Sequentialization is as fun as bungee jumping

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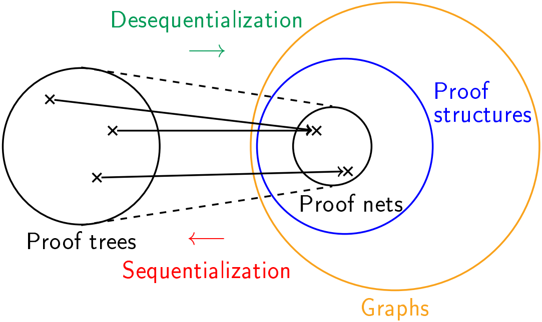
*Lyon, †Rome, ‡Marseille

TLLA 2023, 1 July 2023

Introduction

Proof nets: graphical, more canonical representation of LL proofs



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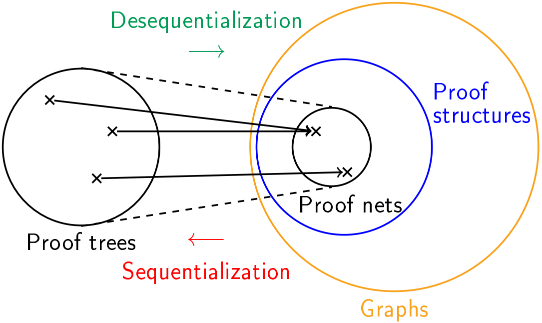

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multiple **correctness criteria**,
proofs of sequentialization

Still sequentialization is not considered easy.

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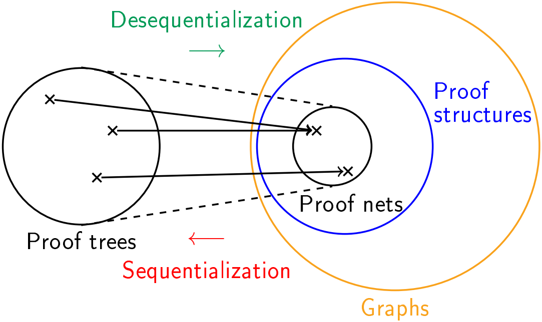

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multiple **correctness criteria**,
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Still sequentialization is not considered easy.

Objective of this talk: present a new simple proof of sequentialization

Outline

Unit-free Multiplicative Linear Logic with Mix

Formulas

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid A \parr A$$

Orthogonality

$$(X^{\perp})^{\perp} = X \qquad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$

Rules

$$\frac{}{\vdash A^{\perp}, A}\ (ax) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr)$$

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Example of proof structure by desequentialization

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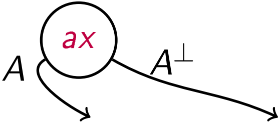

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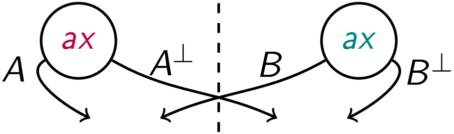

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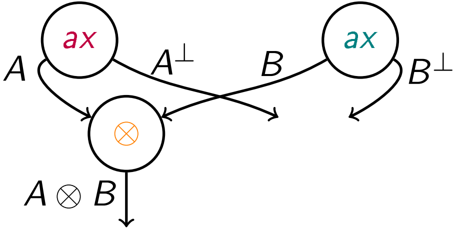

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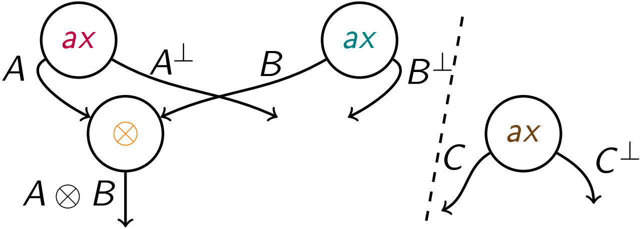

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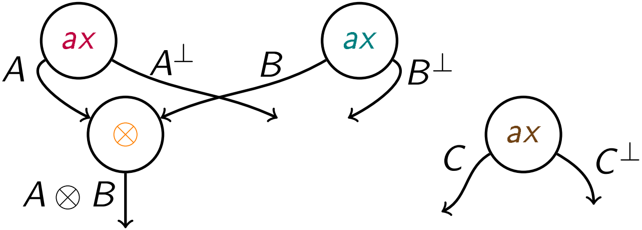

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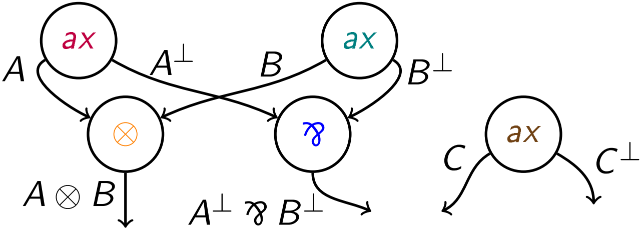

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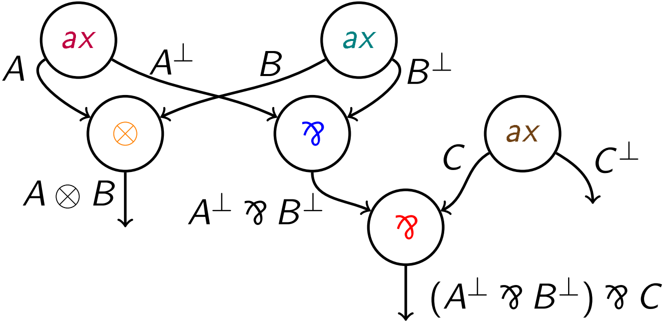

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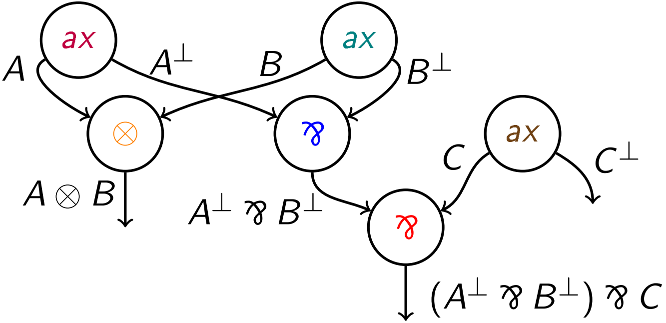


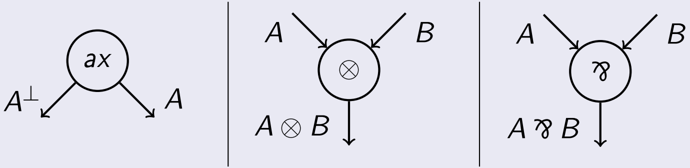
Proof structure

Definition

Partial labeled directed multigraph

vertices $\rightarrow$ ax / $\otimes$ / $⅋$

edges $\rightarrow$ formula



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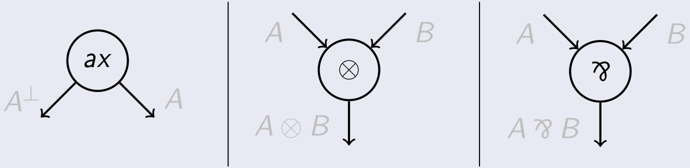

We will not care about edge labels.

Vocabulary on paths

We consider *non-oriented simple* paths.

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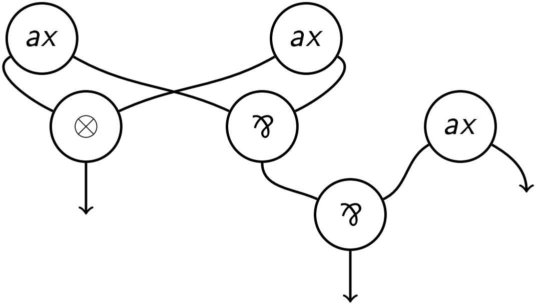

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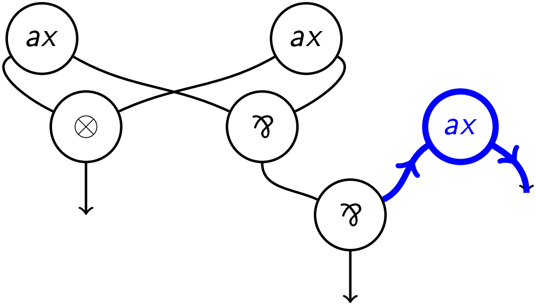
- *Switching* path: does not contain the two premises of any ⅋



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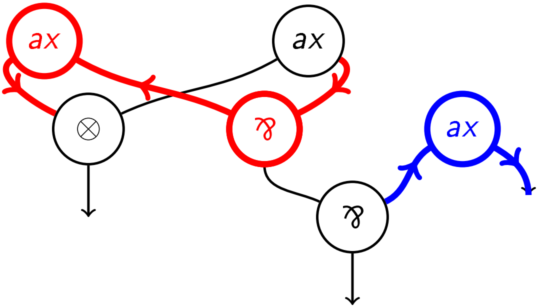

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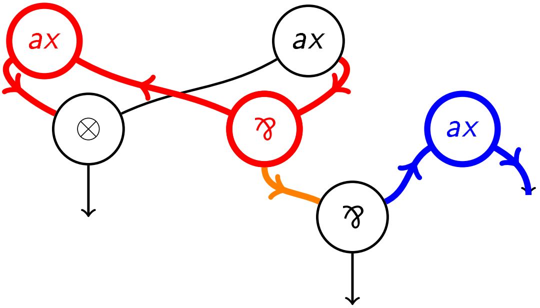

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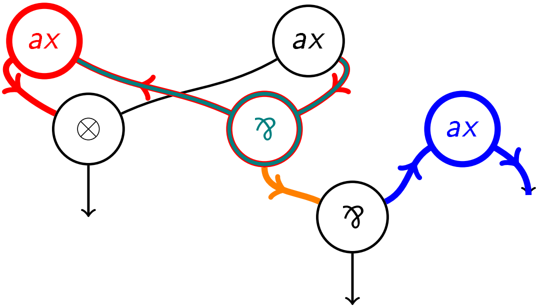

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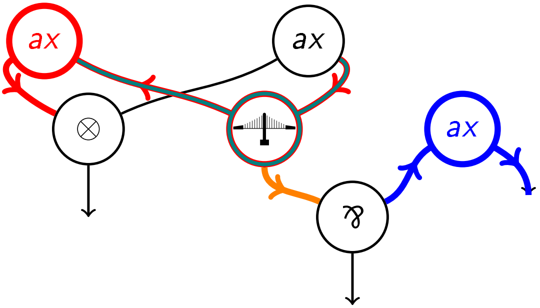
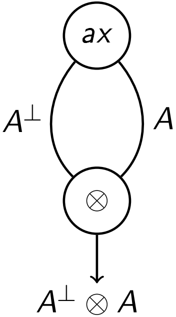


Correctness



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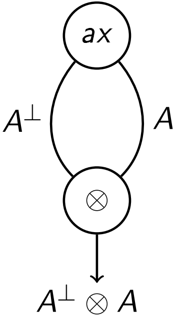
Danos-Regnier Correctness Criterion

A proof structure is *correct* if it does not contain any switching cycle.



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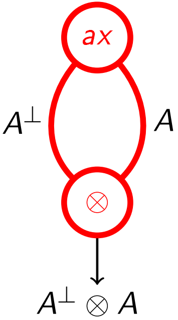

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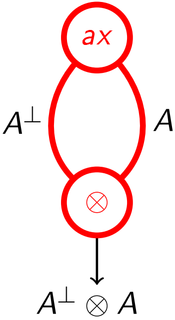

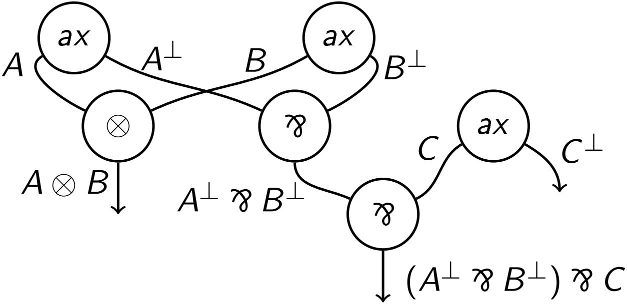

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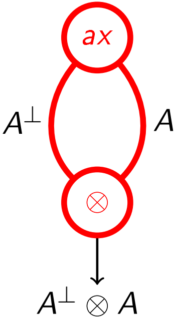

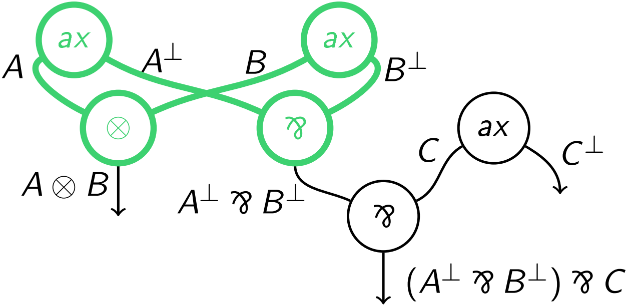

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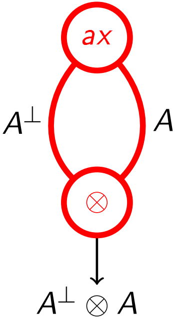
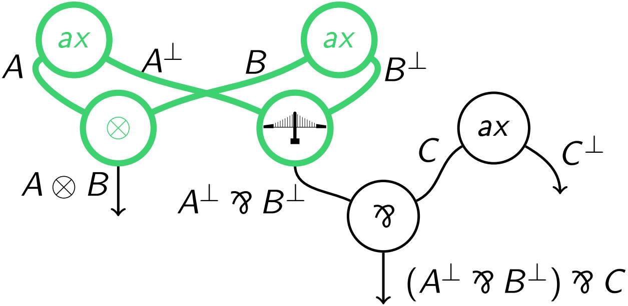




Outline

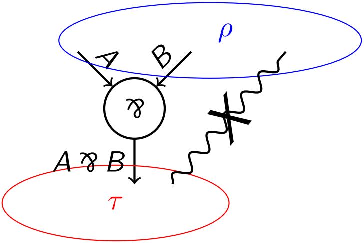
Destination Sequentialization

Sequentialization

Given a correct proof structure, there is a proof desequentializing to it.

Splitting ⅋-node

A ⅋-node is *splitting* if there is no cycle containing its conclusion.



Order $\prec$ on $⅋$-nodes

Definition

$v \prec u$ means v and u are distinct $⅋$-nodes and there is a path p such that:

(1) p is a strong-weak bridge-free path from v to u

(2) there is no strong bridge-free path starting from u and going back on p

Order $\prec$ on $\parr$-nodes

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Lemma

$\prec$ is a strict partial order relation.

Proof.

Irreflexivity: by definition. Transitivity: assume $v \prec u \prec w$.

(1) ?

(2) ?

□

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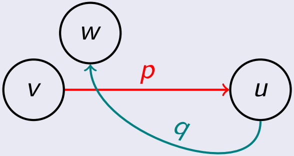

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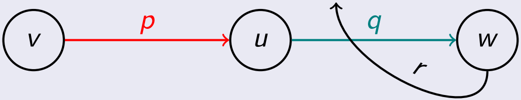

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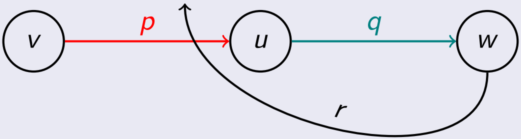


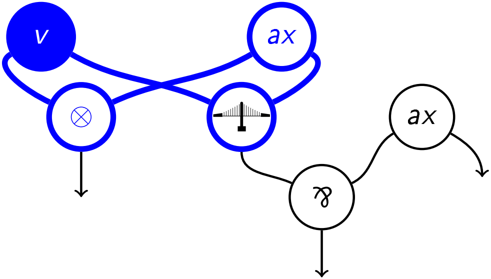
□

Bungee Jumping

Definition

For a vertex v, $\mathcal{M}_v$ is the set of cycles with source v, containing a conclusion of v, and with $\underbrace{\text{a minimal number of bridges}}_{\geq 1 \text{ by correctness}}$ among such cycles.

For v a $\parr$-node, $\mathcal{M}_v = \emptyset$ if and only if v is splitting.



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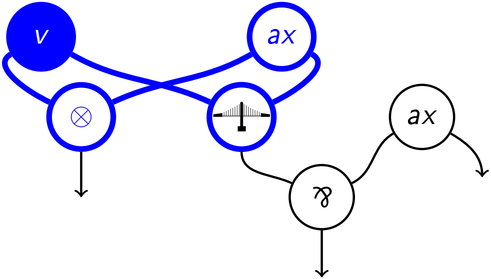
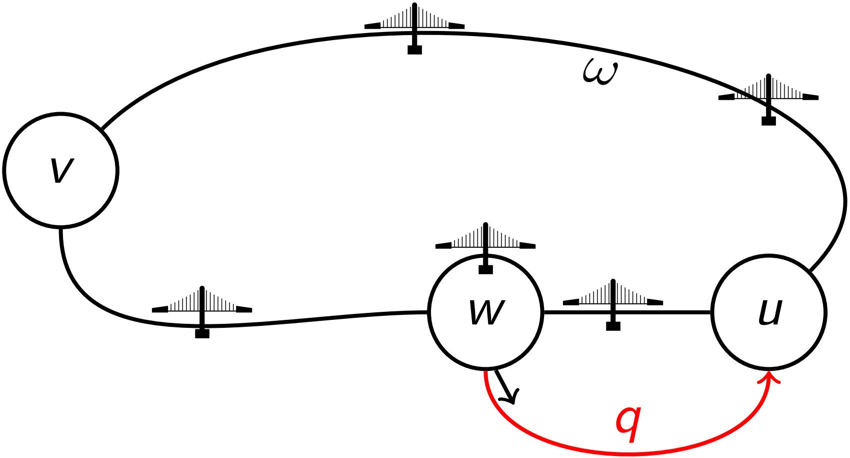


Bungee Jumping

Let ω be a cycle in $\mathcal{M}_v$. There is no strong bridge-free path q with source w the pier of a bridge of ω and going back on ω.

Proof of Bungee Jumping



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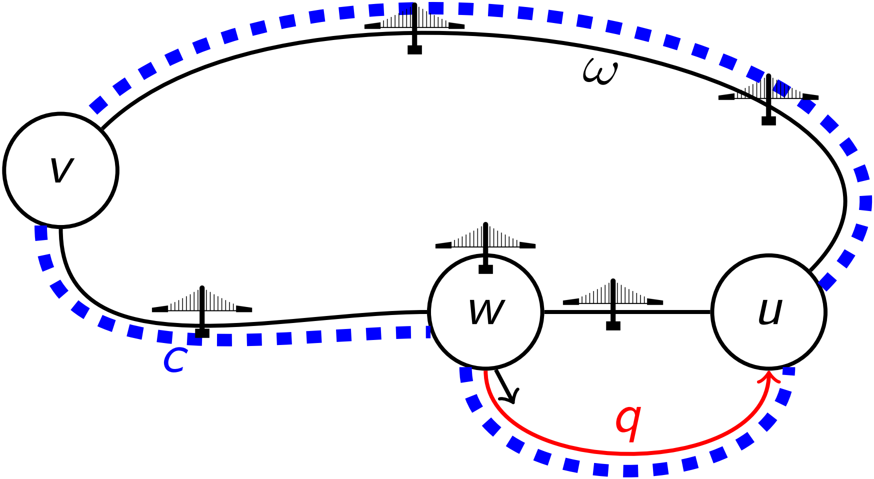

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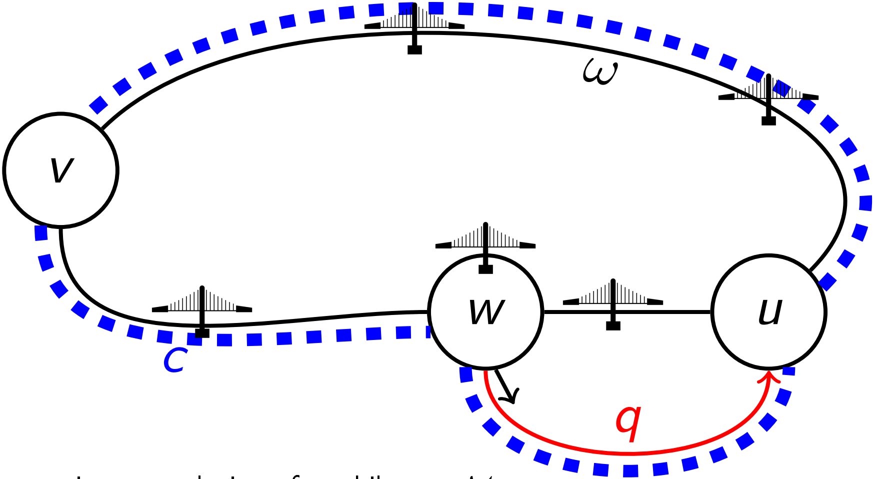

c contains a conclusion of v while $\omega \in \mathcal{M}_v$

$\implies$ c has at least as many bridges as ω

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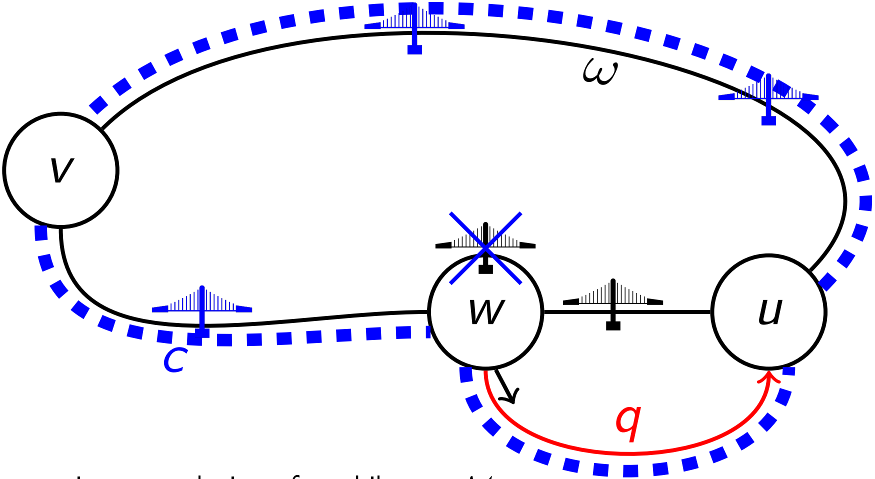

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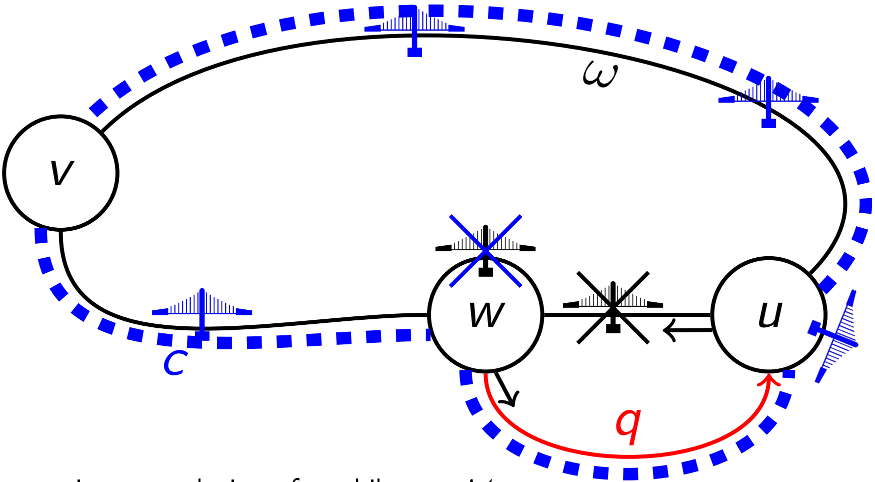

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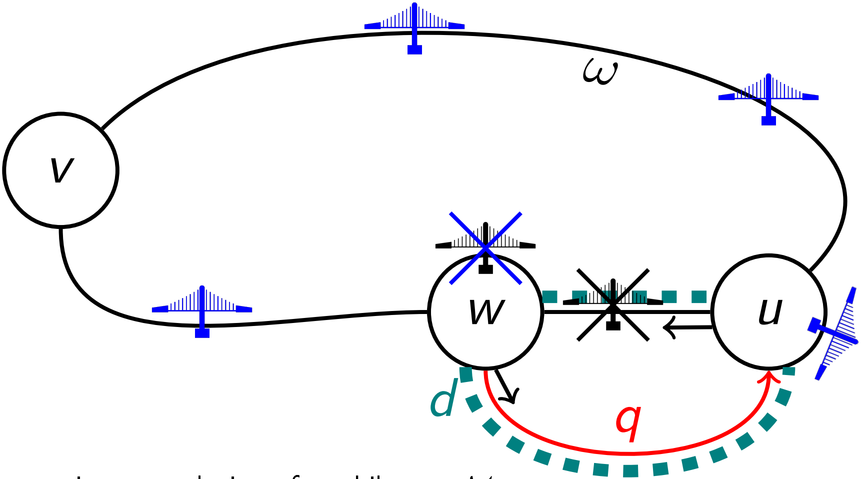
Proof of Bungee Jumping



c contains a conclusion of v while $\omega \in \mathcal{M}_v$

$\implies$ c has at least as many bridges as ω

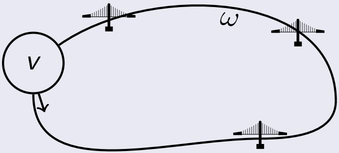
Splitting ⅋

Lemma

Let v be a non-splitting ⅋-node. There exists w such that $v \prec w$.

Proof.

Take $\omega \in \mathcal{M}_v \neq \emptyset$.



□

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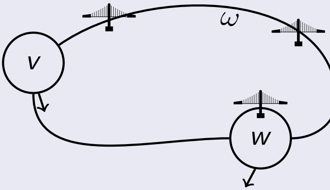
It contains some pier: set w the **first** one going from the conclusion of v.



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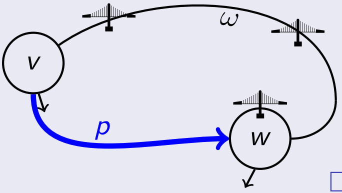

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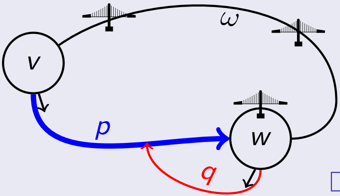

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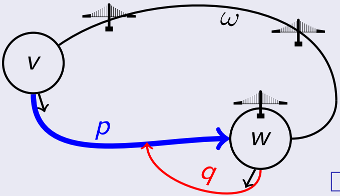


□

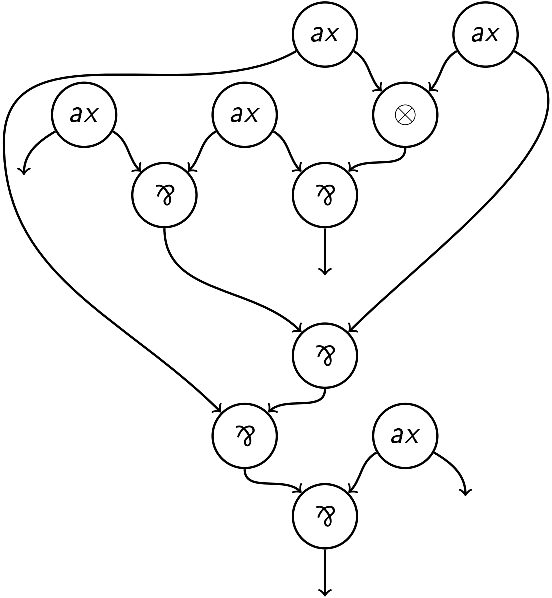
Splitting ⅋

A correct proof structure is ⅋-free or contains a splitting ⅋-node.

Proof.

A maximal ⅋ for the strict partial order $\prec$ is splitting. □

Finding a splitting ⅋ on an example



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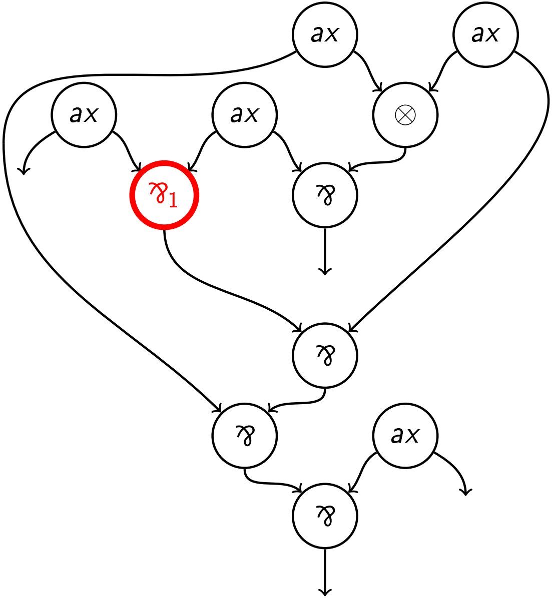

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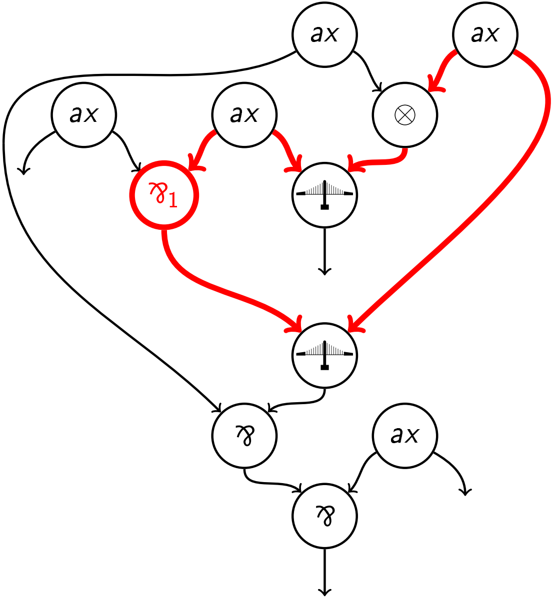

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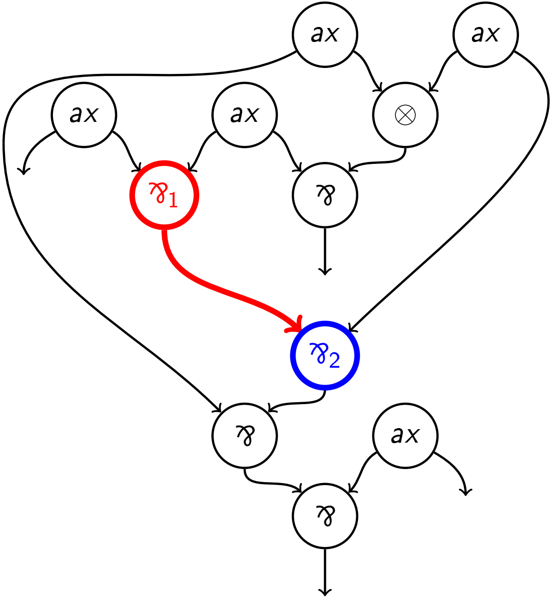

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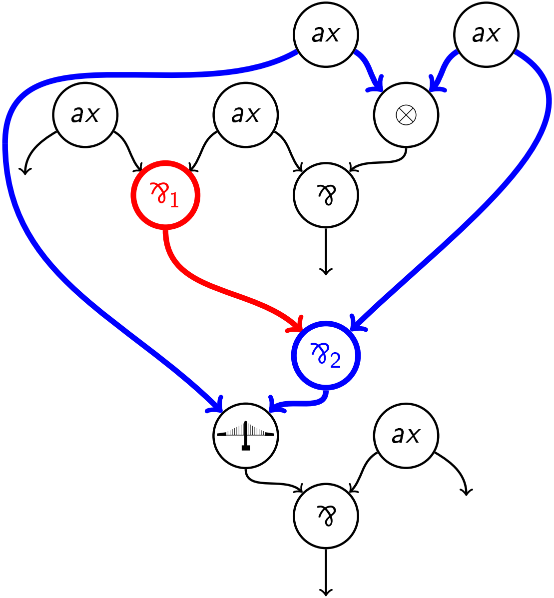

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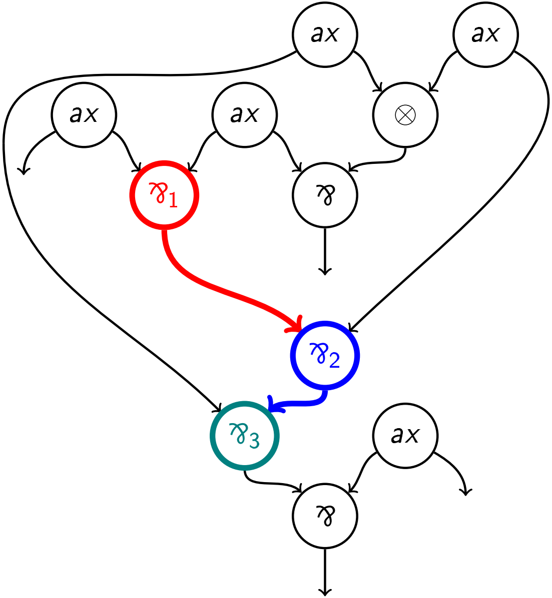

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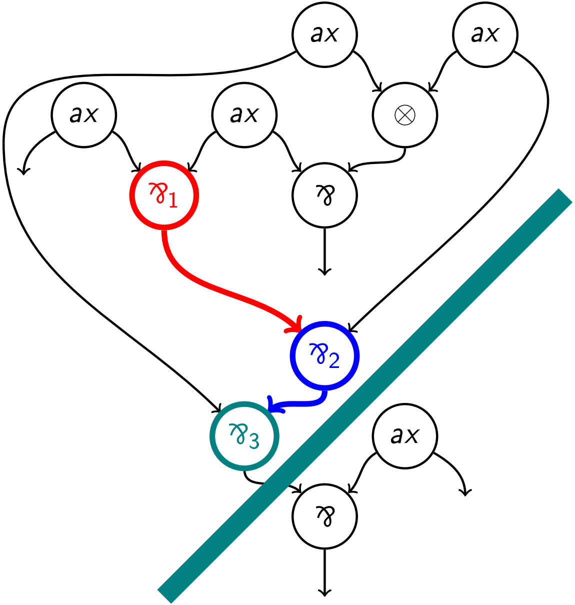

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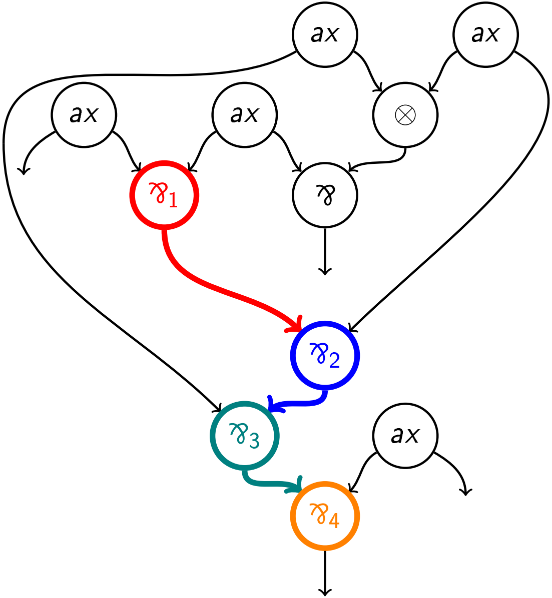

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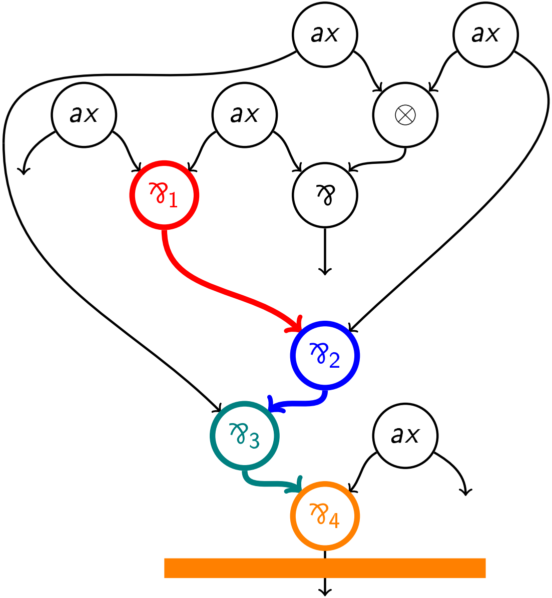

Conclusion

Sequentialization

Given a correct proof structure, there is a proof desequentializing to it.

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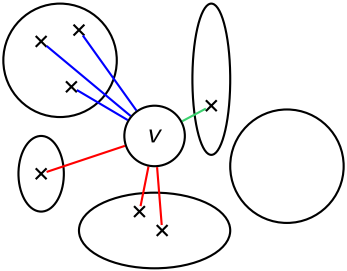
- Splitting $\parr$ formalized in Coq.
- Can be extended to proof nets with additives (from Hugues & Van Glabbeek [HvG05]).

Equivalent in graph theory [Ngu20]

This proof can be generalized to colored graphs.
Alternating cycle: with consecutive edges of different colors

Yeo's Theorem [Yeo97]

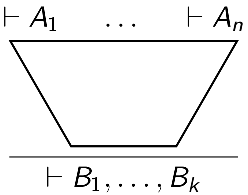
Let G be a (non-empty) edge-colored unoriented graph ***with no alternating cycle****. There exists a vertex v of G such that no component of $G - v$ is joint to v with edges of more than one color.*



Thank you!

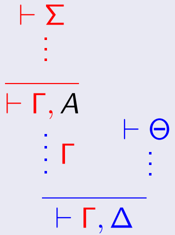
Open proofs

$$\frac{}{\vdash A}\ (hyp)$$

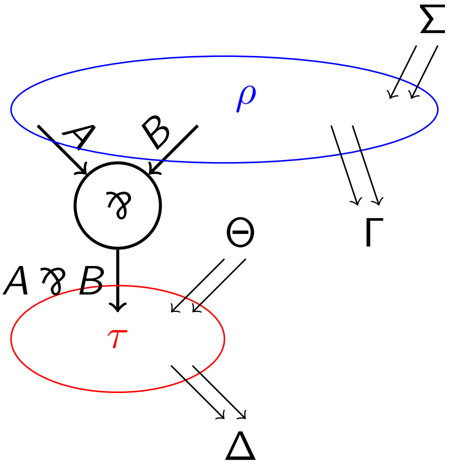


Substitution

$\pi_2 [\pi_1 / A] =$



Sequentialization with hypotheses



By induction:

- $\rho \to \pi$
- $\tau \to \sigma$

Proof of the whole structure:

$$\sigma\left[\frac{\pi}{\vdash A \parr B, \Gamma}\ (\parr) \Big/ A \parr B\right]$$

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