Commands	$c ::= \langle v \mid e \rangle \parallel c[\sigma]$
Expressions	$v ::= V^{\Diamond} \parallel \mu\alpha.c \parallel v[\sigma]$
Values	$V ::= x \parallel (V, V) \parallel inl(V) \parallel inr(V) \parallel e^{\bullet} \parallel V[\sigma]$
Contexts	$e ::= \alpha \parallel \tilde{\mu}x.c \parallel e[\sigma] \parallel$
	$\tilde{\mu}\alpha^{\bullet}.c \parallel \tilde{\mu}(x_1, x_2).c \parallel \tilde{\mu}[inl(x_1).c_1 \vert inr(x_2).c_2]$

(control) $\langle \mu\alpha.c \mid e \rangle \longrightarrow c[e/\alpha]$

$\langle V^{\Diamond} \mid \tilde{\mu}x.c \rangle \longrightarrow c[V/x]$

(logical) $\langle (e^{\bullet})^{\Diamond} \mid \tilde{\mu}\alpha^{\bullet}.c \rangle \longrightarrow c[e/\alpha]$

$\langle (V_1, V_2)^{\Diamond} \mid \tilde{\mu}(x_1, x_2).c \rangle \longrightarrow c[V_1/x_1, V_2/x_2]$

$\langle inl(V_1)^{\Diamond} \mid \tilde{\mu}[inl(x_1).c_1 \vert inr(x_2).c_2] \rangle \longrightarrow c_1[V_1/x_1]$

(commutation) $\langle v \mid e \rangle[\sigma] \longrightarrow \langle v[\sigma] \mid e[\sigma] \rangle$ etc...

System LKQ

$$\frac{}{\Gamma\,,\ x : P \vdash x : P\ ;\ \Delta} \qquad \frac{}{\Gamma \,|\, \alpha : P \vdash \alpha : P\ ,\ \Delta} \qquad \frac{\Gamma \vdash v : P \,|\, \Delta \qquad \Gamma \,|\, e : P \vdash \Delta}{\langle v \,|\, e \rangle : (\Gamma \vdash \Delta)}$$

$$\frac{c : (\Gamma\,,\ x : P \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}x.c : P \vdash \Delta} \qquad \frac{c : (\Gamma \vdash \alpha : P\ ,\ \Delta)}{\Gamma \vdash \mu\alpha.c : P \,|\, \Delta} \qquad \frac{\Gamma \vdash V : P\ ;\ \Delta}{\Gamma \vdash V^{\diamond} : P \,|\, \Delta}$$

$$\frac{\Gamma \,|\, e : P \vdash \Delta}{\Gamma \vdash e^{\bullet} : \neg^{+}P\ ;\ \Delta} \qquad \frac{\Gamma \vdash V_1 : P_1\ ;\ \Delta \qquad \Gamma \vdash V_2 : P_2\ ;\ \Delta}{\Gamma \vdash (V_1, V_2) : P_1 \otimes P_2\ ;\ \Delta} \qquad \frac{\Gamma \vdash V_1 : P_1\ ;\ \Delta}{\Gamma \vdash inl(V_1) : P_1 \oplus P_2\ ;\ \Delta}$$

$$\frac{c : (\Gamma \vdash \alpha : P, \Delta)}{\Gamma \,|\, \tilde{\mu}\alpha^{\bullet}.c : \neg^{+}P \vdash \Delta} \qquad \frac{c : (\Gamma, x_1 : P_1, x_2 : P_2 \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}(x_1, x_2).c : P_1 \otimes P_2 \vdash \Delta} \qquad \frac{c_1 : (\Gamma, x_1 : P_1 \vdash \Delta) \qquad c_2 : (\Gamma, x_2 : P_2 \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}[inl(x_1).c_1 | inr(x_2).c_2] : P_1 \oplus P_2 \vdash \Delta}$$

$$\frac{\ldots \quad \Gamma \vdash V : P\ ;\ \Delta\ \ \ldots\ \Gamma \,|\, e : Q \vdash \Delta\ \ \ldots\ c : (\Gamma\ \ldots, q : P, \ldots \vdash \Delta, \ldots, \alpha : Q, \ldots)}{c[\ldots, V/q, \ldots, e/\alpha] : (\Gamma \vdash \Delta)}$$

(idem $v[\sigma], V[\sigma], e[\sigma]$)

Completeness of LKQ

LKQ and LK prove the same sequents

Cut-elimination in LK is non-confluent (inconsistency at the proof level)

Cut-elimination in LKQ is confluent (makes it an appropriate vehicle for Curry-Howard)

Remarks

Focalisation arose in the context of **proof search** (logic programming), not of **proof transformation** (functional programming).

But the two domains are related :

- Having fewer proofs around can help to make them behave better.
- More fundamentally, proof-search can be understood / guided by interactions with counterproofs (Saurin).

PART III

Synthetic system L

Motivations : two related goals 1/2

First, we want to account for full (or strong) focalisation : carrying the phases maximally, all the way up to the atoms on the left, up to atomic axioms on the right. This is of interest in a proof search perspective, since the stronger discipline further reduces the search space.

Motivations : two related goals 2/2

Second, we would like our syntax to quotient proofs over the order of decomposition of negative formulas (irrelevant, by reversibility). Towards this goal, we shall use **structured pattern-matching.** We can describe the construction of a proof of

$$(\Gamma, x : (P_1 \otimes P_2) \otimes (P_3 \otimes P_4) \vdash \Delta)$$

out of a proof of

$$c : (\Gamma \;,\; x_1 : P_1 \;,\; x_2 : P_2 \;,\; x_3 : P_3 \;,\; x_4 : P_4 \vdash \Delta)$$

“synthetically”, by writing

$$\langle x^\diamond \mid \tilde{\mu}((x_1, x_2), (x_3, x_4)).c\rangle$$

standing for an abbreviation of either of the following two commands :

$$\langle x^\diamond \mid \tilde{\mu}(y, z).\langle y^\diamond \mid \tilde{\mu}(x_1, x_2).\langle z^\diamond \mid \tilde{\mu}(x_3, x_4).c\rangle\rangle\rangle$$
$$\langle x^\diamond \mid \tilde{\mu}(y, z).\langle z^\diamond \mid \tilde{\mu}(x_3, x_4).\langle y^\diamond \mid \tilde{\mu}(x_1, x_2).c\rangle\rangle\rangle$$

The two goals are connected, since applying strong focalisation will forbid the formation of these two terms (because y, z are values appearing with non atomic types), keeping the synthetic form only... provided we make it first class.

First step : introducing first-class counterpatterns

Simple commands	$c ::= \langle v \mid e \rangle$	Commands	$C ::= c \parallel [C \ ^{q,q}\ C]$
Expressions	$v ::= V^{\diamond} \parallel \mu\alpha.C$	Values	$V ::= x \parallel (V,V) \parallel inl(V) \parallel inr(V) \parallel e^{\bullet}$
Contexts	$\boxed{e ::= \alpha \parallel \tilde{\mu}q.C}$	Counterpatterns	$\boxed{q ::= x \parallel \alpha^{\bullet} \parallel (q,q) \parallel [q,q]}$

Let $\Xi = x_1 : X_1, \ldots, x_n : X_n$ denote a left context consisting of *atomic formulas only*.

The rules are as follows :

$$\frac{}{\Xi\,,\ x : X \vdash x : X\ ;\ \Delta} \qquad \frac{C : (\Xi\,,\ q : P \vdash \Delta)}{\Xi \mid \tilde{\mu}q.C : P \vdash \Delta} \qquad \frac{C : (\Xi \vdash \alpha : P\,,\ \Delta)}{\Xi \vdash \mu\alpha.C : P \mid \Delta}$$

$$\frac{C : (\Gamma \vdash \alpha : P\,,\ \Delta)}{C : (\Gamma\,,\ \alpha^{\bullet} : \neg^{+}P \vdash \Delta)} \qquad \frac{C : (\Gamma\,,\ q_1 : P_1\,,\ q_2 : P_2 \vdash \Delta)}{C : (\Gamma\,,\ (q_1,q_2) : P_1 \otimes P_2 \vdash \Delta)}$$

$$\frac{C_1 : (\Gamma\,,\ q_1 : P_1 \vdash \Delta) \quad C_2 : (\Gamma\,,\ q_2 : P_2 \vdash \Delta)}{[C_1\ ^{q_1,q_2}\ C_2] : (\Gamma\,,\ [q_1,q_2] : P_1 \oplus P_2 \vdash \Delta)}$$

(all the other rules as before, with Ξ in place of Γ)

But...

We introduced a new mess, in the form of ugly new compound commands $[C_1 \ ^{q_1,q_2} \ C_2]$. We did a good job for tensors on the left, but not for plus on the left.

If $c_{ij} : (\Gamma \, , \, x_i : P_i \, , \, x_j : P_j \vdash_S \Delta)$ $(i = 1, 2, j = 3, 4)$, we want to identify

$$[[c_{13} \ ^{x_3,x_4} \ c_{14}] \ ^{x_1,x_2} \ [c_{23} \ ^{x_3,x_4} \ c_{24}]]$$
$$[[c_{13} \ ^{x_1,x_2} \ c_{23}] \ ^{x_3,x_4} \ [c_{14} \ ^{x_1,x_2} \ c_{24}]]$$

both proving $\ldots P_1 \oplus P_2 \, , \, P_3 \oplus P_4 \vdash \ldots$, and to write simply

$$\{c_{i,j} \mid i = 1, 2, j = 3, 4\}$$

To make this work, we need a second (and last) ingredient : **patterns**.

Towards the second step : introducing first-class patterns

We redefine the syntax of values, as follows :

Undecomposable values $\mathcal{V} ::= x \parallel e^\bullet$
Values $V ::= p \, \langle \mathcal{V}_i / i \mid i \in p \rangle$ Patterns $p ::= x \parallel \alpha^\bullet \parallel (p,p) \parallel inl(p) \parallel inr(p)$

where $i \in p$ is defined by :

$$\frac{}{x \in x} \quad \frac{}{\alpha^\bullet \in \alpha^\bullet} \quad \frac{i \in p_1}{i \in (p_1, p_2)} \quad \frac{i \in p_2}{i \in (p_1, p_2)} \quad \frac{i \in p_1}{i \in inl(p_1)} \quad \frac{i \in p_2}{i \in inr(p_2)}$$

Moreover, $\mathcal{V}_i$ must be of the form y (resp. $e^\bullet$) if $i = x$ (resp. $i = \alpha^\bullet$).

Patterns are required to be linear, as well as the counterpatterns, for which the definition of "linear" is adjusted in the case $[q_1, q_2]$, in which a variable can occur (but recursively linearly so) in both q_1 and q_2

Values are defined up to α-conversion, e.g. $\alpha^\bullet \langle e^\bullet / \alpha^\bullet \rangle = \beta^\bullet \langle e^\bullet / \beta^\bullet \rangle$

Pattern-counterpattern interaction

We rephrase the logical reduction rules in terms of pattern/counterpattern interaction :

$$\frac{V = p\,\langle \ldots y/x, \ldots, e^{\bullet}/\alpha^{\bullet}, \ldots\rangle \qquad C[p/q] \longrightarrow^{*} c}{\langle V^{\Diamond} \mid \tilde{\mu} q.C\rangle \longrightarrow c\{\ldots, y/x, \ldots, e/\alpha, \ldots\}}$$

where $c\{\sigma\}$ is the usual, implicit substitution, and where c (see the next proposition) is the normal form of $C[p/q]$ with respect to the following set of rules :

$$\begin{array}{l}
C[(p_1,p_2)/(q_1,q_2),\sigma] \longrightarrow C[p_1/q_1,p_2/q_2,\sigma] \\
C[\alpha^{\bullet}/\alpha^{\bullet},\sigma] \longrightarrow C[\sigma] \\
C[x/x,\sigma] \longrightarrow C[\sigma] \\
[C_1 \; {}^{q_1,q_2} \; C_2][inl(p_1)/[q_1,q_2],\sigma] \longrightarrow C_1[p_1/q_1,\sigma] \\
[C_1 \; {}^{q_1,q_2} \; C_2][inr(p_2)/[q_1,q_2],\sigma] \longrightarrow C_2[p_2/q_2,\sigma]
\end{array}$$

Logically, this means that we now consider each formula as made of blocks of *synthetic* connectives.

An example

Patterns for $P = X \otimes (Y \oplus \neg^{+}Q)$. Focusing on the right yields two possible proof searches :

$$\frac{\Gamma \vdash x\{\mathcal{V}_x\} : X \;;\; \Delta \quad \Gamma \vdash y\{\mathcal{V}_y\} : Y \;;\; \Delta}{\Gamma \vdash (x, inl(y))\{\mathcal{V}_x, \mathcal{V}_y\} : X \otimes (Y \oplus \neg^{+}Q) \;;\; \Delta}$$

$$\frac{\Gamma \vdash x\{\mathcal{V}_x\} : X \;;\; \Delta \quad \Gamma \vdash \alpha^{\bullet}\{\mathcal{V}_{\alpha^{\bullet}}\} : \neg^{+}Q \;;\; \Delta}{\Gamma \vdash (x, inr(\alpha^{\bullet}))\{\mathcal{V}_x, \mathcal{V}_{\alpha^{\bullet}}\} : X \otimes (Y \oplus \neg^{+}Q) \;;\; \Delta}$$

Counterpattern for $P = X \otimes (Y \oplus \neg^{+}Q)$. The counterpattern describes the tree structure of P :

$$\frac{c_1 : (\Gamma \,,\, x : X \,,\, y : Y \vdash \Delta) \qquad c_2 : (\Gamma \,,\, x : X \,,\, \alpha^{\bullet} : \neg^{+}Q \vdash \Delta)}{[c_1 \;{}^{y,\alpha^{\bullet}}\; c_2] : (\Gamma \,,\, (x, [y, \alpha^{\bullet}]) : X \otimes (Y \oplus \neg^{+}Q) \vdash \Delta)}$$

We observe that the leaves of the decomposition of P on the left are in one-to-one correspondence with the patterns p for the (irreversible) decomposition of P on the right :

$$[c_1 \;{}^{y,\alpha^{\bullet}}\; c_2][p_1/q] \longrightarrow^{*} c_1 \qquad [c_1 \;{}^{y,\alpha^{\bullet}}\; c_2][p_2/q] \longrightarrow^{*} c_2$$

where $q = (x, [y, \alpha^{\bullet}])$, $p_1 = (x, inl(y))$, $p_2 = (x, inr(\alpha^{\bullet}))$.

A key one-to-one correspondence

We define two predicates $c \in C$ and $q \perp p$ ("q is orthogonal to p") as follows :

$$\frac{}{c \in c} \qquad \frac{c \in C_1}{c \in [C_1 \;{}^{q_1,q_2}\; C_2]} \qquad \frac{c \in C_2}{c \in [C_1 \;{}^{q_1,q_2}\; C_2]}$$

$$\boxed{\begin{array}{ccc} \dfrac{}{x \perp x} & \dfrac{}{\alpha^\bullet \perp \alpha^\bullet} & \dfrac{q_1 \perp p_1 \quad q_2 \perp p_2}{(q_1, q_2) \perp (p_1, p_2)} \\ \multicolumn{2}{c}{\dfrac{q_1 \perp p_1}{[q_1, q_2] \perp inl(p_1)}} & \dfrac{q_2 \perp p_2}{[q_1, q_2] \perp inr(p_2)} \end{array}}$$

Proposition Let $C : (\Xi\,,\; q : P \vdash \Delta)$ and let p be such that q is orthogonal to p. Then the normal form c of $C[p/q]$ is a simple command, and the mapping $p \mapsto c$ $(q, C$ fixed) from $\{p \mid q \perp p\}$ to $\{c \mid c \in C\}$ is one-to-one and onto.

Synthetic system L

$$c ::= \langle v \mid e \rangle \qquad v ::= V^{\Diamond} \parallel \mu\alpha.c$$
$$V ::= p\,\langle \mathcal{V}_i/i \mid i \in p \rangle \qquad \mathcal{V} ::= x \parallel e^{\bullet} \qquad p ::= x \parallel \alpha^{\bullet} \parallel (p,p) \parallel inl(p) \parallel inr(p)$$
$$e ::= \alpha \parallel \tilde{\mu}q.\{p \mapsto c_p \mid q \perp p\} \qquad q ::= x \parallel \alpha^{\bullet} \parallel (q,q) \parallel [q,q]$$

$$\begin{array}{c} \langle (p\,\langle \ldots, y/x, \ldots, e^{\bullet}/\alpha^{\bullet} \ldots \rangle)^{\Diamond} \mid \tilde{\mu}q.\{p \mapsto c_p \mid q \perp p\} \rangle \\ \downarrow \\ c_p\,\{\ldots, y/x, \ldots, e/\alpha, \ldots \rangle\} \end{array}$$

The idea is that the pattern-counterpattern interaction has been “precomputed” and stored in record format.

The computation rule is thus a packaging of

– **field selection** (object-oriented programming)

– **substitution** (functional programming)

Related work : Zeilberger’s *unity of duality*, Girard’s ludics (revisited by Terui)

PART II

Focalising system L

as intermediate language

PART II A

Translating from...

Encoding CBV λ-calculus into LKQ 1/2

We define the following derived CBV implication and terms :

$$P \to^v Q = \neg^+(P \otimes \neg^+ Q)$$

$$\lambda x.v = ((\tilde{\mu}(x, \alpha^\bullet).\langle v \mid \alpha \rangle)^\bullet)^\lozenge \qquad v_1 v_2 = \mu\alpha.\langle v_2 \mid \tilde{\mu}x.\langle v_1 \mid (x, \alpha^\bullet)^\blacklozenge \rangle\rangle$$

where $\tilde{\mu}(x, \alpha^\bullet).c$ is an abbreviation for $\tilde{\mu}(x, y).\langle y^\lozenge \mid \tilde{\mu}\alpha^\bullet.c\rangle$ and where $V^\blacklozenge$ stands for $\tilde{\mu}\alpha^\bullet.\langle V^\lozenge \mid \alpha\rangle$, and hence enjoys the reduction

$$\langle (e^\bullet)^\lozenge \mid V^\blacklozenge \rangle \longrightarrow \langle V^\lozenge \mid e \rangle$$

We simulate β_v (Plotkin's CBV β-reduction) as follows :

$$\begin{array}{ll} (\lambda x.v_1)v_2 & \longrightarrow \mu\alpha.\langle v_2 \mid \tilde{\mu}x.\langle (x, \alpha^\bullet)^\lozenge \mid \tilde{\mu}(x, \alpha^\bullet).\langle v_1 \mid \alpha\rangle\rangle\rangle \\ & \longrightarrow \mu\alpha.\langle v_2 \mid \tilde{\mu}x.\langle v_1 \mid \alpha \rangle\rangle \end{array}$$

which then forces v_2 to reduce to some V_2, allowing to reach $\mu\alpha.\langle v_1[V_2/x]|\alpha\rangle$.

Encoding CBV λ-calculus into LKQ 2/2

The translation extends to (call-by-value) $\lambda\mu$-calculus

The translation makes also sense in the untyped setting

Encoding CBN $\lambda(\mu)$-calculus 1/2

What about CBN ? We can translate it to LKQ, but at the price of translating terms to contexts, which is kind of a violence...

But keeping the *same* term language, we can type sequents of negative formulas, giving rise to a dual logic LKT :

$$N := \overline{X} \parallel N \parr N \parallel N \& N \parallel \neg^{-} N$$

Four kinds of judgements :

$$c : (\Gamma \vdash \Delta) \quad \Gamma\,;\, E : N \vdash \Delta \quad \Gamma \,|\, e : N \vdash \Delta \quad \Gamma \vdash v : N \,|\, \Delta$$

We would have arrived to this logic naturally if we had chosen to present LK with a reversible disjunction on the right and an irreversible conjunction on the left (cf. above)

Focalising system L (negatively-minded repainting)

Commands	$c ::= \langle v \mid e \rangle$
Covalues	$E ::= \alpha \parallel [E, E] \parallel E[fst] \parallel E[snd] \parallel v^{\bullet}$
Contexts	$e ::= E^{\Diamond} \parallel \tilde{\mu}x.c$
Expressions	$v ::= x \parallel \mu\alpha.c \parallel \mu x^{\bullet}.c \parallel \ldots$

$$\langle v \mid \tilde{\mu}x.c \rangle \longrightarrow c[v/x]$$
$$\langle \mu\alpha.c \mid E^{\Diamond} \rangle \longrightarrow c[E/\alpha]$$
$$\langle \mu x^{\bullet}.c \mid (v^{\bullet})^{\Diamond} \rangle \longrightarrow c[v/x]$$
$$\vdots$$

The system LKT

$$\frac{}{\Gamma\,;\,\alpha : N \vdash \Delta\,,\,\alpha : N} \qquad \frac{\Gamma \vdash v : N \mid \Delta}{\Gamma\,;\,v^{\bullet} : \neg^{-} N \vdash \Delta}$$

$$\frac{\Gamma\,;\,E_1 : N_1 \vdash \Delta \qquad \Gamma\,;\,E_2 : N_2 \vdash \Delta}{\Gamma\,;\,[E_1, E_2] : N_1 \parr N_2 \vdash \Delta} \qquad \frac{\Gamma\,;\,E_1 : N_1 \vdash \Delta}{\Gamma\,;\,E_1[fst] : N_1 \& N_2 \vdash \Delta}$$

$$\frac{\Gamma\,;\,E : N \vdash \Delta}{\Gamma \mid E^{\diamond} : N \vdash \Delta} \qquad \frac{c : (\Gamma\,,\,x : N \vdash \Delta)}{\Gamma \mid \tilde{\mu}x.c : N \vdash \Delta}$$

$$\frac{}{\Gamma\,,\,x : N \vdash x : N \mid \Delta} \qquad \frac{c : (\Gamma \vdash \alpha : N\,,\,\Delta)}{\Gamma \vdash \mu\alpha.c : N \mid \Delta} \qquad \frac{c : (\Gamma\,,\,x : N \vdash \Delta)}{\Gamma \vdash \mu x^{\bullet}.c : \neg^{-} N \mid \Delta} \quad \ldots$$

$$\frac{\Gamma \vdash v : N \mid \Delta \qquad \Gamma \mid e : N \vdash \Delta}{\langle v \mid e \rangle : (\Gamma \vdash \Delta)}$$

A judgement is provable in LKT iff the dual judgement in provable in LKQ

Encoding CBN $\lambda(\mu)$-calculus 2/2

In LKT we can define the following derived CBN implication and terms :

$$M \to^n N = (\neg^- M) \parr N$$

$$\lambda x.v = \mu(x^\bullet, \alpha).\langle v \mid \alpha^\Diamond \rangle \qquad v_1 v_2 = \mu\alpha.\langle v_1 \mid (v_2^\bullet, \alpha)^\Diamond \rangle$$

The translation extends to $\lambda\mu$-calculus, and also to left introduction of implication :

$$\frac{\Gamma \vdash v : N_1 \mid \Delta \quad \Gamma\,;\, E : N_2 \vdash \Delta}{\Gamma\,;\, v \cdot E : N_1 \Rightarrow N_2 \vdash \Delta}$$

with $v \cdot E = (v^\bullet, E)$ (read covalues as stacks, and this one as obtained by pushing v on top of E)

With these definitions, we have :

$$\langle \lambda x.v_1 \mid (v_2 \cdot E)^\Diamond \rangle = \langle \mu(x^\bullet, \alpha).\langle v_1 \mid \alpha^\Diamond \rangle \mid (v_2^\bullet, E)^\Diamond \rangle \longrightarrow \langle v_1[v_2/x] \mid E^\Diamond \rangle$$

$$\langle v_1 v_2 \mid E^\Diamond \rangle = \langle \mu\alpha.\langle v_1 \mid (v_2^\bullet, \alpha)^\Diamond \rangle \mid E^\Diamond \rangle \longrightarrow \langle v_1 \mid (v_2^\bullet, E)^\Diamond \rangle = \langle v_1 \mid (v_2 \cdot E)^\Diamond \rangle$$

(Krivine CBN abstract machine)

What about mixed CBN/CBV ?

LKQ and LKT are in fact the positive and negative fragments of a larger system (described in an appendix of Munch-Maccagnoni CSL09), in which positive and negative formulas can freely appear both on the left and on the right of sequents.

Cf. Girard's LC.

PART II B

Translating to...

Translating LKQ to intuitionistic logic 1/3

Our target language will be intuitionistic logic with the following connectives :

$\neg^i$ (negation) $\quad \times$ (conjunction) $\quad +$ (disjunction)

$$
\begin{array}{lcl}
c & ::= & t\,t \\
t & ::= & x \parallel (t,t) \parallel inl(t) \parallel inr(t) \\
 & & \lambda x.c \parallel \lambda(x_1,x_2).c \parallel \lambda z.\textit{case}\ z\ [inl(x_1)\cdot c_1, inr(x_2)\cdot c_2]
\end{array}
$$

Two typing judgements :

$$c : (\Gamma \vdash) \qquad \Gamma \vdash t : A$$

System NJ_0

N for Natural, J for Intuitionistic, $_0$ for not having full implication : think of $\neg^i A$ as $A \Rightarrow R$ for some fixed R, considered as “false”, or as “the type of final results”

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash t_1 : \neg^i A \quad \Gamma \vdash t_2 : A}{t_1 t_2 : (\Gamma \vdash)} \qquad \frac{c : (\Gamma, x : A \vdash)}{\Gamma \vdash \lambda x.c : \neg^i A}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2} \qquad \frac{\Gamma \vdash t_1 : A_1}{\Gamma \vdash inl(t_1) : A_1 + A_2}$$

$$\frac{c : (\Gamma, x_1 : A_1, x_2 : A_2 \vdash)}{\Gamma \vdash \lambda(x_1, x_2).c : \neg^i (A_1 \times A_2)}$$

$$\frac{c_1 : (\Gamma, x_1 : A_1 \vdash) \quad c_2 : (\Gamma, x_2 : A_2 \vdash)}{\Gamma \vdash \lambda z.\textit{case}\ z\ [inl(x_1) \cdot c_1, inr(x_2) \cdot c_2] : \neg^i (A_1 + A_2)}$$

Translating LKQ to intuitionistic logic 2/3

Translation of formulas :

$$X_{cps} = X \qquad (\neg^{+} P)_{cps} = \neg^{i}(P_{cps})$$
$$(P \otimes Q)_{cps} = (P_{cps}) \times (Q_{cps}) \quad (P \oplus Q)_{cps} = (P_{cps}) + (Q_{cps})$$

Translation of terms :

$$\langle v \mid e \rangle_{cps} = (v_{cps})(e_{cps})$$
$$(V^{\Diamond})_{cps} = \lambda k.k(V_{cps}) \qquad (\mu\alpha.c)_{cps} = \lambda k_{\alpha}.(c_{cps}) = (\tilde{\mu}\alpha^{\bullet}.c)_{cps}$$
$$x_{cps} = x \qquad (V_1, V_2)_{cps} = ((V_1)_{cps}, (V_2)_{cps})$$
$$inl(V_1)_{cps} = inl((V_1)_{cps}) \qquad (e^{\bullet})_{cps} = e_{cps}$$
$$\alpha_{cps} = k_{\alpha} \qquad (\tilde{\mu}x.c)_{cps} = \lambda x.(c_{cps}) \qquad (\tilde{\mu}(x_1, x_2).c)_{cps} = \lambda(x_1, x_2).(c_{cps})$$
$$(\tilde{\mu}[inl(x_1).c_1 | inr(x_2).c_2])_{cps} = \lambda z.\textit{case}\ z\ [inl(x_1) \cdot (c_1)_{cps}, inr(x_2) \cdot (c_2)_{cps}]$$

Translating LKQ to intuitionistic logic 3/3

We set

$$\Gamma_{cps} = \{x : P_{cps} \mid x : P \in \Gamma\}$$
$$\neg^i(\Delta_{cps}) = \{k_\alpha : \neg^i(P_{cps}) \mid \alpha : P \in \Delta\}$$

We have :

$$\begin{array}{rcl} c : (\Gamma \vdash \Delta) & \Rightarrow & c_{cps} : (\Gamma_{cps} \, , \, \neg^i(\Delta_{cps}) \vdash) \\ \Gamma \vdash V : P \, ; \, \Delta & \Rightarrow & \Gamma_{cps} \, , \, \neg^i(\Delta_{cps}) \vdash V_{cps} : P_{cps} \\ \Gamma \vdash v : P \mid \Delta & \Rightarrow & \Gamma_{cps} \, , \, \neg^i(\Delta_{cps}) \vdash v_{cps} : \neg^i(\neg^i(P_{cps})) \\ \Gamma \mid e : P \vdash \Delta & \Rightarrow & \Gamma_{cps} \, , \, \neg^i(\Delta_{cps}) \vdash e_{cps} : \neg^i(P_{cps}) \end{array}$$

Moreover, the translation preserves reduction

A three-step decomposition of CPS

The translation from LKQ to NJ does actually two transformations for the price of one : *from classical to intuitionistic*, and *from sequent calculus style to natural deduction style*. This yields the following overall decomposition

$$\begin{array}{ccc} \lambda\mu & -I\rightarrow & \text{LKQ} \\ & & \Big| \\ & & II \\ & & \downarrow \\ \text{NJ}_0 & \leftarrow III - & \text{LJ}_0 \end{array}$$

– step I specifies the order of evaluation
– step II blurs the distinction between continuation variables and ordinary variables
– step III blurs the distinction between contexts and expressions

(See Munch-Maccagnoni 2010 (submitted), with an extension to delimited continuations)

A lozenge of translations (work in progress)

CBN $\lambda\mu$

$|$

LKT

$\diagup$ $\diagdown$

LJ_0

$\diagdown$ $\diagup$

LL

$\diagup$ translations = “Girard / Lafont-Reus-Streicher”

Lower $\diagdown$ translation = “reversing / Krivine”

$\diagdown$ (resp. $\diagup$) allows us to recover contraction on neg. (resp. pos.) formulas

(see also Laurent-Regnier LICS 2003)