Bottom-Up Sequentialization of Unit-Free MALL Proof Nets

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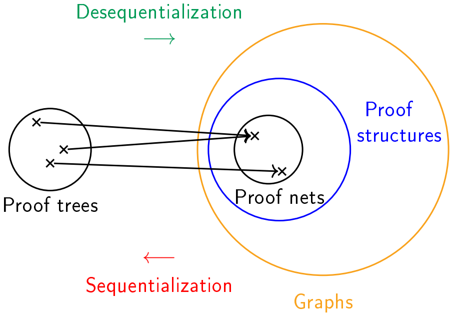
Linearity & TLLA 2022
31 July - 1 August 2022

Introduction

Proof nets: graphical, more canonical representation of LL proofs



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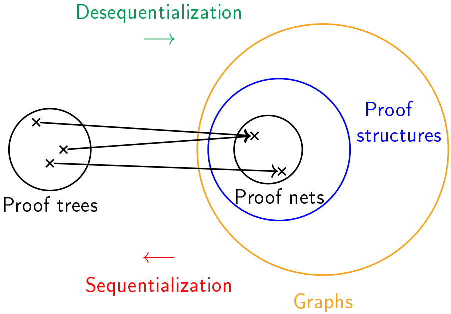

MALL Proof Net:

- Girard [Gir96]
- Hughes & Van Glabbeek [HvG05]

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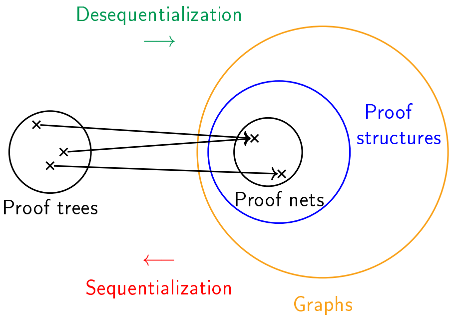

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MLL: multiple correctness criteria, proofs of sequentialization
MALL: only one proof of sequentialization for HVG proof nets

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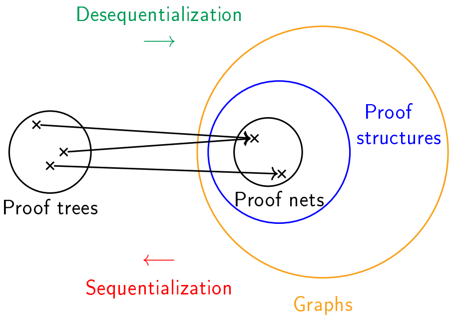

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MLL: multiple correctness criteria, proofs of sequentialization
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Objectives of this talk:

- main ideas on MALL proof nets
- another proof of sequentialization, extending one from MLL

Plan

1. Introduction
2. Sequentialization for MLL Proof Nets
   - Presentation of MLL
   - Sequentialization proof in MLL
3. Sequentialization for MALL Proof Nets
   - Presentation of MALL
   - Sequentialization proof in MALL
4. Conclusion

Example of proof structure

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash A^{\perp}, A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash B^{\perp}, B$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash A^{\perp}, A \otimes B^{\perp}, B$}
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$$

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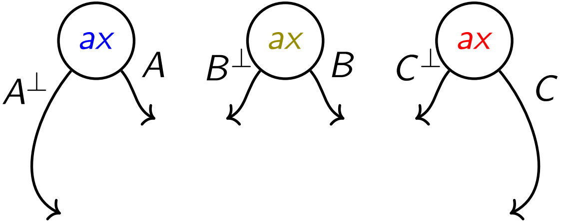
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,ax \quad \dfrac{}{\vdash B^{\perp}, B}\,ax}{\vdash A^{\perp}, A \otimes B^{\perp}, B}\,\otimes \quad \dfrac{}{\vdash C^{\perp}, C}\,ax}{\vdash A^{\perp}, A \otimes B^{\perp}, B \otimes C^{\perp}, C}\,\otimes}{\vdash A^{\perp} \parr (A \otimes B^{\perp}), B \otimes C^{\perp}, C}\,\parr}{\vdash A^{\perp} \parr (A \otimes B^{\perp}), (B \otimes C^{\perp}) \parr C}\,\parr$$



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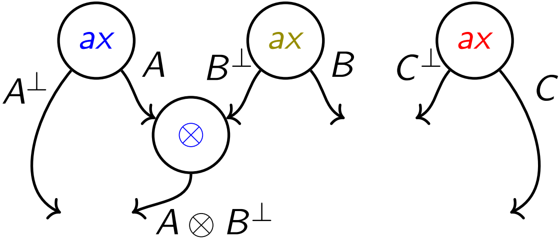

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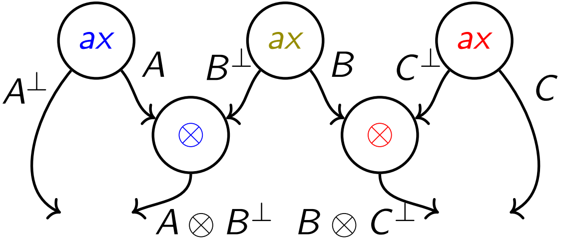

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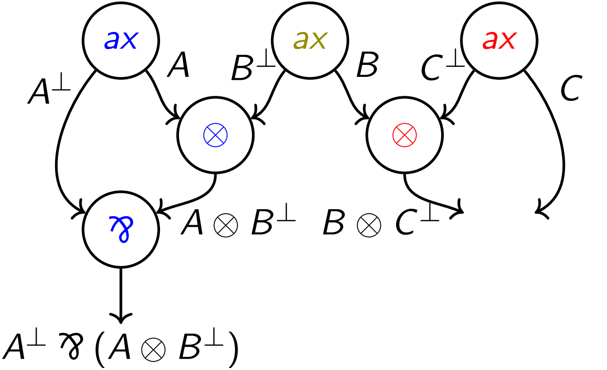

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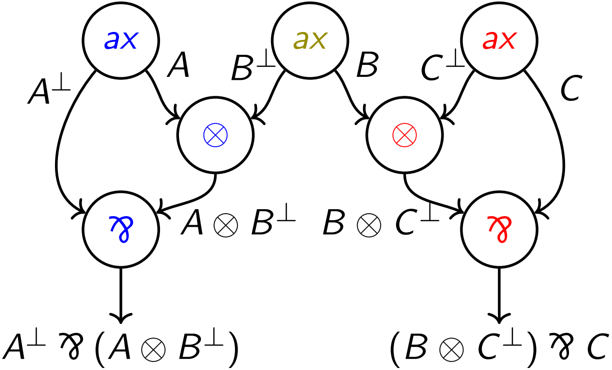

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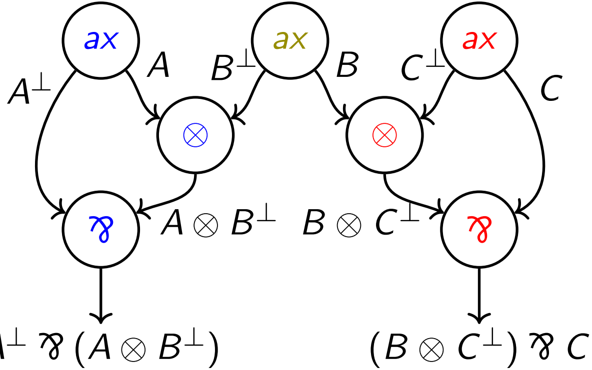

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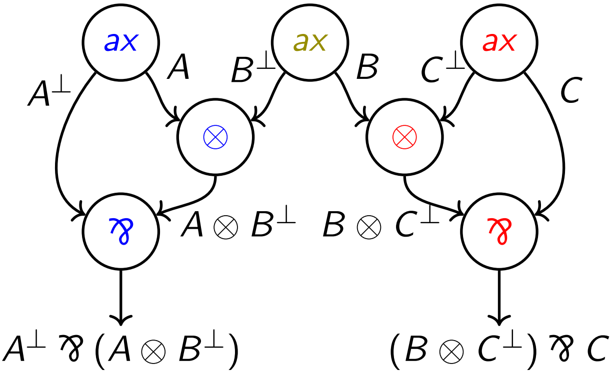

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\infer0[ax]{\vdash C^\perp, C}
\infer2[$\otimes$]{\vdash B^\perp, B \otimes C^\perp, C}
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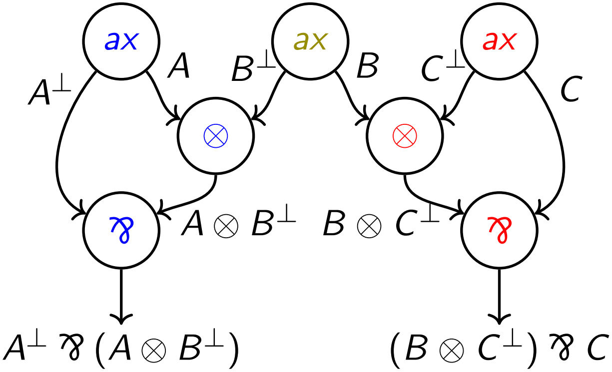
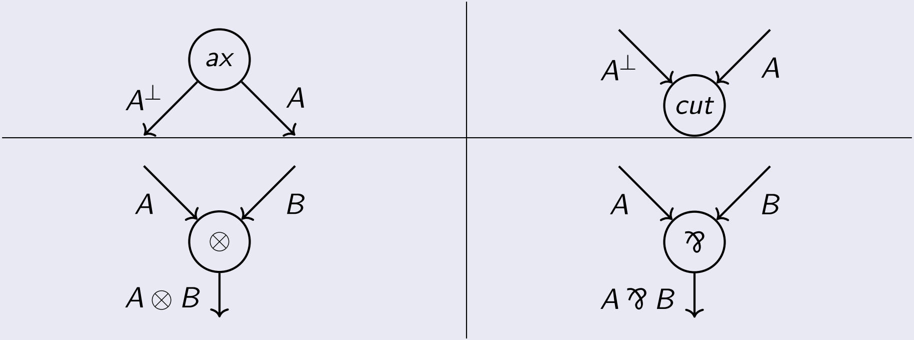


MLL Proof structure

MLL Proof structure

Multigraph, labeled and directed:
vertices $\to$ $ax/cut/\otimes/⅋$; edges $\to$ formula



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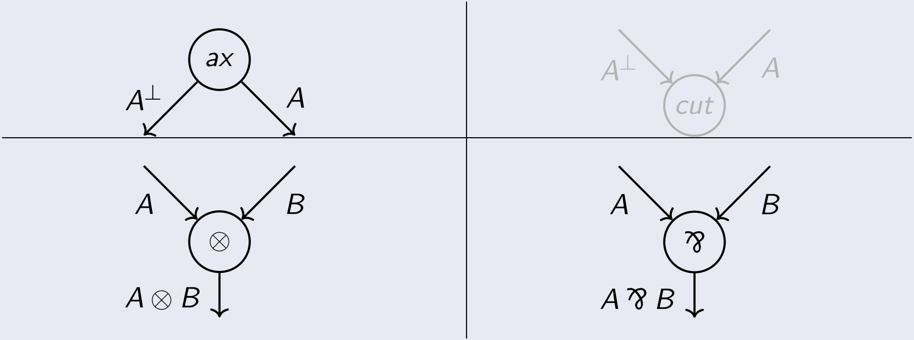
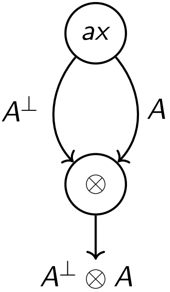


Structure & Net



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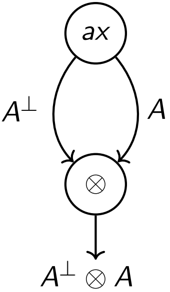
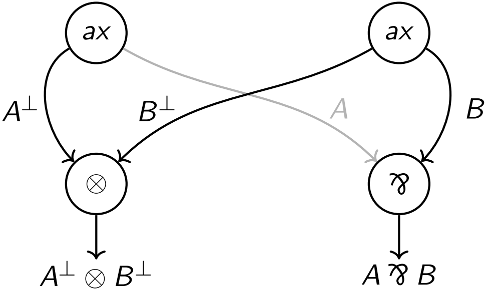


Danos-Regnier correctness criterion

Correctness graph:
Remove one in-edge of each $⅋$-vertex
Correctness criterion:
Correctness graphs are **acyclic** and **connected**



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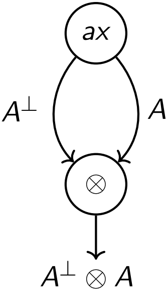

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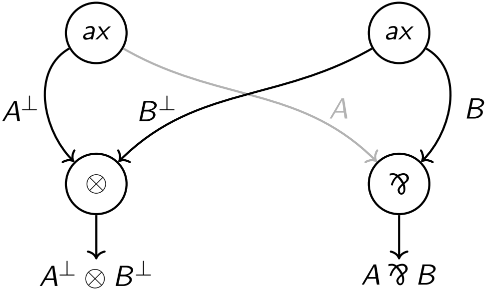

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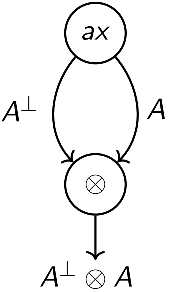

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Correctness graph:
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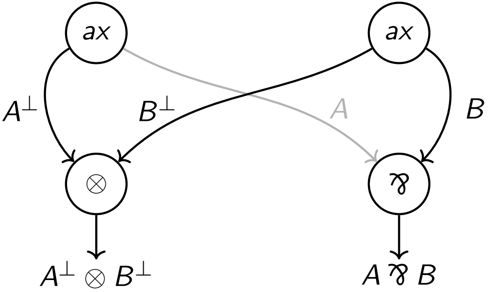
$\Longrightarrow$ no **switching cycle** (with < 2 in-edges of a $\parr$)



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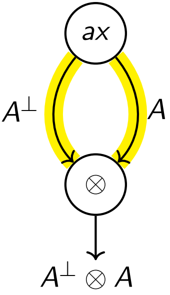

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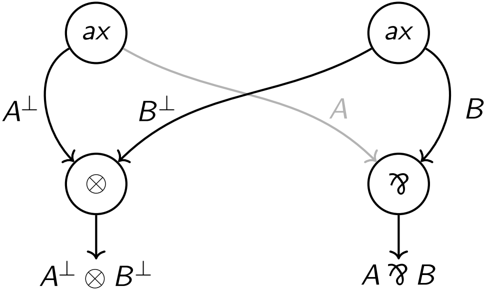

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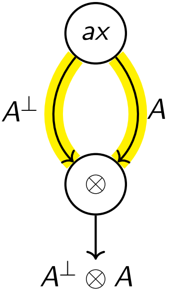
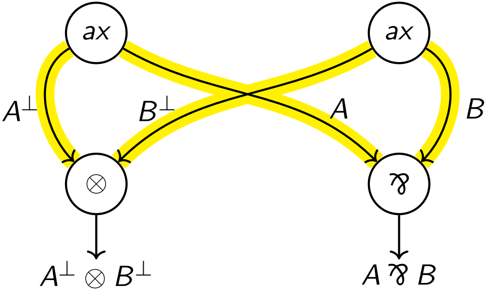
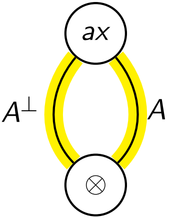




Structure & Net



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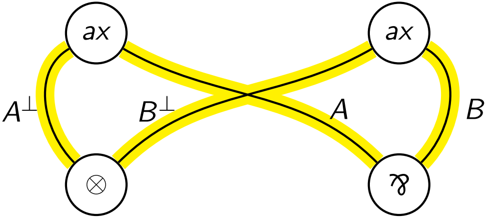
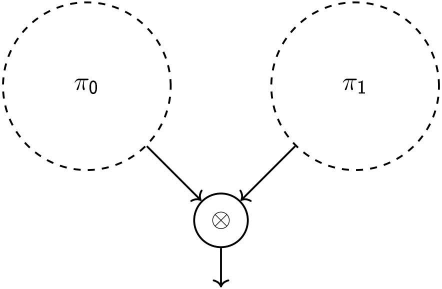


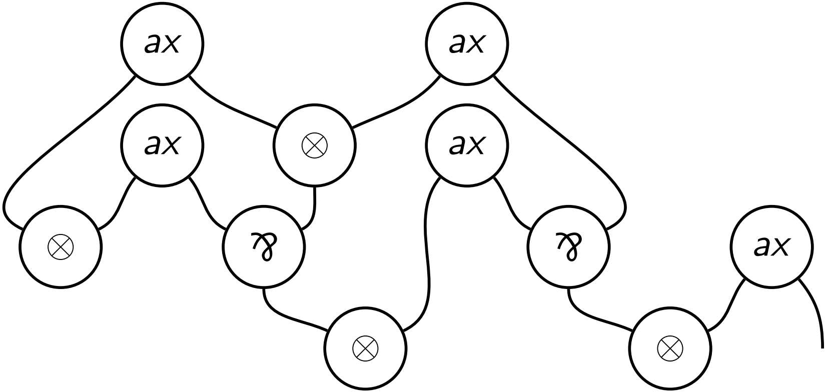
Desequentialization

$$\dfrac{\dfrac{\pi_0}{\vdash\ A, \Gamma} \qquad \dfrac{\pi_1}{\vdash\ B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta} \otimes$$



Plan

Leading example



Towards sequentialization

Looking for a vertex corresponding to a possible last rule
How to sequentialize then? Inverse of desequentialization!

Terminal vertex

No vertex below

Towards sequentialization

Looking for a vertex corresponding to a possible last rule
How to sequentialize then? Inverse of desequentialization!

Terminal vertex

No vertex below

Sequentializing vertex

Terminal, non-leaf such that:

- $\parr$-vertex: always

Towards sequentialization

Looking for a vertex corresponding to a possible last rule
How to sequentialize then? Inverse of desequentialization!

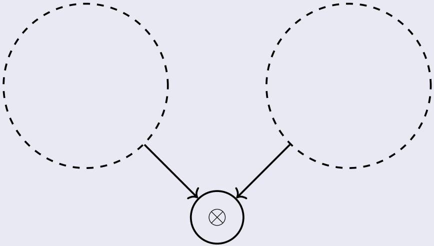
Terminal vertex

No vertex below

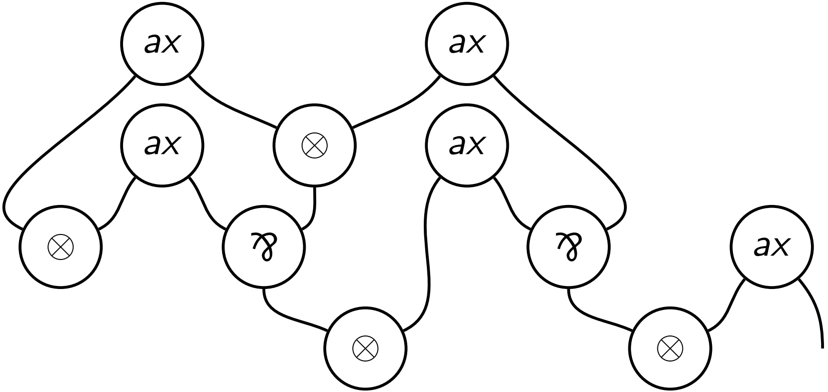
Sequentializing vertex

Terminal, non-leaf such that:

- $\parr$-vertex: always
- $\otimes$-vertex: remove it yields two connected components



Leading example



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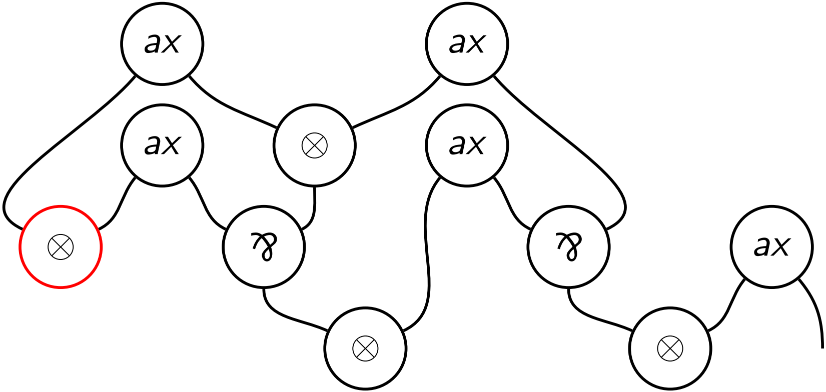

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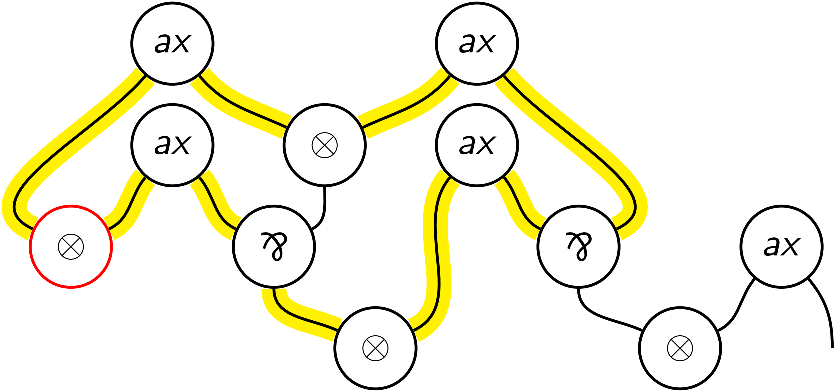

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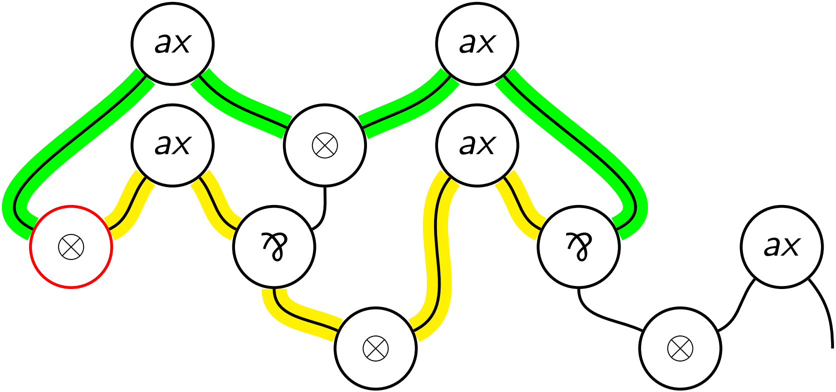
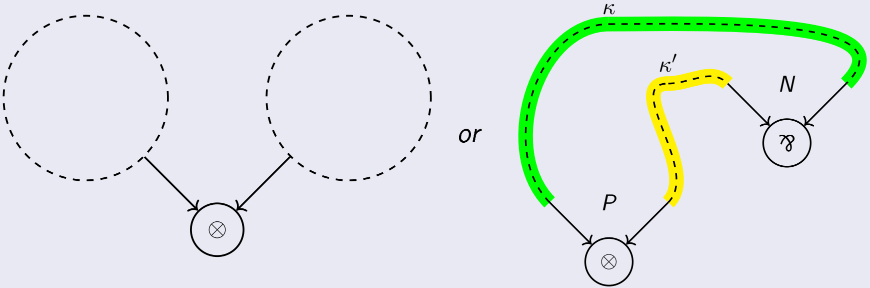


Correctness path

Dependencies for non-sequentializing $\otimes$

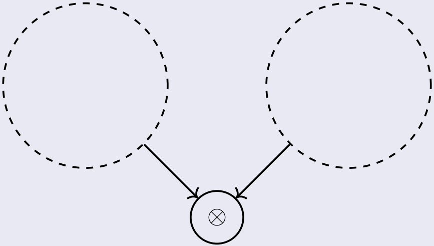
Lemma

For a terminal $\otimes$ either

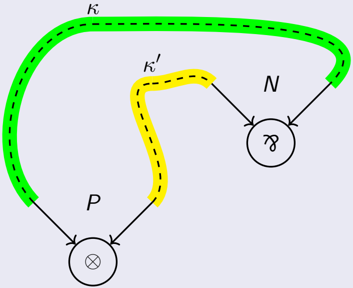


Lemma

For a terminal $\otimes$ either



or



Sequentialization constraint: must sequentialize this $\parr$ before the $\otimes$!

Correctness path

Dependencies for non-sequentializing $\otimes$

Switching path

Simple path without both in-edges of a $⅋$

Path surviving in some correctness graph

Strong path

Switching path not starting from a $⅋$ through an in-edge

Switching paths which are easy to concatenate

Correctness path

Dependencies for non-sequentializing $\otimes$

Switching path

Simple path without both in-edges of a $⅋$

Path surviving in some correctness graph

Strong path

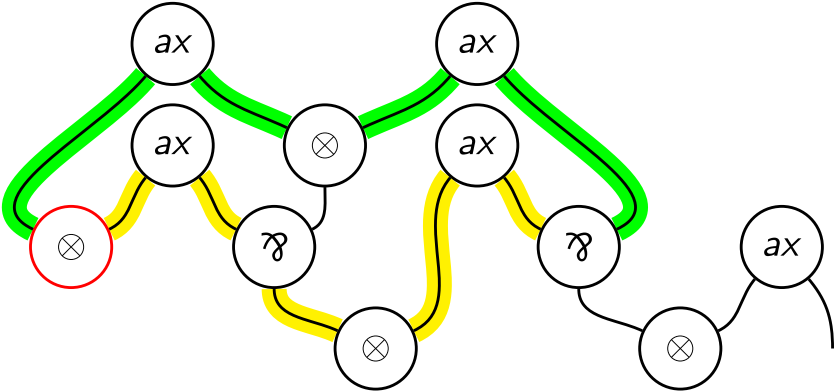
Switching path not starting from a $⅋$ through an in-edge

Switching paths which are easy to concatenate

Correctness $⅋$ for a $\otimes$ P

$⅋$-vertex N with two disjoint strong paths κ and κ' starting with an in-edge of P and ending with an in-edge of N

Leading example



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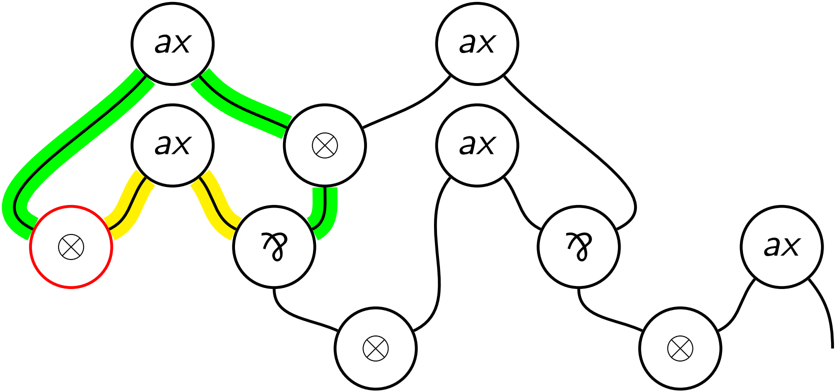

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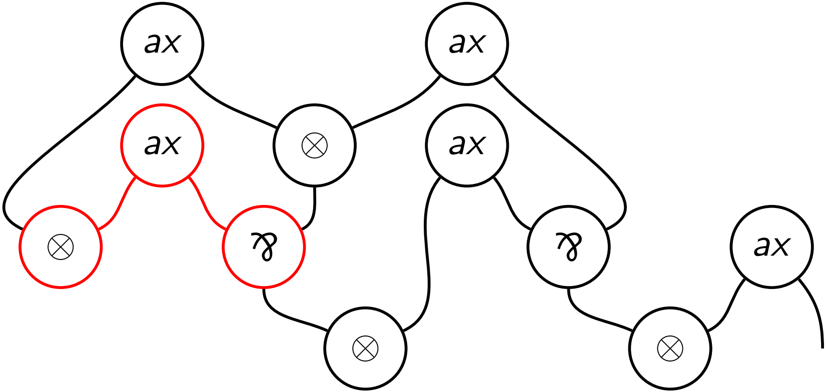
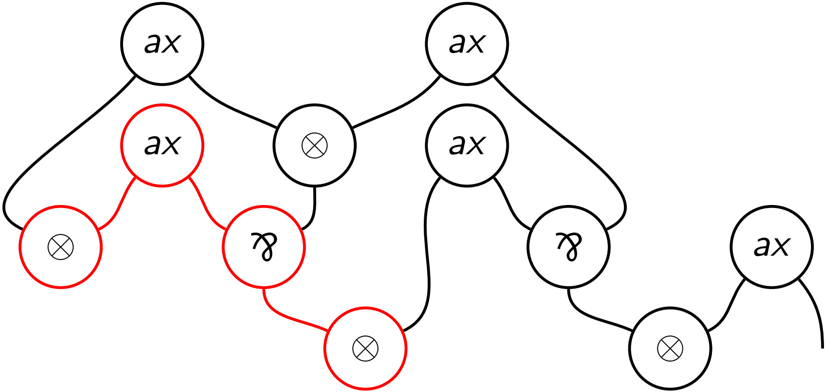


Descending path

Logical dependencies



Descending path

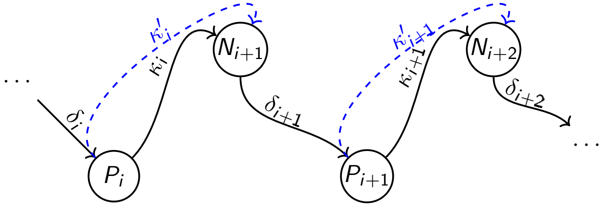
From a vertex to the root of its syntax tree; strong path

Dependency directly from the syntax

Critical path
Chain of dependencies

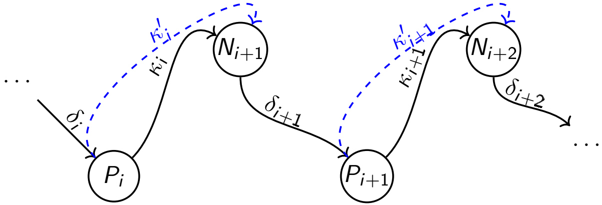
Critical path

Concatenation of correctness path, descending path, ...



Critical path

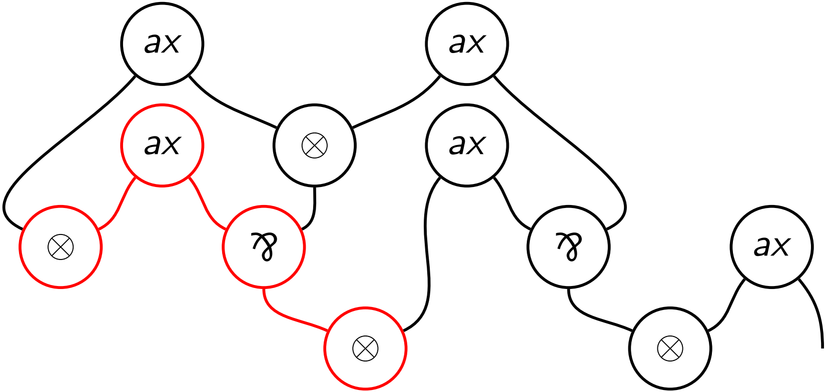
Concatenation of correctness path, descending path, ...



Properties:

- strong path
- cannot go back
- can keep going until reaching a sequentializing vertex

Leading example



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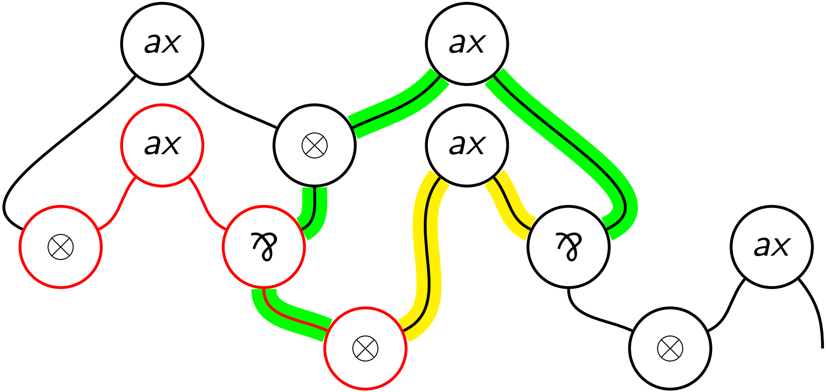

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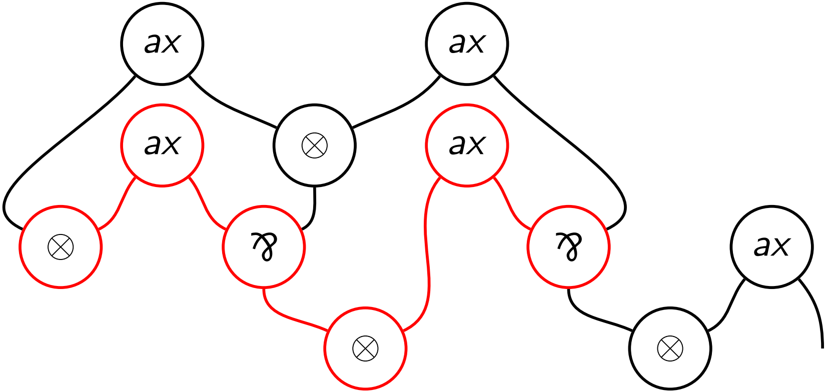
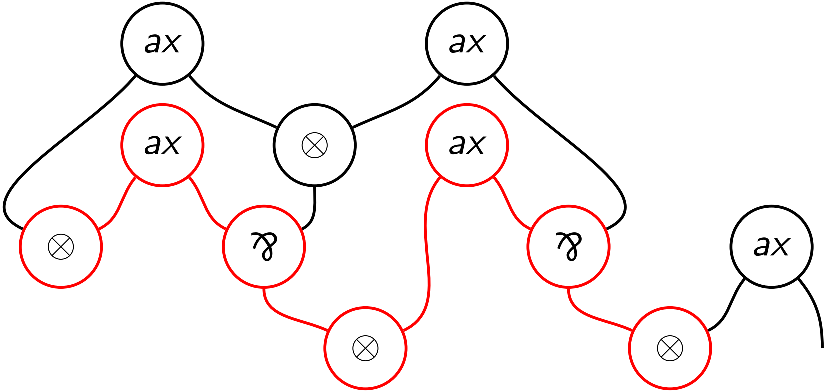


Leading example



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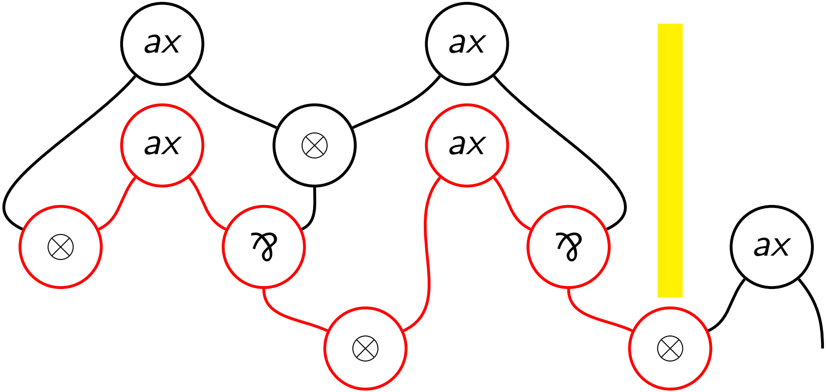

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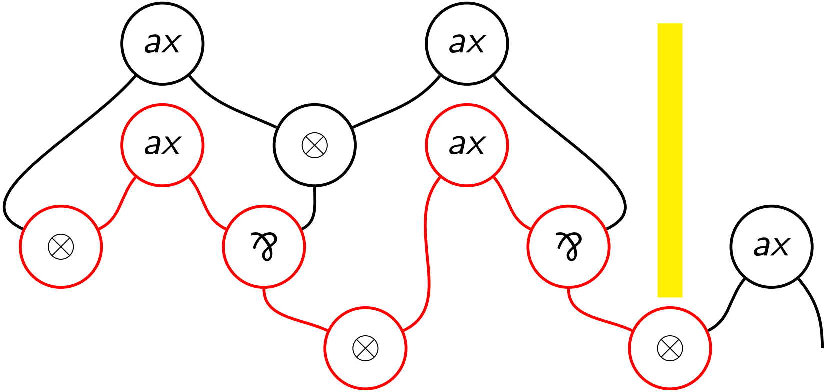
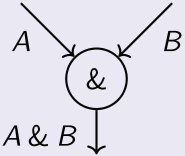


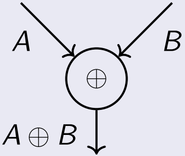
Plan

MALL Proof structure

MALL Proof structure

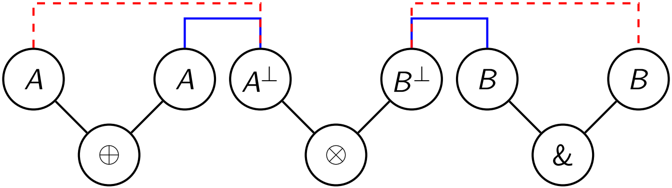
Multigraph, labeled and directed:
vertices $\rightarrow$ $\ldots / \,\&\, / \oplus$; edges $\rightarrow$ formula





Example of MALL proof net

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,ax \qquad \dfrac{}{\vdash B^{\perp}, B}\,ax}{\vdash A, A^{\perp} \otimes B^{\perp}, B}\,\otimes}{\vdash A \oplus A, A^{\perp} \otimes B^{\perp}, B}\,\oplus_2 \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,ax \qquad \dfrac{}{\vdash B^{\perp}, B}\,ax}{\vdash A, A^{\perp} \otimes B^{\perp}, B}\,\otimes}{\vdash A \oplus A, A^{\perp} \otimes B^{\perp}, B}\,\oplus_1}{\vdash A \oplus A, A^{\perp} \otimes B^{\perp}, B \,\&\, B}\,\&$$



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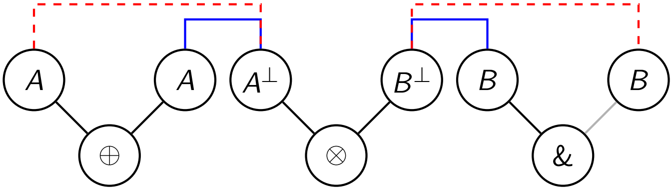

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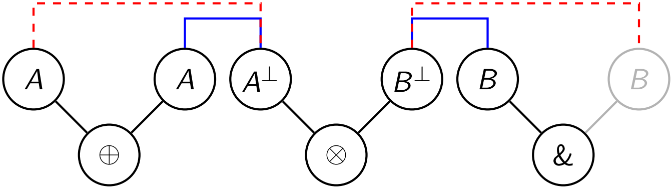

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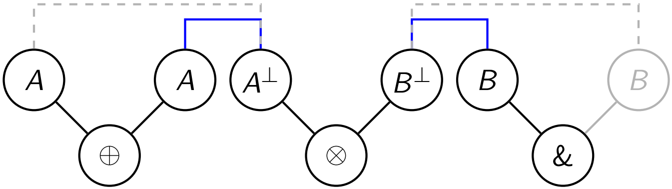

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$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash A^{\perp}, A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash B^{\perp}, B$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash A, A^{\perp} \otimes B^{\perp}, B$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash A \oplus A, A^{\perp} \otimes B^{\perp}, B$}
\AxiomC{}
\RightLabel{ax}
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\RightLabel{$\otimes$}
\BinaryInfC{$\vdash A, A^{\perp} \otimes B^{\perp}, B$}
\RightLabel{$\oplus_1$}
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\RightLabel{$\&$}
\BinaryInfC{$\vdash A \oplus A, A^{\perp} \otimes B^{\perp}, B \,\&\, B$}
\end{prooftree}
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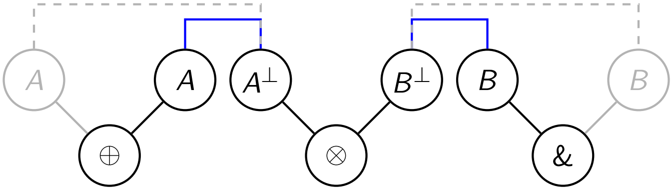

Slice-correctness

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \mathbin{\&} B, \Gamma}\ \&$$

Slice = choice of premise for each &

Resolution (ALL correctness):
Exactly one linking (= axioms set) by slice

Danos-Regnier (MLL correctness):
Each linking is MLL-correct

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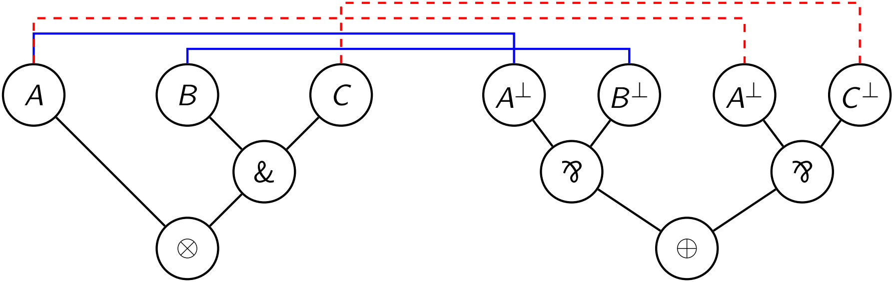
DOES NOT WORK!

Slice-correctness: MLL and ALL side-by-side

Slice-correctness is not sufficient



But $(A \otimes B) \& (A \otimes C) \nvdash A \otimes (B \& C)$

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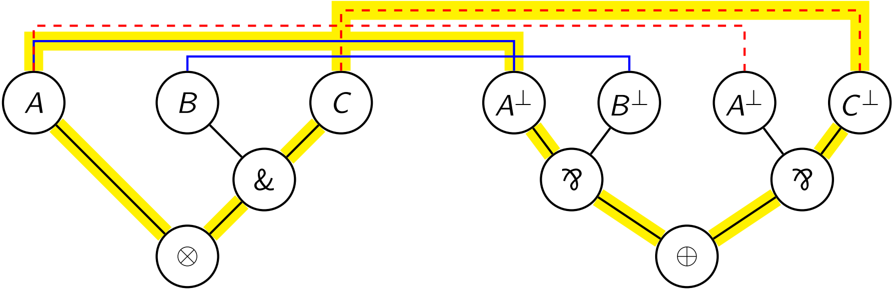

But $(A \otimes B) \& (A \otimes C) \nvdash A \otimes (B \& C)$

Some switching cycles are legal

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,ax}{\vdash A^{\perp} \oplus A^{\perp}, A}\,\oplus_2}{\vdash A^{\perp} \oplus A^{\perp}, A \oplus A}\,\oplus_1 \qquad \dfrac{}{\vdash B, B^{\perp}}\,ax}{\vdash A^{\perp} \oplus A^{\perp}, (A \oplus A) \otimes B, B^{\perp}}\,\otimes
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,ax}{\vdash A^{\perp} \oplus A^{\perp}, A}\,\oplus_1}{\vdash A^{\perp} \oplus A^{\perp}, A \oplus A}\,\oplus_2 \qquad \dfrac{}{\vdash B, B^{\perp}}\,ax}{\vdash A^{\perp} \oplus A^{\perp}, (A \oplus A) \otimes B, B^{\perp}}\,\otimes
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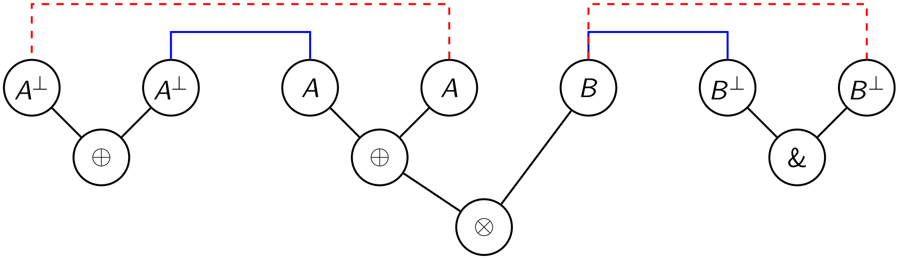
$$



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\dfrac{
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\dfrac{}{\vdash A^\perp, A}\,ax
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}{\vdash A^\perp \oplus A^\perp, A \oplus A}\,\oplus_1
\qquad
\dfrac{}{\vdash B, B^\perp}\,ax
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\qquad
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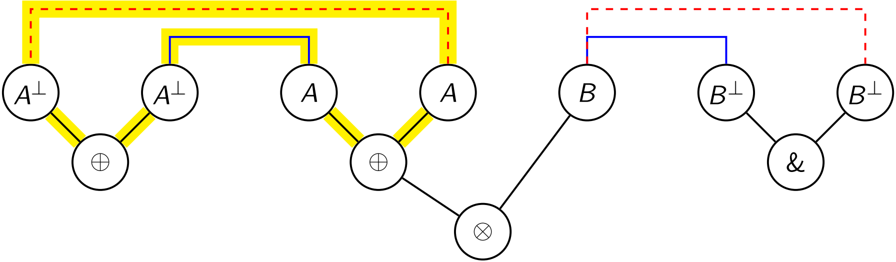
$$



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\AxiomC{}
\RightLabel{ax}
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\RightLabel{$\otimes$}
\BinaryInfC{$\vdash A^\perp \oplus A^\perp, (A \oplus A) \otimes B, B^\perp$}
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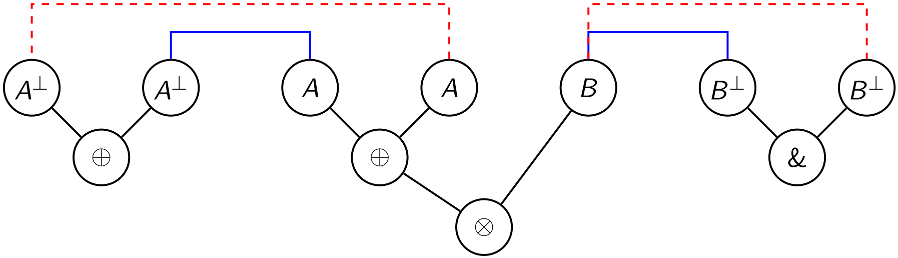

The $\otimes$ is not sequentializing!

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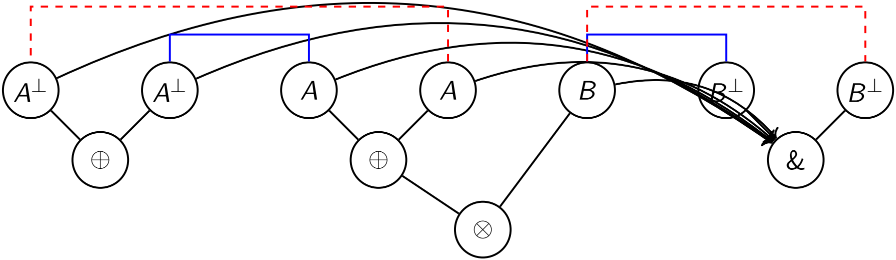

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$\longrightarrow$ jump edges

Correctness criterion (Hughes & Van Glabbeek)

Correctness criterion (HVG)

1. **Resolution**: ALL-criterion (exactly one linking by &-resolution)
2. **Danos-Regnier**: MLL-criterion (all linking give a MLL proof net)
3. **Toggling**: criterion for interaction between multiplicative and additive; forbids cycles caused by no &, which would be impossible to break

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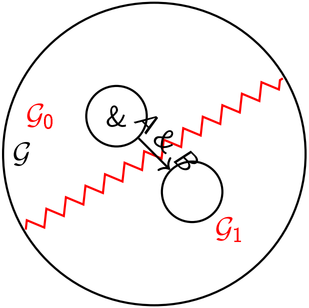
Jump edges: from an axiom to a & on which it depends
Toggling: for a set of ≥ 2 linkings, there is a & with both premises taken and in no switching cycle of these linkings

$A, B \in$ switching cycle $\iff$ in some slices A before B, in others B before
Correct if there is a & whose dependencies are independent of slices

Plan

1. Introduction
2. Sequentialization for MLL Proof Nets
   - Presentation of MLL
   - Sequentialization proof in MLL
3. **Sequentialization for MALL Proof Nets**
   - Presentation of MALL
   - **Sequentialization proof in MALL**
4. Conclusion

Sequentialization - Splitting ⅋\& lemma (HVG)



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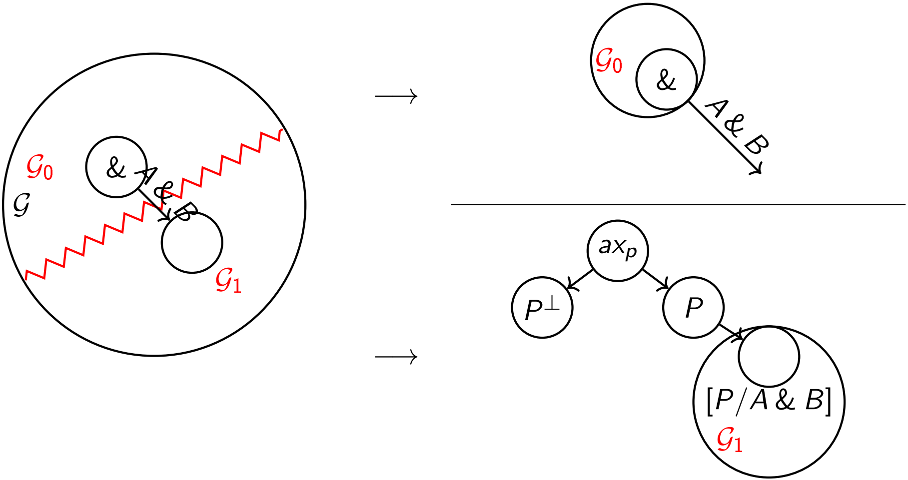

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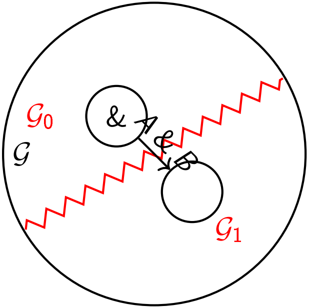
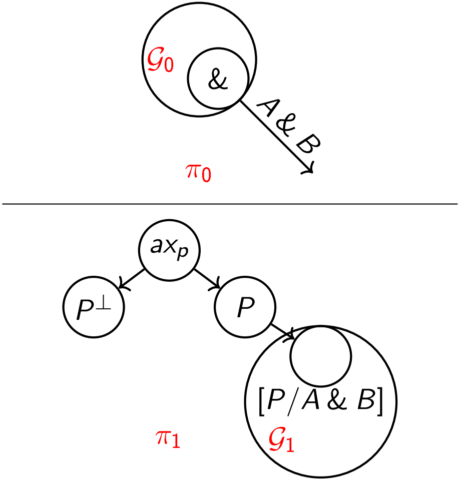


$$\pi \;=\; \pi_1[\pi_0/ax_P]$$

$\longrightarrow$



Sequentializing vertex - Updated

Sequentializing vertex

Terminal, non-leaf such that:

- $\parr\backslash\&$-vertex: always

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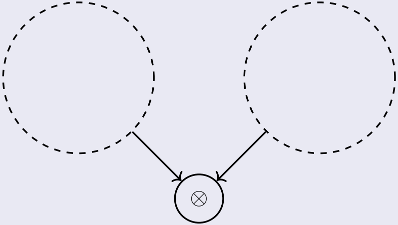
Sequentializing vertex

Terminal, non-leaf such that:

- $\parr\backslash\&$-vertex: always
- $\otimes$-vertex: remove it yields two connected components (with jumps)

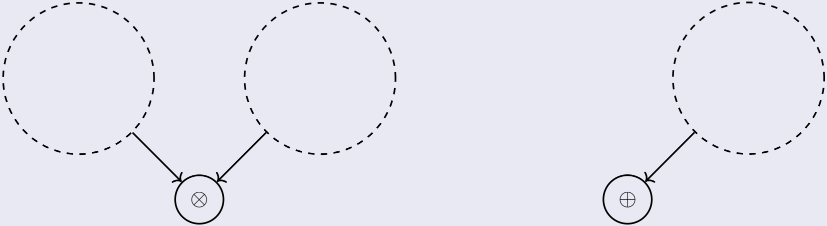


Sequentializing vertex - Updated

Sequentializing vertex

Terminal, non-leaf such that:

- $⅋\backslash\&$-vertex: always
- $\otimes$-vertex: remove it yields two connected components (with jumps)
- $\oplus$-vertex: no left or right formula tree for it



Correctness $\parr \backslash \&$

Terminal non-sequentializing $\otimes \backslash \oplus \longrightarrow$ correctness $\parr \backslash \&$

Finding them:

- Easy for $\oplus$, always depend on a $\&$
- $\otimes - \parr$ as in MLL
- $\otimes - \&$ harder: need to find a "good" slice, where the dependency is visible

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Correctness = dependency inside one slice
If no switching cycle then dependency inside all slices

Critical path

Critical path as in MLL . . . but not with all its properties

Switching cycles are now allowed $\implies$ may go back on our path

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Need to follow dependencies holding across all slices, to not run in cycles!
Simple solution: (almost) forbid switching cycles in the critical path

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Halting condition

Reaching a sequentializing vertex **or a switching cycle**

We need a way to continue our critical path through these cycles!

Going out of switching cycles - Aim

Lemma 4.32 in [HvG05]

Every non-empty union S of switching cycles of $\mathcal{G}_\theta$ has a jump out of it: for some leaf $l \in S$ and &-vertex $W \notin S$, there is a jump $l \to W$ in $\mathcal{G}_\theta$.

Given switching cycles, can find a & that created some of them

Proposition

Let γ be a maximal critical path, not ending on a sequentializing vertex. Then there exists a non-empty non-cyclic strong path $\xi = \gamma\chi$ such that:

- *χ is a non-empty non-cyclic strong path*
- *$t(\xi)$ is a &-vertex to which ξ arrives through a jump*
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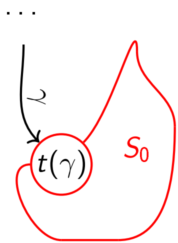
Going out of switching cycles - Proof

Start with S_0 the cycle at the end of the path
Invariant: S_i = connected union of switching cycles, $S_{i-1} \subsetneq S_i$
Result: connected union of switching cycles $\implies$ connected w.r.t. strong paths



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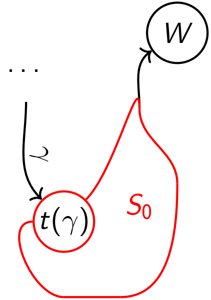

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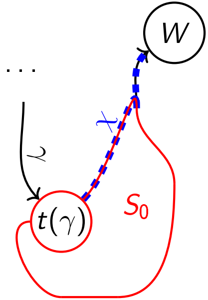

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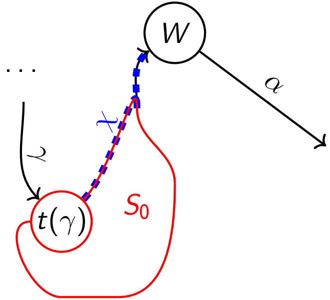

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3a. If no strong path α from W to $\gamma \cup S_i$: victory!

Going out of switching cycles - Proof

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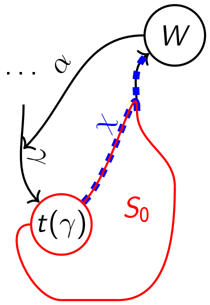

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Going out of switching cycles - Proof

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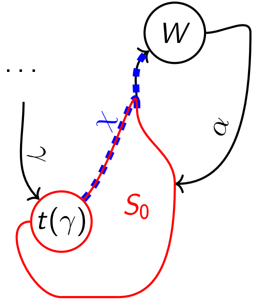

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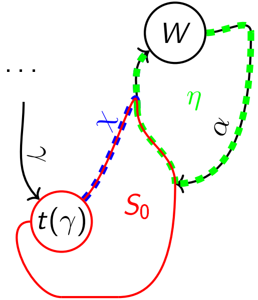

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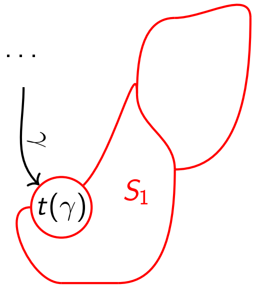

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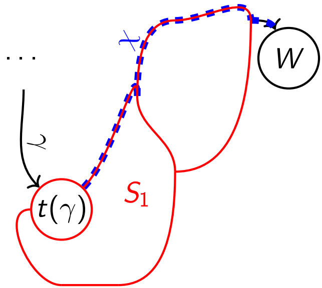

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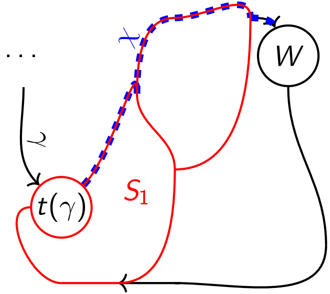

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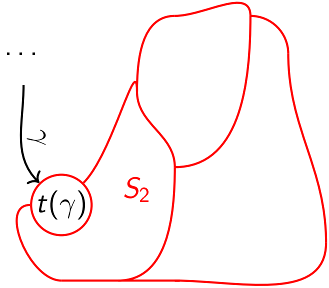

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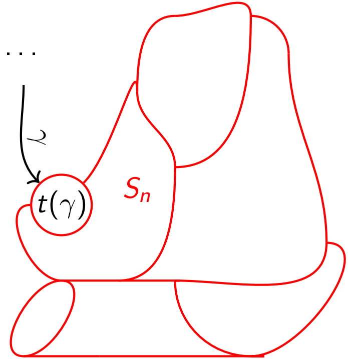

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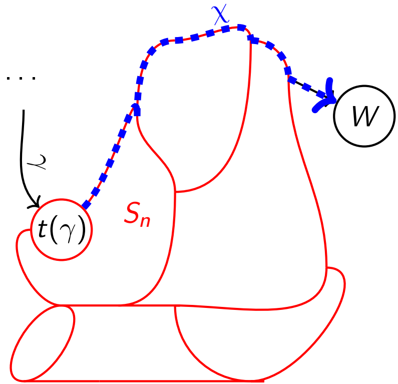

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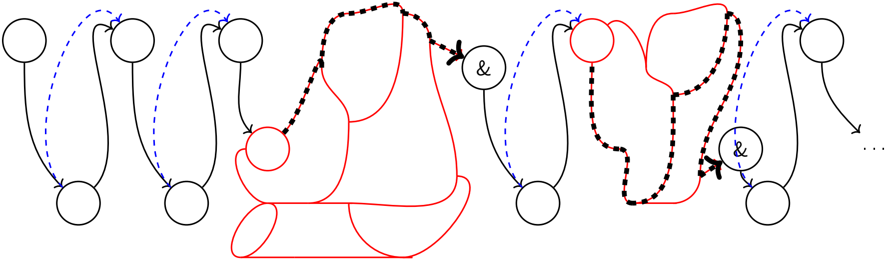
Global picture



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Sequentialization

Proof for MALL = proof for MLL + taking care of cycles

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Ongoing and future work:

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Questions?

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Multiplicative Additive Linear Logic - MALL

Formulas

$$A, B := X \mid X^\perp \mid A \otimes B \mid A \parr B \mid A \,\&\, B \mid A \oplus B$$

$$X^{\perp\perp} = X \qquad (A \otimes B)^\perp = A^\perp \parr B^\perp \qquad (A \parr B)^\perp = A^\perp \otimes B^\perp$$

$$(A \,\&\, B)^\perp = A^\perp \oplus B^\perp \qquad (A \oplus B)^\perp = A^\perp \,\&\, B^\perp$$

Sequents

$$\vdash A_1, \ldots, A_n$$

Rules

$$\frac{}{\vdash A^\perp, A}\; ax \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\; cut \qquad \frac{\Gamma}{\vdash \sigma(\Gamma)}\; ex(\sigma)$$

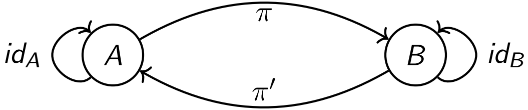
$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\; \otimes \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\; \parr$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\; \& \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\; \oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\; \oplus_2$$

Type isomorphisms in MALL

Linear isomorphism

Two formulas A and B are **isomorphic** if there exists proofs π of $\vdash A^\perp, B$ and π' of $\vdash B^\perp, A$ whose composition by cut over B (resp. A) is equal to the axiom on A (resp. B) up to cut elimination and axiom expansion.

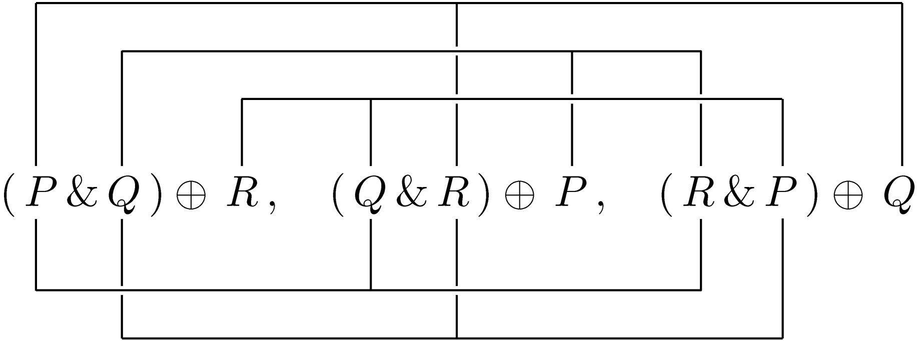


Commutativity		Associativity	
$A \otimes B = B \otimes A$	$A \parr B = B \parr A$	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$	$A \parr (B \parr C) = (A \parr B) \parr C$
$A \oplus B = B \oplus A$	$A \,\&\, B = B \,\&\, A$	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$	$A \,\&\, (B \,\&\, C) = (A \,\&\, B) \,\&\, C$

Multiplicative-additive distributivity	
$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$	$A \parr (B \,\&\, C) = (A \parr B) \,\&\, (A \parr C)$

Neutrality				Absorption	
$A \otimes 1 = A$	$A \parr \perp = A$	$A \oplus 0 = A$	$A \,\&\, \top = A$	$A \otimes 0 = 0$	$A \parr \top = \top$

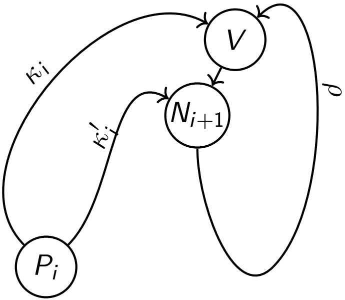
Toggling condition



$$R = P^{\perp} \otimes Q^{\perp}$$

Difficulties for sequentializing

- Not passing through cycles? Does not seem possible in every case
- Local / Global correction (N before P in one slice / all slices); switching cycles $\rightarrow$ correction &
- Problematic case:



- Finally: concentrate all difficulties in connected unions of switching cycles

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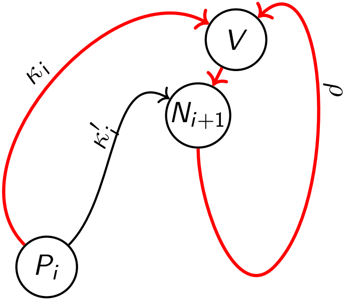

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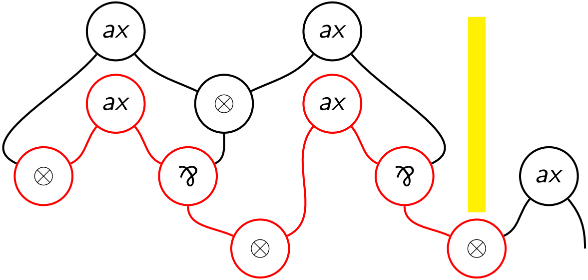
Leading example - Tree



$$
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