Identity of Proofs and Formulas using Proof-Nets in Multiplicative-Additive Linear Logic

Rémi Di Guardia

supervised by Olivier Laurent

23 September 2024

Cooking

Mousse Recipe

Ingredients: chocolate, eggs

1. Warm the chocolate
2. Beat the whites
3. Add the yolks in the chocolate
4. Add the whites in the mixture

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1. Get the eggs
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Another? Mousse Recipe

Ingredients: chocolate, eggs

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Same Mousse Recipe

Ingredients: chocolate, eggs

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$\longrightarrow$ The order of *independent* steps is meaningless for the result!
Different writings for a unique recipe

Another representation of recipes

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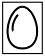

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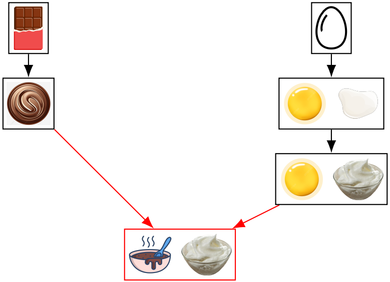

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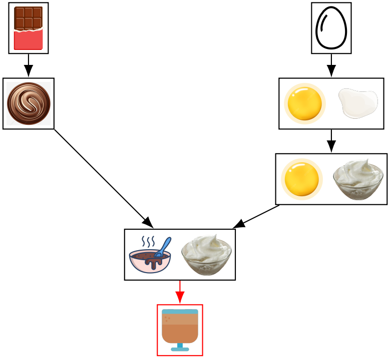

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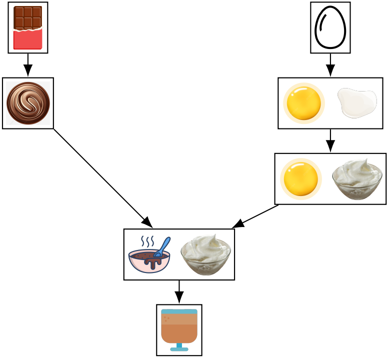

⟶ Better representation by following **causality**

Another representation of recipes

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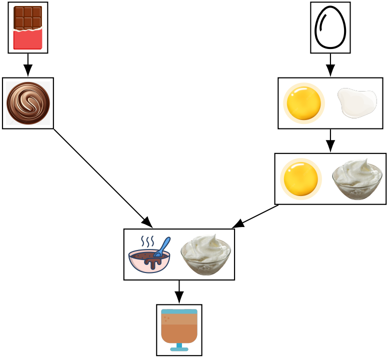

⟶ Better representation by following **causality**

Diagrams solve the equality of recipes?

Recipe 1

1. Get eggs
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Recipe 2

1. Get eggs
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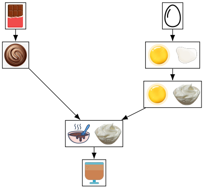
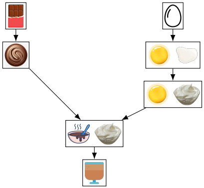
5. Beat whites
2. Get chocolate
3. Warm chocolate
6. Add yolks
7. Add whites





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Recipe 2

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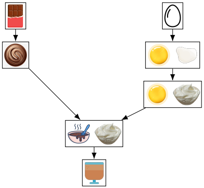
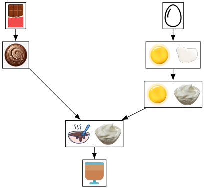
7 Add whites

Recipe 3

1 Get eggs
2 Get chocolate
3 Warm chocolate
4 Separate eggs
5 Beat whites
7 Add whites
6 Add yolks





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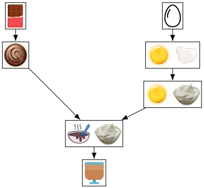

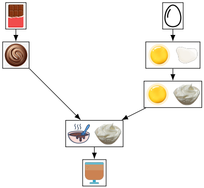

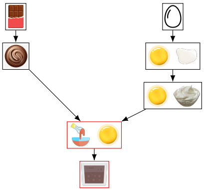

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Recipe 4

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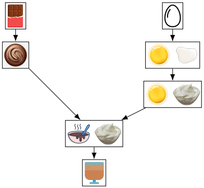
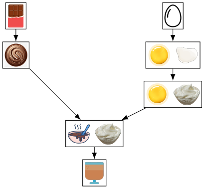
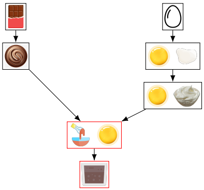
5 Beat whites
7 Add whites

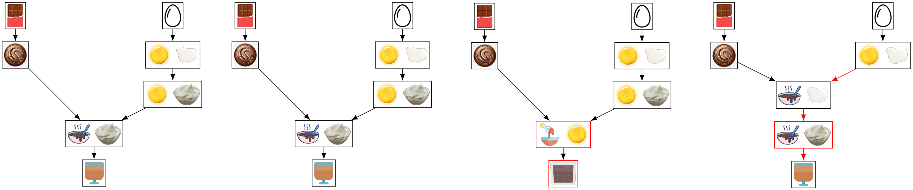






Diagrams solve the equality of recipes?

Recipe 1	Recipe 2	Recipe 3	Recipe 4
1 Get eggs	1 Get eggs	1 Get eggs	1 Get eggs
2 Get chocolate	4 Separate eggs	2 Get chocolate	2 Get chocolate
3 Warm chocolate	5 Beat whites	3 Warm chocolate	3 Warm chocolate
4 Separate eggs	2 Get chocolate	4 Separate eggs	4 Separate eggs
5 Beat whites	3 Warm chocolate	5 Beat whites	6 Add yolks
6 Add yolks	6 Add yolks	7 Add whites	5 Beat whites
7 Add whites	7 Add whites	6 Add yolks	7 Add whites



⟶ Some order in diagrams is still meaningless for causality!
Some **commutations** give different graphs but are the **same** recipe

Yet another representation of recipes

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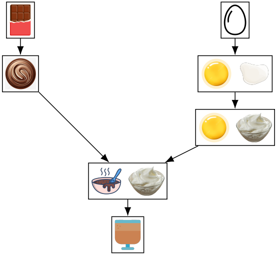

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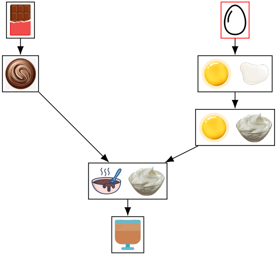

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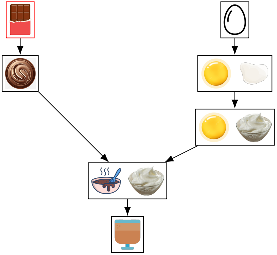

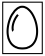

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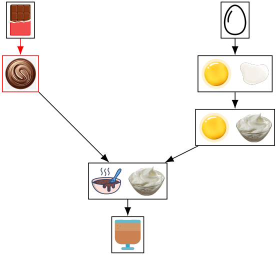

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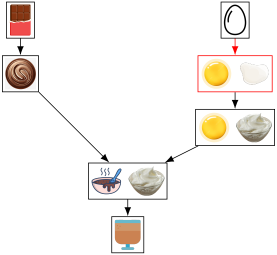

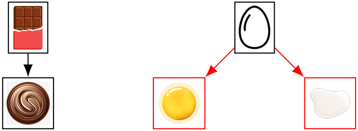

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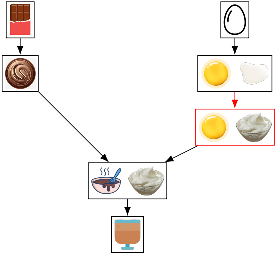

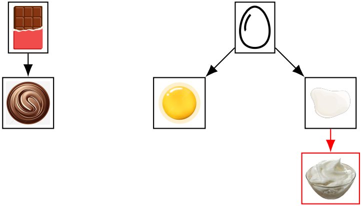

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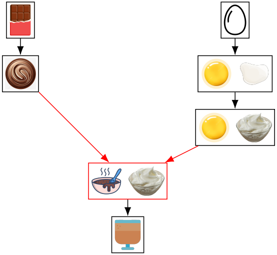

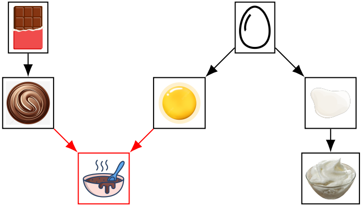

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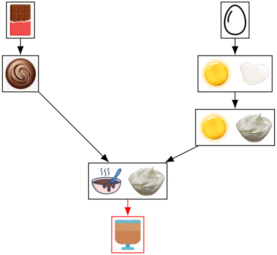

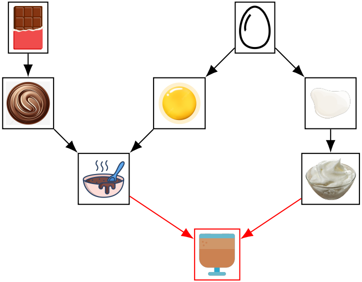

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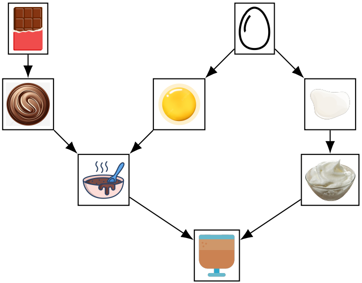

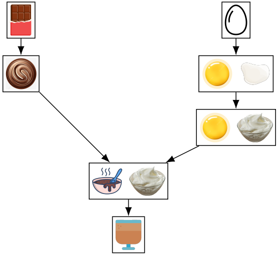

⟶ Parallelize everything
The remaining order is causality!
Unique writing for a recipe: **canonicity**

Yet another representation of recipes

Recipe 4

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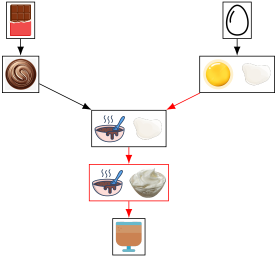

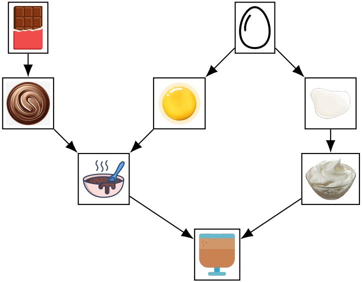

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Three representations of recipes or proofs

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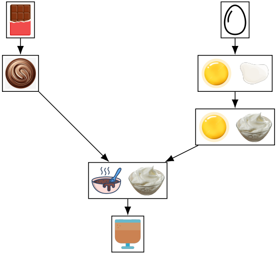

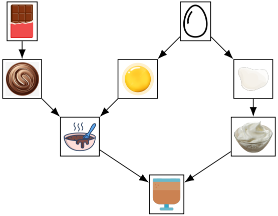

Three representations of recipes or proofs

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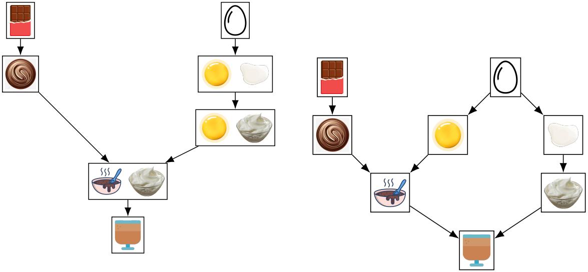

Hilbert System

1. E
2. C
3. hC
4. Y W
5. bW
6. P
7. M

Three representations of recipes or proofs

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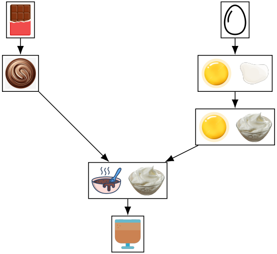

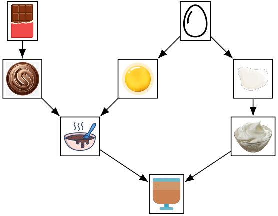

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Sequent Calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash hC} \qquad \dfrac{\dfrac{\dfrac{}{E \vdash E}}{E \vdash Y\ W}}{E \vdash Y\ bW}}{C\ E \vdash P\ bW}}{C\ E \vdash M}$$

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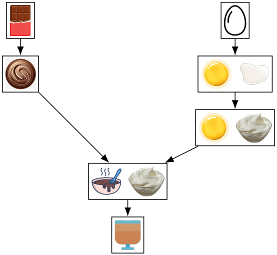

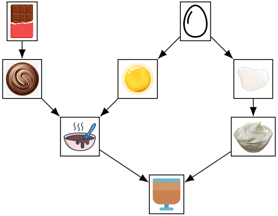

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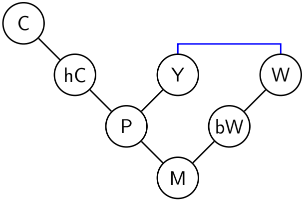
Proof-Net



Formulas and Connectives

Mousse Recipe

Ingredients: chocolate, eggs

$\vdots$

Recipe: produce mousse from chocolate and eggs

Formulas and Connectives

Mousse Recipe

Ingredients: chocolate, eggs

⋮

C and E

Formulas and Connectives

Mousse Recipe

Ingredients: chocolate, eggs

⋮

$C \wedge E$

$\wedge$ and, take both

Formulas and Connectives

Dark or Milk Mousse

Ingredients: dark or milk chocolate, eggs

⋮

$(dC \vee mC) \wedge E$

$\wedge$ and, take both

$\vee$ or, take one

Formulas and Connectives

Organic or Regular, Dark or Milk Mousse

Ingredients: dark or milk chocolate, organic or regular eggs

⋮

$$(dC \vee mC) \wedge (oE \vee rE)$$

$\wedge$ and, take both

$\vee$ or, take one

Formulas and Connectives

Organic or Regular, Dark or Milk Mousse

Ingredients: dark or milk chocolate, organic or regular eggs

⋮

$$(dC \vee mC) \otimes (oE \vee rE)$$

$\otimes$ and, take both

$\vee$ or, take one

Formulas and Connectives

Mousses at a Restaurant

Desserts

Dark Chocolate Mousse

or

Milk Chocolate Mousse

(organic or regular eggs according to supplies)

$$(dC \vee mC) \otimes (oE \vee rE)$$

- $\otimes$ and, take both
- $\vee$ or, take one

Formulas and Connectives

Mousses at a Restaurant

Desserts

Dark Chocolate Mousse

or

Milk Chocolate Mousse

(organic or regular eggs according to supplies)

$$(dC \mathbin{\&} mC) \otimes (oE \oplus rE)$$

- $\otimes$ and, take both
- $\&$ or, **you** choose one
- $\oplus$ or, **someone else** chooses one

Equality of Formulas

- $\otimes$ and, take both
- $\&$ or, **you** choose one
- $\oplus$ or, **someone else** chooses one

- compare $C \otimes E$ and $E \otimes C$

Equality of Formulas

$\otimes$ and, take both

$\&$ or, **you** choose one

$\oplus$ or, **someone else** chooses one

- compare $C \otimes E$ and $E \otimes C$
- compare $C \otimes (Y \otimes W)$ and $(C \otimes Y) \otimes W$

Equality of Formulas

$\otimes$ and, take both

$\&$ or, **you** choose one

$\oplus$ or, **someone else** chooses one

- compare $C \otimes E$ and $E \otimes C$
- compare $C \otimes (Y \otimes W)$ and $(C \otimes Y) \otimes W$
- compare $C \otimes (oE \oplus rE)$ and $(C \otimes oE) \oplus (C \otimes rE)$

Equality of Formulas

$\otimes$ and, take both

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- compare $C \otimes E$ and $E \otimes C$
- compare $C \otimes (Y \otimes W)$ and $(C \otimes Y) \otimes W$
- compare $C \otimes (oE \oplus rE)$ and $(C \otimes oE) \oplus (C \otimes rE)$

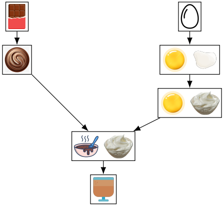
Again, **different syntaxes / writings** for a **same** underlying object

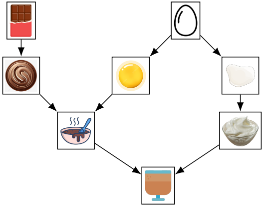
Transparent ways to go from one formula to the other, without losses
$\rightarrow$ isomorphism

$$
\begin{aligned}
C \otimes E &\simeq E \otimes C && \text{(associativity)} \\
C \otimes (Y \otimes W) &\simeq (C \otimes Y) \otimes W && \text{(commutativity)} \\
C \otimes (oE \oplus rE) &\simeq (C \otimes oE) \oplus (C \otimes rE) && \text{(distributivity)}
\end{aligned}
$$

Recipe 1

1. Get eggs
2. Get chocolate
3. Warm chocolate
4. Separate eggs
5. Beat whites
6. Add yolks
7. Add whites





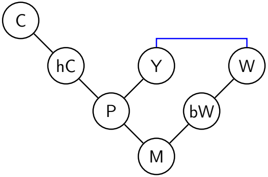
Hilbert System

1. E
2. C
3. hC
4. Y W
5. bW
6. P
7. M

Sequent Calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash hC} \qquad \dfrac{\dfrac{\dfrac{}{E \vdash E}}{E \vdash Y\ W}}{E \vdash Y\ bW}}{C\ E \vdash P\ bW}}{C\ E \vdash M}$$

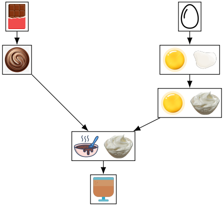
Proof-Net

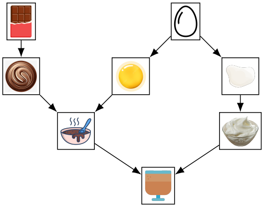


Multiplicative-Additive Linear Logic

Recipe 1

1. Get eggs
2. Get chocolate
3. Warm chocolate
4. Separate eggs
5. Beat whites
6. Add yolks
7. Add whites





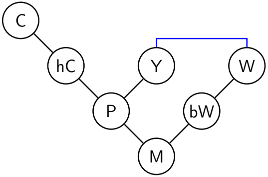
Hilbert System

1. E
2. C
3. hC
4. Y W
5. bW
6. P
7. M

Sequent Calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash C}}{C \vdash hC} \qquad \dfrac{\dfrac{\dfrac{}{E \vdash E}}{E \vdash Y\ W}}{E \vdash Y\ bW}}{C\ E \vdash P\ bW}}{C\ E \vdash M}$$

Proof-Net



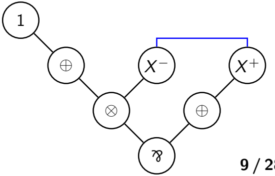
Hilbert System

1. X^-, X^+
2. 1
3. $1 \oplus H$
4. $X^-, B \oplus X^+$
5. $((1 \oplus H) \otimes X^-), B \oplus X^+$
6. $((1 \oplus H) \otimes X^-) \parr (B \oplus X^+)$

Sequent Calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}}{\vdash 1 \oplus H} \qquad \dfrac{\dfrac{}{\vdash X^-, X^+}}{\vdash X^-, B \oplus X^+}}{\vdash (1 \oplus H) \otimes X^-, B \oplus X^+}}{\vdash ((1 \oplus H) \otimes X^-) \parr (B \oplus X^+)}$$

Proof-Net



Multiplicative-Additive Linear Logic

$$A, B ::= \overbrace{X^+ \mid X^-}^{\textit{atoms}} \mid \overbrace{A \underset{\textit{and}}{\otimes} B \mid A \underset{\textit{or}}{\parr} B \mid \underset{\textit{true}}{1} \mid \underset{\textit{false}}{\bot}}^{\textit{multiplicative}}$$

$$\mid \underbrace{A \overset{\textit{and}}{\&} B \mid A \overset{\textit{or}}{\oplus} B \mid \overset{\textit{true}}{\top} \mid \overset{\textit{false}}{0}}_{\textit{additive}}$$

Hilbert System

1. X^-, X^+
2. 1
3. $1 \oplus H$
4. $X^-, B \oplus X^+$
5. $((1 \oplus H) \otimes X^-), B \oplus X^+$
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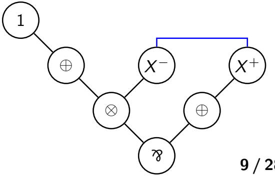
Sequent Calculus

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}}{\vdash 1 \oplus H} \qquad \dfrac{\dfrac{}{\vdash X^-, X^+}}{\vdash X^-, B \oplus X^+}}{\vdash (1 \oplus H) \otimes X^-, B \oplus X^+}}{\vdash ((1 \oplus H) \otimes X^-) \parr (B \oplus X^+)}$$

Proof-Net



Multiplicative-Additive Linear Logic

$$A, B \;::=\; \overbrace{X^{+} \mid X^{-}}^{\textit{atoms}} \mid \overbrace{A \underset{\textit{and}}{\otimes} B \mid A \underset{\textit{or}}{⅋} B \mid \underset{\textit{true}}{1} \mid \underset{\textit{false}}{\bot}}^{\textit{multiplicative}}$$

$$\mid \underbrace{A \overset{\textit{and}}{\&} B \mid A \overset{\textit{or}}{\oplus} B \mid \overset{\textit{true}}{\top} \mid \overset{\textit{false}}{0}}_{\textit{additive}}$$

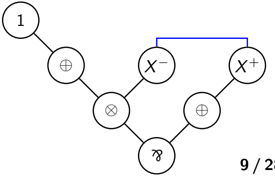
Hilbert System

1. X^{-}, X^{+}
2. 1
3. $1 \oplus H$
4. $X^{-}, B \oplus X^{+}$
5. $((1 \oplus H) \otimes X^{-}), B \oplus X^{+}$
6. $((1 \oplus H) \otimes X^{-}) ⅋ (B \oplus X^{+})$

Sequent Calculus

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}}{\vdash 1 \oplus H} \qquad \dfrac{\dfrac{}{\vdash X^{-}, X^{+}}}{\vdash X^{-}, B \oplus X^{+}}}{\dfrac{\vdash (1 \oplus H) \otimes X^{-}, B \oplus X^{+}}{\vdash ((1 \oplus H) \otimes X^{-}) ⅋ (B \oplus X^{+})}}$$

Proof-Net



Plan

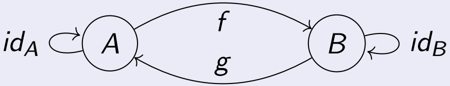
- Isomorphisms in Multiplicative-Additive Linear Logic

- Retractions in Multiplicative Linear Logic

Isomorphisms

$C \otimes E$ behaves the same as $E \otimes C$ $\rightarrow$ isomorphism $C \otimes E \simeq E \otimes C$

In category theory: isomorphism $A \simeq B$

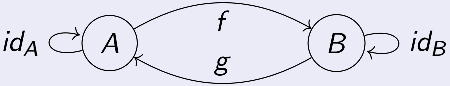


$$g \circ f = id_A$$
$$f \circ g = id_B$$

Isomorphisms

$C \otimes E$ behaves the same as $E \otimes C$ $\rightarrow$ isomorphism $C \otimes E \simeq E \otimes C$

In category theory: isomorphism $A \simeq B$



$$g \circ f = id_A$$
$$f \circ g = id_B$$

In λ-calculus: isomorphism $A \simeq B$

Terms $M : A \rightarrow B$ and $N : B \rightarrow A$ such that

$$\lambda x^A.N(Mx) =_{\beta\eta} \lambda x^A.x \qquad \text{and} \qquad \lambda y^B.M(Ny) =_{\beta\eta} \lambda y^B.y$$

Isomorphisms

$C \otimes E$ behaves the same as $E \otimes C$ $\rightarrow$ isomorphism $C \otimes E \simeq E \otimes C$

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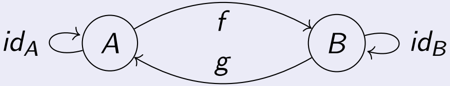

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In (linear) logic: isomorphism $A \simeq B$

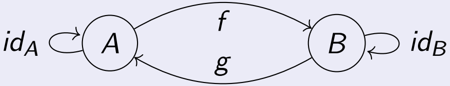
Proofs π of $A \vdash B$ and ϕ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\phi}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\phi}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta} \dfrac{}{B \vdash B}\,(ax)$$

Isomorphisms

$C \otimes E$ behaves the same as $E \otimes C$ $\to$ isomorphism $C \otimes E \simeq E \otimes C$

In category theory: isomorphism $A \simeq B$



$$g \circ f = id_A$$
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In λ-calculus: isomorphism $A \simeq B$

Terms $M : A \to B$ and $N : B \to A$ such that

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Proofs π of $A \vdash B$ and ϕ of $B \vdash A$ such that

$$\frac{\overset{\pi}{A \vdash B} \quad \overset{\phi}{B \vdash A}}{A \vdash A}\ (cut) =_{\beta\eta} \frac{}{A \vdash A}\ (ax) \quad \text{and} \quad \frac{\overset{\phi}{B \vdash A} \quad \overset{\pi}{A \vdash B}}{B \vdash B}\ (cut) =_{\beta\eta} \frac{}{B \vdash B}\ (ax)$$

rule commutations $\overset{r}{\vdash\!\dashv} \subseteq =_{\beta\eta}$

Literature on isomorphisms

Goal: obtain an equational theory

Syntactic Method

Analyze pairs of proofs of isos
$\rightarrow$ *get information on their formulas*

Semantic Method

Find a model with the same isos but where computation/equality is easy

Literature on isomorphisms

Goal: obtain an equational theory

Syntactic Method	Semantic Method
Analyze pairs of proofs of isos $\rightarrow$ *get information on their formulas*	*Find a model with the same isos but where computation/equality is easy*

λ-calculus with products and unit type
Cartesian closed categories

Semantic (finite sets) [Sol83]

$A \times (B \times C) \simeq (A \times B) \times C$	$A \times B \simeq B \times A$	$1 \times A \simeq A$
$(A \times B) \rightarrow C \simeq A \rightarrow (B \rightarrow C)$		$1 \rightarrow A \simeq A$
$A \rightarrow (B \times C) \simeq (A \rightarrow B) \times (A \rightarrow C)$		$A \rightarrow 1 \simeq 1$

Reduces to Tarski's High School Algebra Problem:
can all equalities involving product, exponential and 1 be found using only

$$a(bc) = (ab)c \quad ab = ba \quad 1a = a$$
$$c^{ab} = (c^b)^a \qquad a^1 = a$$
$$(bc)^a = b^a c^a \qquad 1^a = 1$$

Literature on isomorphisms

Goal: obtain an equational theory

Syntactic Method

Analyze pairs of proofs of isos
$\rightarrow$ get information on their formulas

Semantic Method

Find a model with the same isos but where computation/equality is easy

$\begin{cases}\text{Multiplicative Linear Logic}\\ \star\text{-autonomous categories}\end{cases}$ Syntactic (proof-nets) [BD99]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \parr (B \parr C) \simeq (A \parr B) \parr C$
Commutativity	$A \otimes B \simeq B \otimes A \quad A \parr B \simeq B \parr A$
Neutrality	$A \otimes 1 \simeq A \quad A \parr \bot \simeq A$

Literature on isomorphisms

Goal: obtain an equational theory

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Analyze pairs of proofs of isos
$\rightarrow$ get information on their formulas

Semantic Method

Find a model with the same isos but where computation/equality is easy

$\begin{cases}\text{Polarized Linear Logic} \\ \text{Control Categories}\end{cases}$ Semantic (games, forest isomorphisms) [Lau05]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A ⅋ (B ⅋ C) \simeq (A ⅋ B) ⅋ C$		$A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A ⅋ B \simeq B ⅋ A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$	$A ⅋ \bot \simeq A$	$A \oplus 0 \simeq A$	$A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A ⅋ (B \& C) \simeq (A ⅋ B) \& (A ⅋ C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A ⅋ \top \simeq \top$	
Seely	$!(A \& B) \simeq !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq ?A ⅋ ?B$ $?0 \simeq \bot$	

Literature on isomorphisms

<u>Goal:</u> obtain an equational theory

Syntactic Method

Analyze pairs of proofs of isos
→ get information on their formulas

Semantic Method

Find a model with the same isos but where computation/equality is easy

{ Polarized Linear Logic
{ Control Categories

Semantic (games, forest isomorphisms) [Lau05]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \parr (B \parr C) \simeq (A \parr B) \parr C$		$A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
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~~$\top \otimes A$~~

Literature on isomorphisms

Goal: obtain an equational theory

Syntactic Method

Analyze pairs of proofs of isos
→ get information on their formulas

Semantic Method

Find a model with the same isos but where computation/equality is easy

{ Multiplicative-Additive Linear Logic
{ ⋆-autonomous categories with finite products

Syntactic [thesis]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \parr (B \parr C) \simeq (A \parr B) \parr C$		$A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
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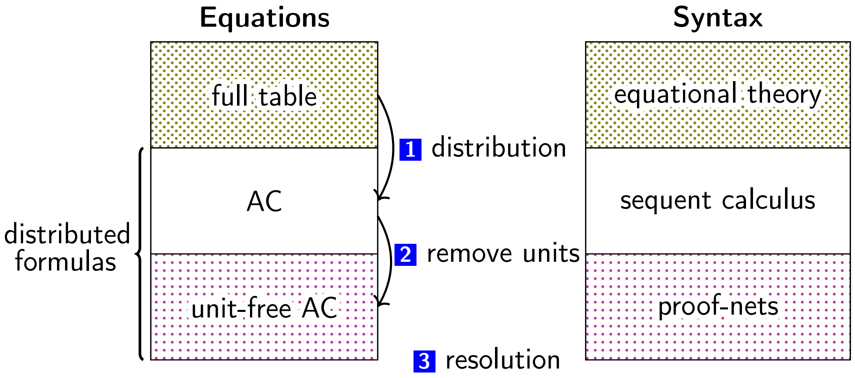
Seely	$!(A \,\&\, B) \simeq !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq ?A \parr ?B$ $?0 \simeq \bot$	

Proof sketch

Syntactic method:

1. Simplify using the **distributivity** equations *(rewriting theory)*
2. Remove the units *(sequent calculus)*
3. Analyze the shape of isomorphisms to conclude *(proof-nets)*



Proof 1/3: Distribution

Distributed Formula

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A ⅋ (B ⅋ C) \simeq (A ⅋ B) ⅋ C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A ⅋ B \simeq B ⅋ A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \rightarrow A$	$A ⅋ \bot \rightarrow A$	$A \oplus 0 \rightarrow A$	$A \& \top \rightarrow A$
Distributivity	$A \otimes (B \oplus C) \rightarrow (A \otimes B) \oplus (A \otimes C)$		$A ⅋ (B \& C) \rightarrow (A ⅋ B) \& (A ⅋ C)$	
Annihilation	$A \otimes 0 \rightarrow 0$		$A ⅋ \top \rightarrow \top$	

Proof 1/3: Distribution

Distributed Formula

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \to A$	$A \parr \bot \to A$	$A \oplus 0 \to A$	$A \& \top \to A$
Distributivity	$A \otimes (B \oplus C) \to (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) \to (A \parr B) \& (A \parr C)$	
Annihilation	$A \otimes 0 \to 0$		$A \parr \top \to \top$	

Proposition

$\mathcal{E}$ *complete for* ***distributed*** *formulas*

$\Downarrow$

$\mathcal{E}$ + *Neut.* + *Dist.* + *Anni.* *complete for* ***all*** *formulas*

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
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Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) \simeq (A \parr B) \& (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	

Proof 2/3: Remove the units

In isomorphisms of *distributed* formulas: units = fresh atoms

1. In the identity: $\dfrac{}{\vdash \top, 0}\ (\top)$

$$\dfrac{\dfrac{}{\vdash 1}\ (1)}{\vdash \bot, 1}\ (\bot)$$

Proof 2/3: Remove the units

In isomorphisms of *distributed* formulas: units = fresh atoms

1. In the identity:

$$\frac{}{\vdash \top, 0}\,(\top) \quad \longrightarrow \quad \frac{}{\vdash X^-, X^+}\,(ax)$$

$$\frac{\frac{}{\vdash 1}\,(1)}{\vdash \bot, 1}\,(\bot) \quad \longrightarrow \quad \frac{}{\vdash Y^-, Y^+}\,(ax)$$

Proof 2/3: Remove the units

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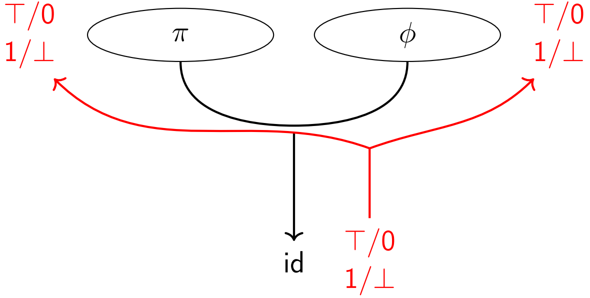

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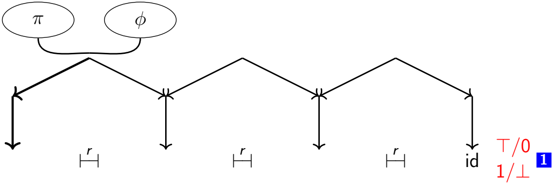

Proof 2/3: Remove the units

In isomorphisms of *distributed* formulas: units = fresh atoms

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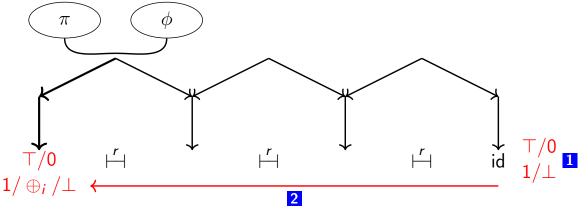
   $\dfrac{\dfrac{}{\vdash 1}\ (1)}{\vdash \bot, 1}\ (\bot) \quad \longrightarrow \quad \dfrac{}{\vdash Y^-, Y^+}\ (ax)$

2. Shape preserved by $\overset{r}{\vdash\!\dashv}$: $\dfrac{\dfrac{\dfrac{}{\vdash 1}\ (1)}{\vdash F}\ \oplus_i}{\vdash \bot, F}\ (\bot)$ using *distributivity*



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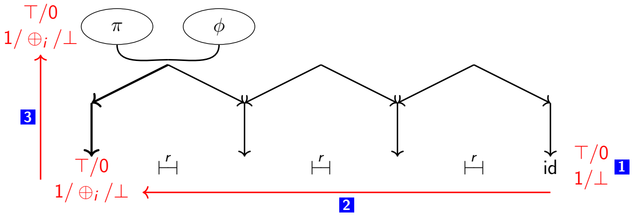
   $$\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash \bot, 1}\,(\bot) \quad \longrightarrow \quad \dfrac{}{\vdash Y^-, Y^+}\,(ax)$$

2. Shape preserved by $\overset{r}{\vdash\!\dashv}$: $\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash F}\,\oplus_i}{\vdash \bot, F}\,(\bot)$ using *distributivity*
3. Cut-elimination in isomorphisms cannot "completely" erase units rules



Proof 2/3: Remove the units

In isomorphisms of *distributed* formulas: units = fresh atoms

1. In the identity: $\dfrac{}{\vdash \top, 0}\,(\top) \longrightarrow \dfrac{}{\vdash X^-, X^+}\,(ax)$

   $\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash \bot, 1}\,(\bot) \longrightarrow \dfrac{}{\vdash Y^-, Y^+}\,(ax)$

2. Shape preserved by $\overset{r}{\longmapsto}$: $\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash F}\,\oplus_i}{\vdash \bot, F}\,(\bot)$ using *distributivity*

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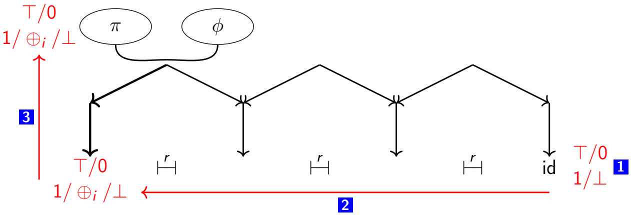
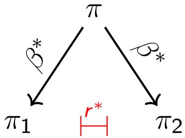


⟹ No more units!

Parenthesis: Confluence up to

Needed:



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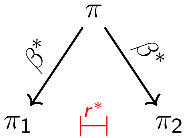
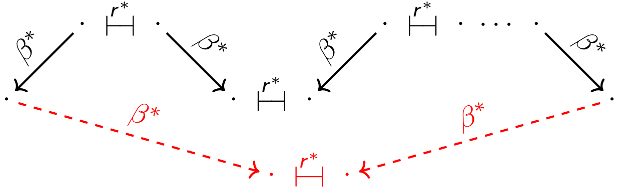


Generalization: Church-Rosser modulo



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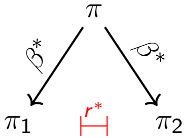

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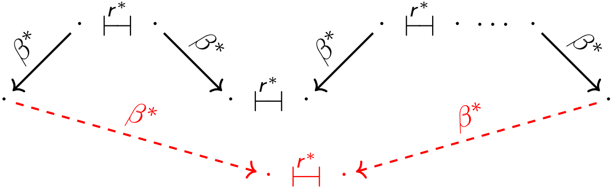

$$\Longrightarrow \quad MALL/=_{\beta\eta} = MALL/\overset{r}{\longmapsto} = \text{proof-nets}$$

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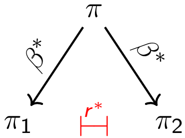

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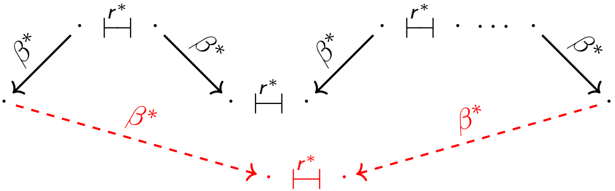
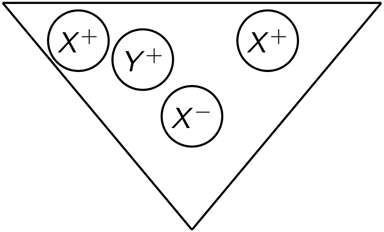


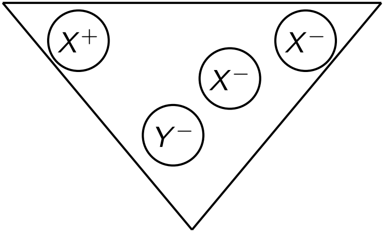
$$\Longrightarrow \quad MALL/ =_{\beta\eta} = MALL/ \overset{r}{\vdash\!\dashv} \;=\; \text{proof-nets}$$

- Already proved for MALL [CP05]
- We reproved it by showing **Strong Normalization** and using a theorem from rewriting theory [Hue80]

Proof 3/3: Shape of distributed isomorphisms

Use **proof-nets**





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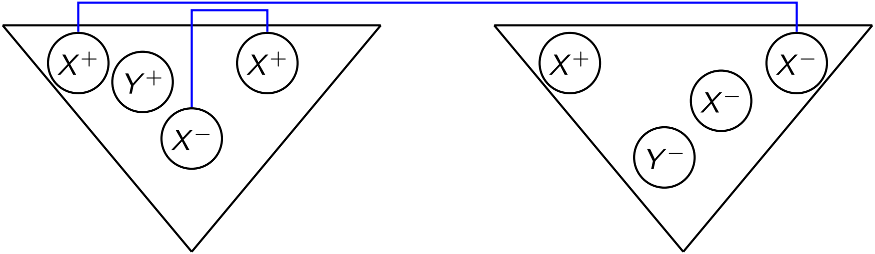

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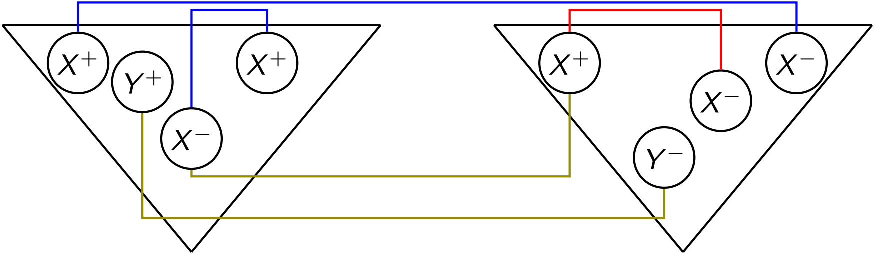
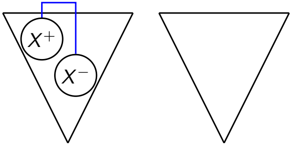
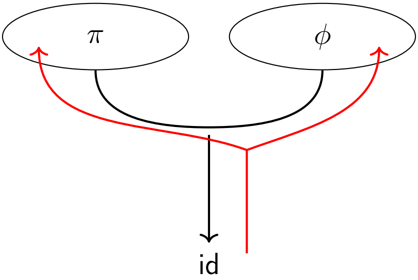


Proof 3/3: Shape of distributed isomorphisms

Forbidden configurations in distributed isomorphisms:





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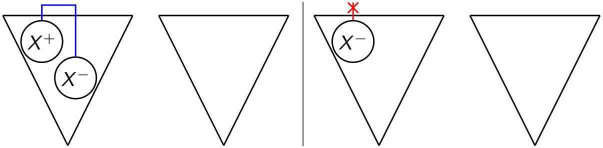

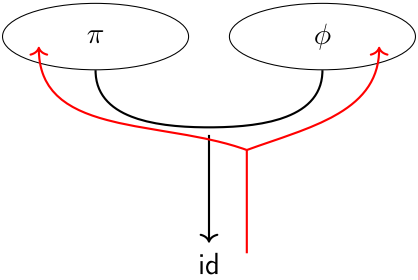

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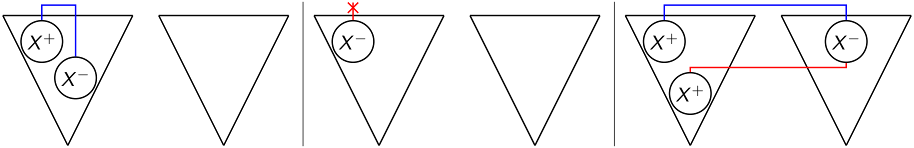

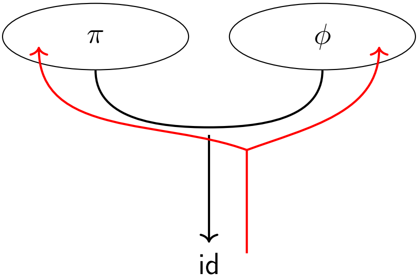

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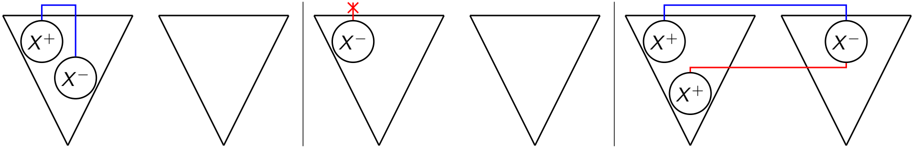
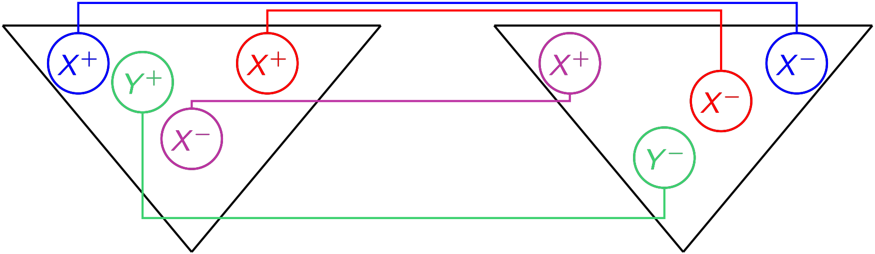


General shape:



$\longrightarrow$ only reordering = AC!

Plan

- **Isomorphisms in Multiplicative-Additive Linear Logic**

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \parr (B \parr C) \simeq (A \parr B) \parr C$		$A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) \simeq (A \parr B) \& (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	

- Retractions in Multiplicative Linear Logic

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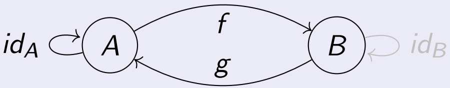
- **Retractions in Multiplicative Linear Logic**

Retractions

In category theory: retraction $A \trianglelefteq B$



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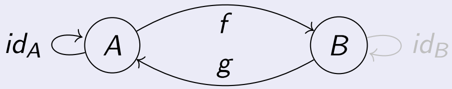

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$f \circ g = id_B$

$\longrightarrow$ Natural notion of sub-typing

Example

bool $\trianglelefteq$ nat with $f(b) := \begin{cases} 0 & \text{if } b \\ 1 & \text{otherwise} \end{cases}$ and $g(n) := \begin{cases} \mathit{true} & \text{if } n \geq 1 \\ \mathit{false} & \text{otherwise} \end{cases}$

Retractions

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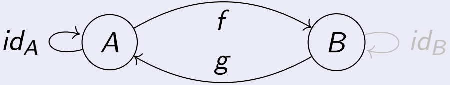

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In (linear) logic: retraction $A \trianglelefteq B$

Proofs π of $A \vdash B$ and ϕ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\phi}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\phi}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta} \dfrac{}{B \vdash B}\,(ax)$$

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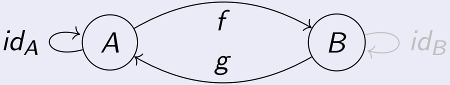

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$$A \trianglelefteq B \iff A^{\perp} \trianglelefteq B^{\perp}$$

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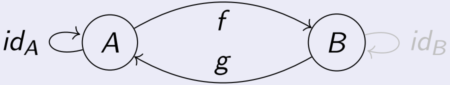

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$$A \trianglelefteq B \iff A^{\perp} \trianglelefteq B^{\perp}$$

Example: Beffara's retraction

$$A \trianglelefteq A \otimes (A^{\perp} \parr A) \qquad \text{also} \qquad \underset{f\ b}{\overset{a}{A}} \ \underset{\longleftarrow}{\overset{\longrightarrow}{\trianglelefteq}} \ A \otimes \underset{(b,f)}{\overset{(a,id)}{(A \multimap A)}}$$

Simplifications in MLL

Syntactic method:

1. Simplify using the **neutrality** equations *(rewriting theory)*
2. Remove the units *(sequent calculus)*
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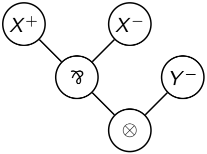
$\longrightarrow$ Problem purely in MLL **proof-nets**!

Proof-nets for MLL

- **Formula**

$$A, B \ ::= \ \mid X^+ \mid X^- \\ \mid A \otimes B \mid A \parr B$$



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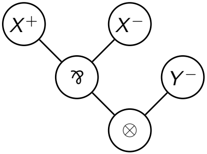
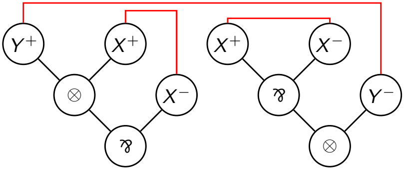


- **Proof Structure**
  formulas +
  axioms partitionning atoms



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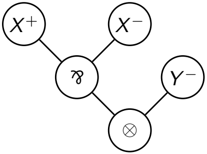

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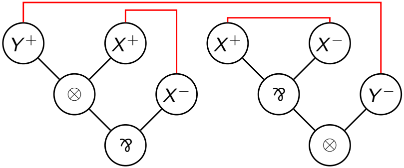
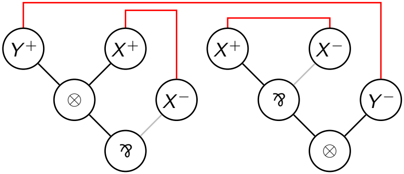


- **Danos-Regnier Criterion**
  <u>acyclic</u> and <u>connected</u>
  correctness graphs



Retractions in proof-nets

In MLL Proof-Nets: retraction $A \trianglelefteq B$

Proof-nets $\mathcal{R}$ of $\vdash A^{\perp}, B$ and $\mathcal{S}$ of $\vdash B^{\perp}, A$ such that
$\mathcal{R}$ cut with $\mathcal{S}$ reduces to id

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General case difficult:

$$R_4 = X_1^+ \otimes X_2^+ \otimes X_3^+ \otimes X_4^+ \trianglelefteq (X_1^+ \otimes X_2^+ \otimes X_3^+ \otimes X_4^+) \parr (X_1^+ \otimes (X_1^- \parr (X_2^+ \otimes (X_2^- \parr (X_3^+ \otimes (X_3^- \parr (X_4^+ \otimes X_4^-)))))))$$

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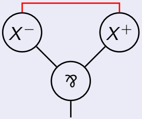
In MLL Proof-Nets: atomic retraction $X^+ \trianglelefteq B$

Proof-nets $\mathcal{R}$ of $\vdash X^-, B$ and $\mathcal{S}$ of $\vdash B^\perp, X^+$ ~~such that~~
~~$\mathcal{R}$ cut with $\mathcal{S}$ reduces to id~~

Key Result: finding a shape

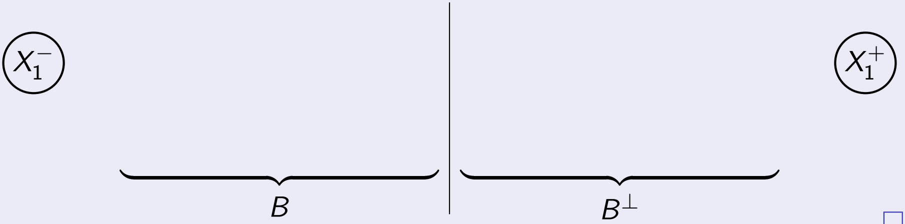
Lemma

In $X^+ \trianglelefteq B$ one of the two proof-nets contains



Proof.

Follow a GOI path until finding it



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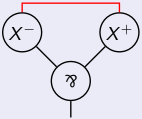

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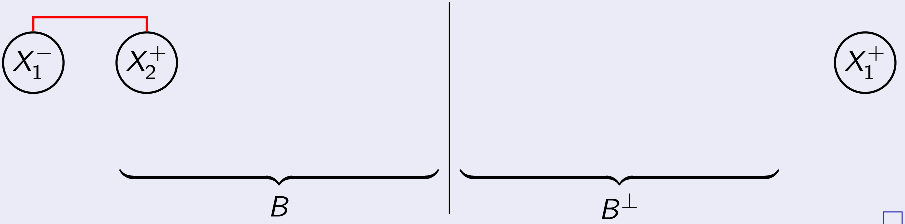

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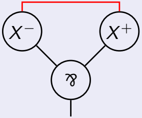

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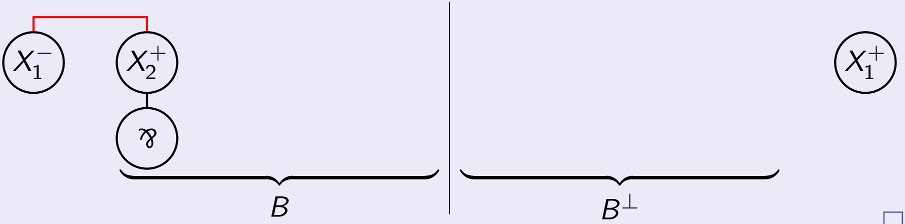

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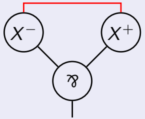

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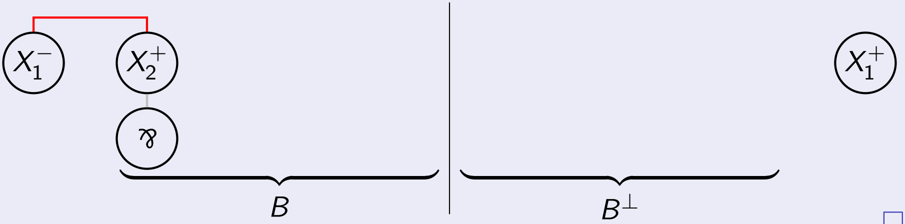

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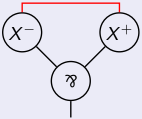

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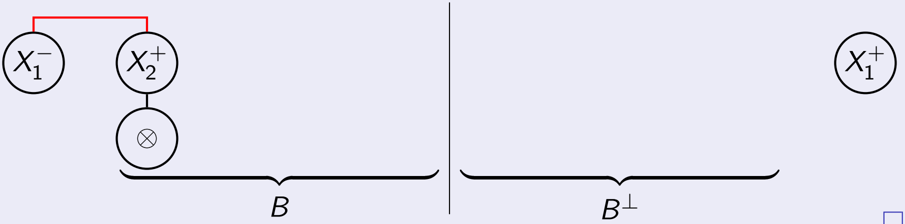

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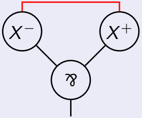

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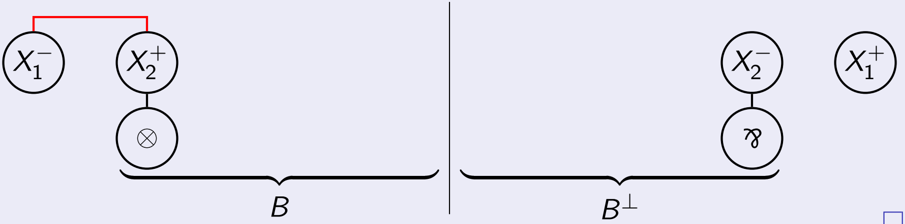
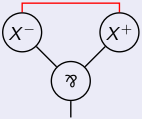


□

Key Result: finding a shape

Lemma

In $X^+ \trianglelefteq B$ one of the two proof-nets contains



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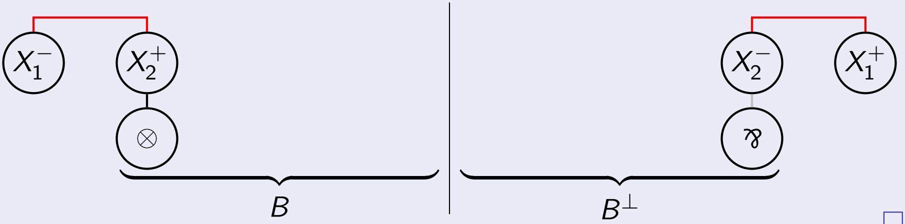

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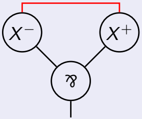

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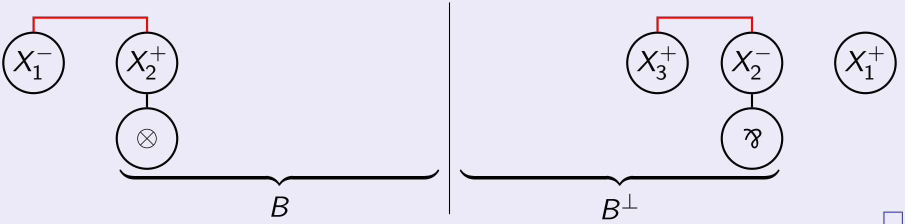

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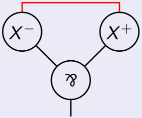

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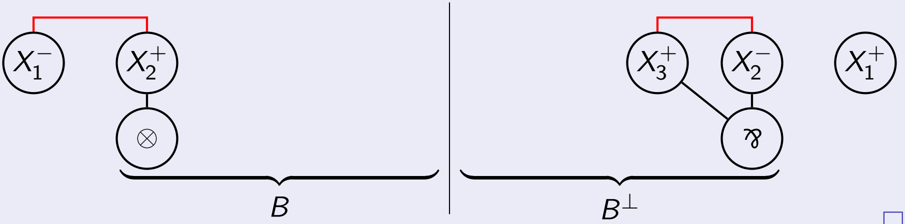

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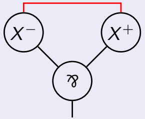

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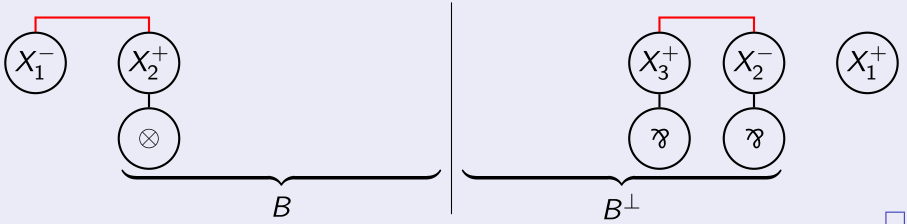

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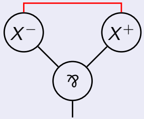

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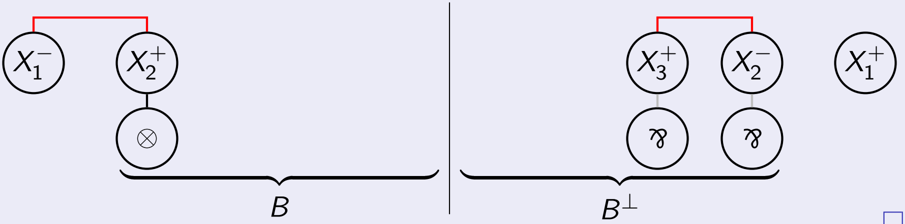

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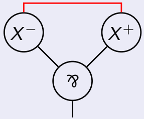

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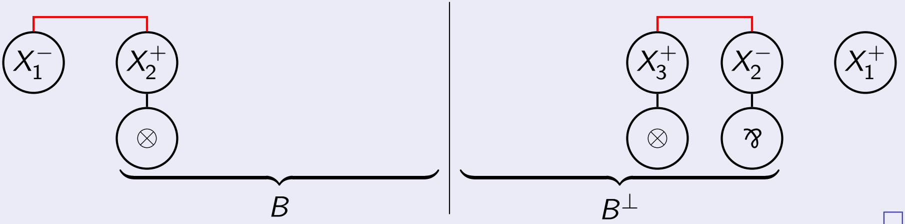
Follow a GOI path until finding it

<u>Invariant:</u> every X^+ of B is above a $\otimes$, and every X^- above a ⅋



□

Key Result: finding a shape

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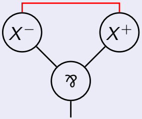

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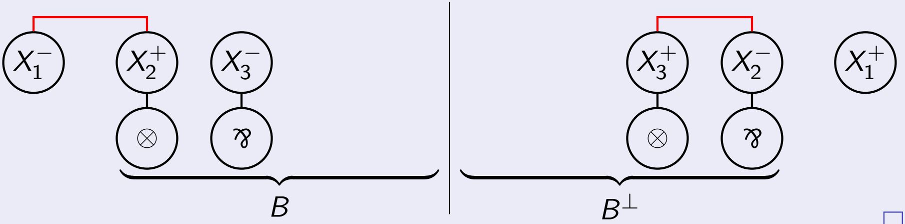

□

Key Result: finding a shape

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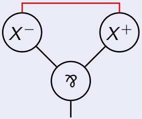
In $X^+ \trianglelefteq B$ *one of the two proof-nets contains*



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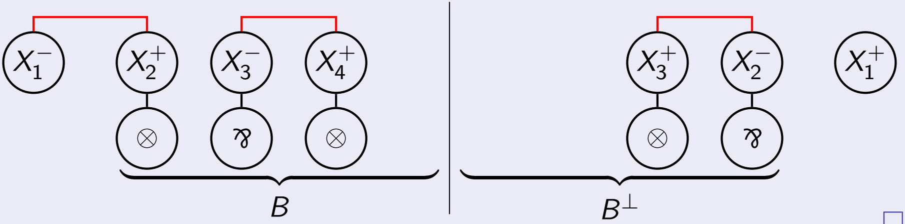

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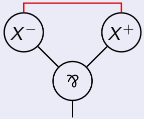

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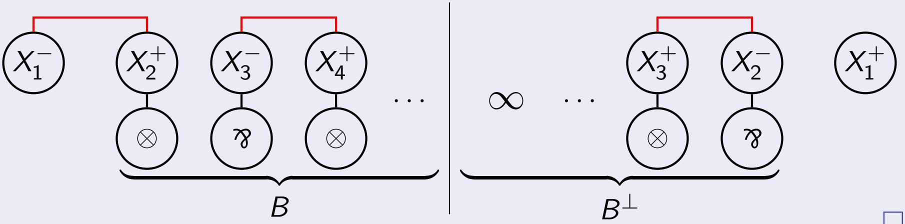
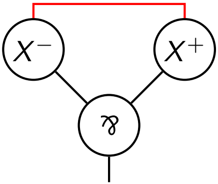


□

Using the found pattern



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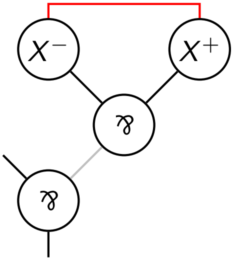

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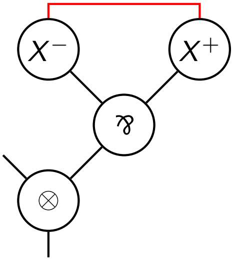
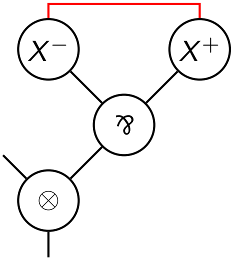
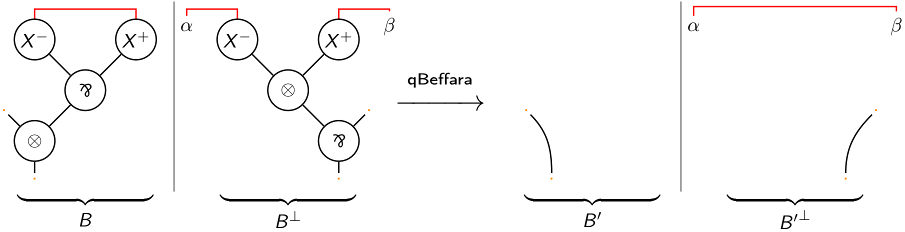


Using the found pattern





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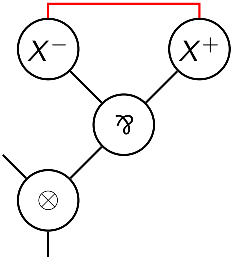
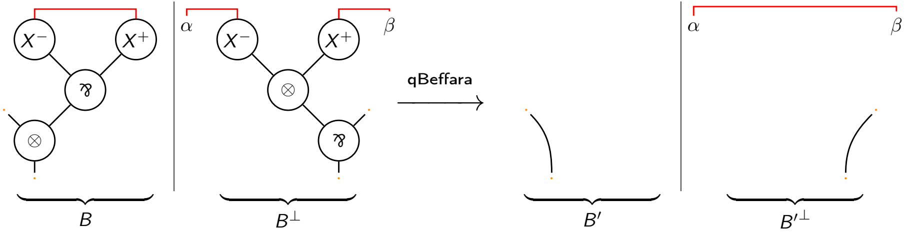




$\longrightarrow$ This rewriting **preserves** being a retraction!

Atomic retractions

Theorem

$X^+ \trianglelefteq B$ iff B is obtained from X^+ by Beffara $A \trianglelefteq A \otimes (A^\perp \parr A)$ and isomorphisms

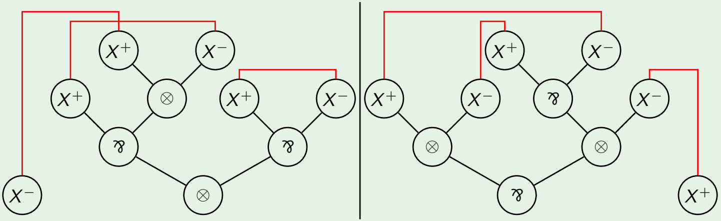
Atomic retractions

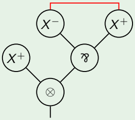
Theorem

$X^+ \trianglelefteq B$ iff B is obtained from X^+ by Beffara $A \trianglelefteq A \otimes (A^\perp \parr A)$ and isomorphisms

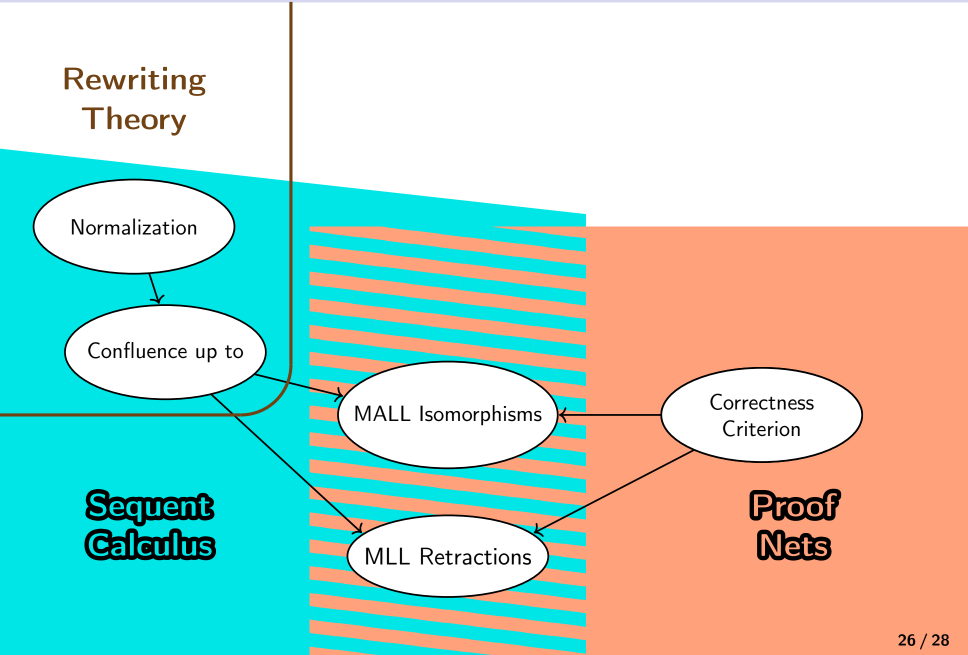
…but only at the level of **formulas**; Beffara does not give all **proofs**!

Proofs of $X^+ \trianglelefteq (X^+ \otimes X^-) \parr ((X^+ \parr X^-) \otimes X^-)$



no  so cannot be generated!

Thesis' Overview



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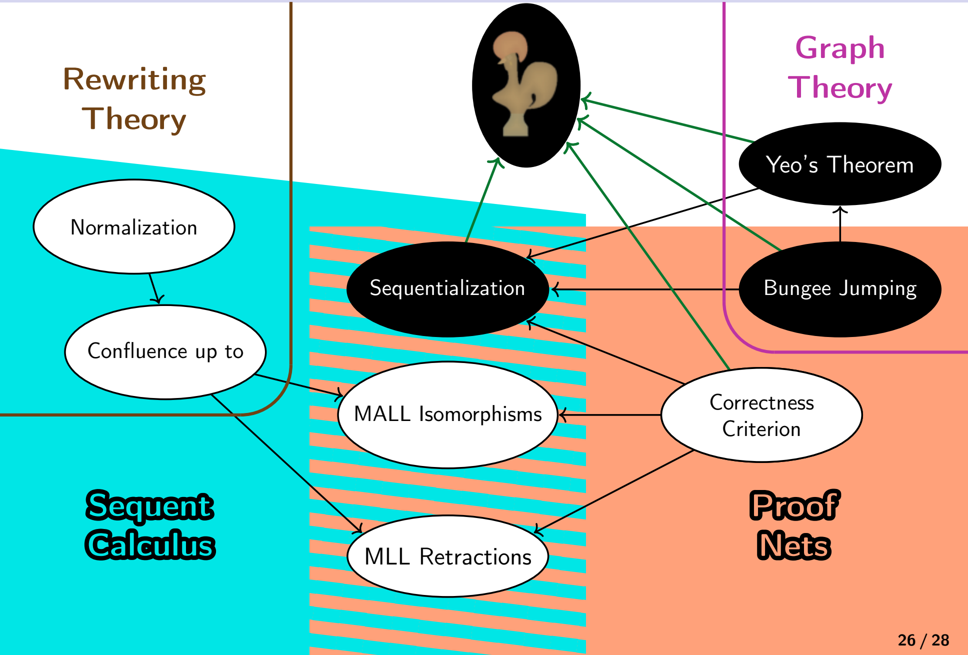
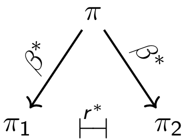


Future Work

- We should have all tools to expend the confluence up to result to LL



- Isomorphisms for MELL or MALL with 1st order quantifiers (proof-nets)
- Characterize all retractions in MLL

Merci !

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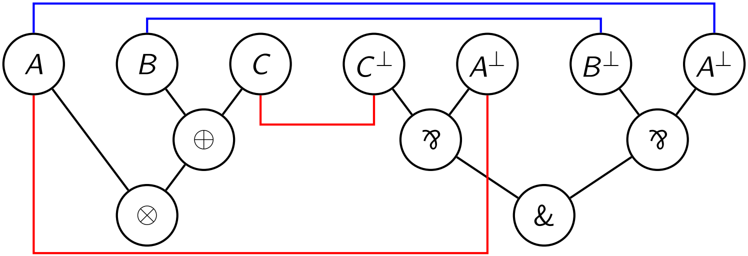
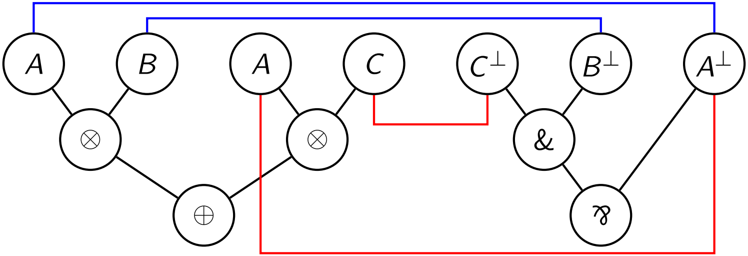
Proof-nets for $A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$

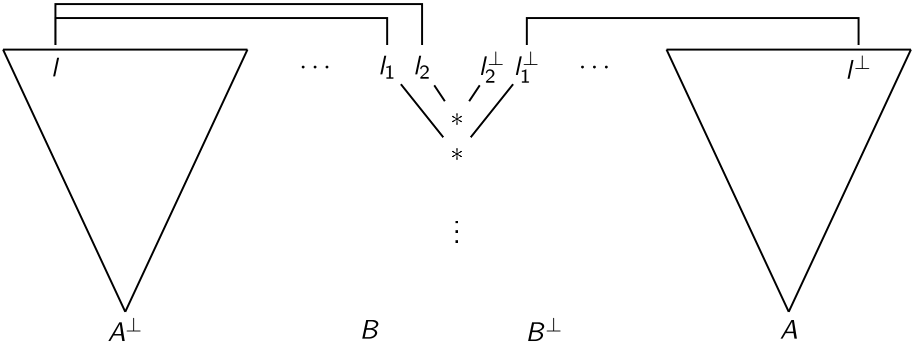




Proof 3/3: Why the distributed shape?

$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$ **not** of this shape

Correctness criterion to get this local shape from global *distributivity*



1. Forbidden configuration

Proof 3/3: Why the distributed shape?

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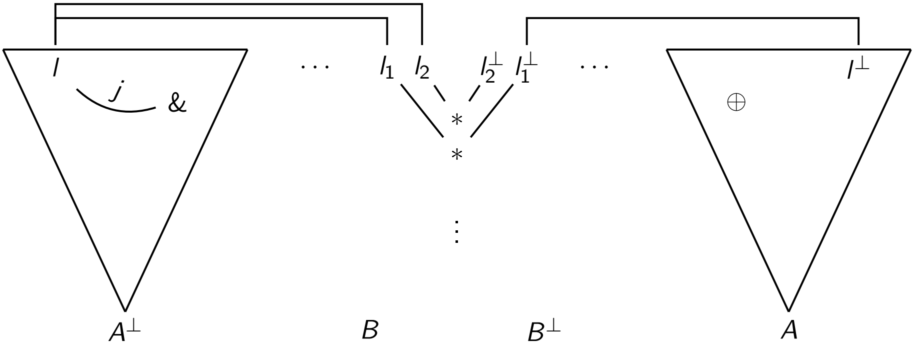
Correctness criterion to get this <u>local</u> shape from <u>global</u> *distributivity*



1. Forbidden configuration
2. Dependence on a &

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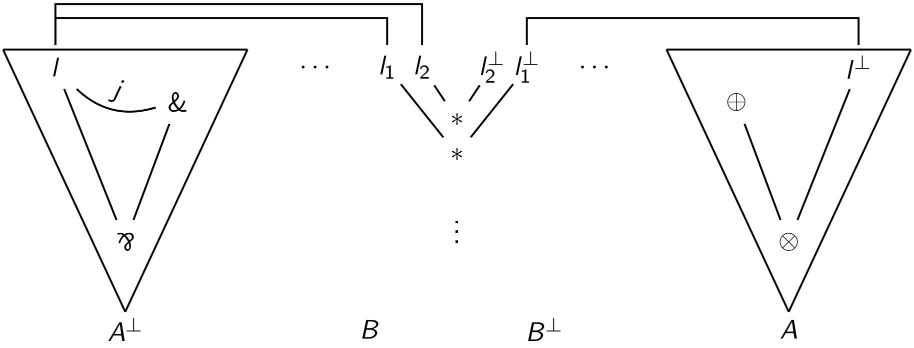

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2. Dependence on a &
3. ⅋ below

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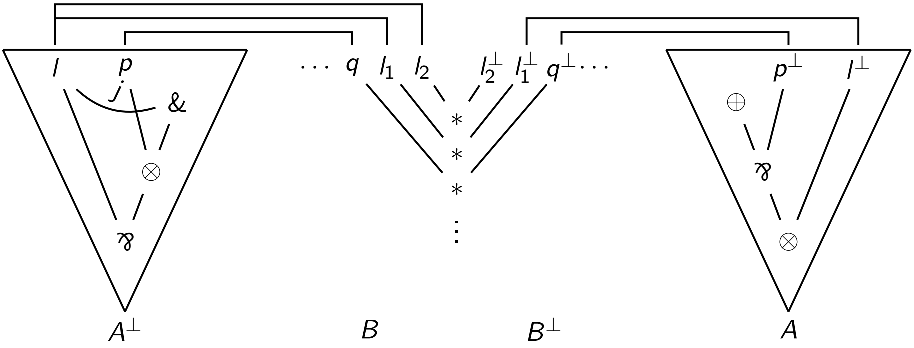

1. Forbidden configuration
2. Dependence on a &
3. ⅋ below

4. Distributivity

Proof 3/3: Why the distributed shape?

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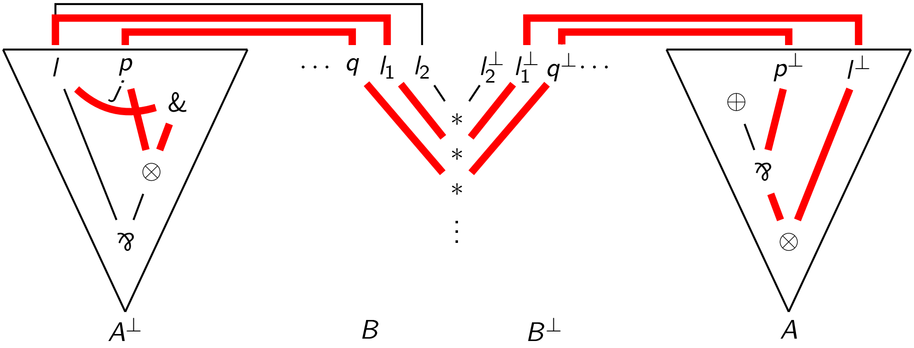
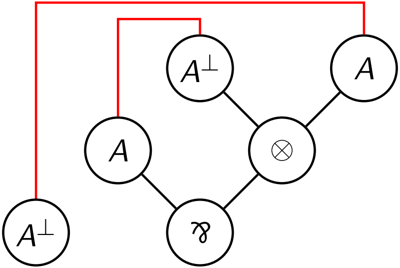
Correctness criterion to get this local shape from global *distributivity*

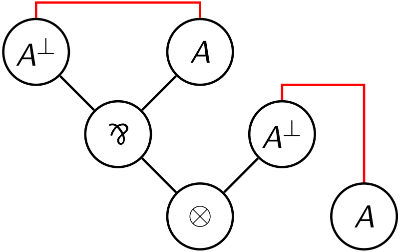


1. Forbidden configuration
2. Dependence on a &
3. ⅋ below
4. Distributivity
5. Forbidden cycle

Beffara $A \trianglelefteq A \otimes (A^\perp \parr A)$ is a retraction



$A \parr (A^\perp \otimes A)$



$(A^\perp \parr A) \otimes A^\perp$

Around Sequentialization

Sequentialization [HG05]

__MALL__ Proof-nets are exactly the images of proofs.

Around Sequentialization

Sequentialization [Gir87]

MLL *Proof-nets are exactly the images of proofs.*

↑

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[Ngu20]
encoding

Kotzig [Kot59]

On perfect matchings

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Proof-nets

Graph Theory

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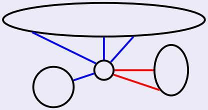
Seymour & Giles [Sey78]

Shoesmith & Smiley [SS79]

Yeo [Yeo97]

A graph with no alternating cycle has a splitting vertex:



Graph Theory

Around Sequentialization

Proof-nets

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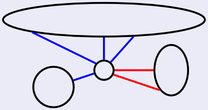
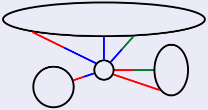


encoding

Yeo with local coloring



(and a parameter)

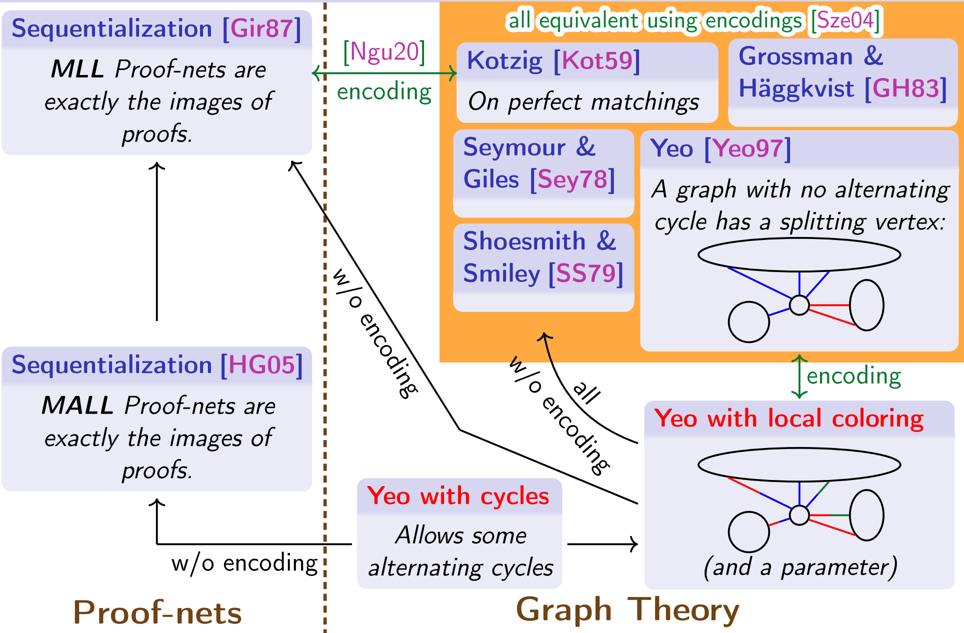
[Ngu20] encoding

w/o encoding

all

w/o encoding

Around Sequentialization



Bungee Jumping