Bifix codes and interval exchanges

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Amiens, 25\textsuperscript{th} November 2014

Joint work with:
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Motivation

\[ x = ababaababaababa \cdots \]

\[ x = \varphi^\omega(a) \]

\[ \varphi : \begin{cases} a &\mapsto ab \\ b &\mapsto a \end{cases} \]
Motivation

\[ x = abaababaabaababa \ldots \]

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Motivation

\[ x = ab \, aa \, ba \, ba \, ab \, aa \, ba \, ba \ldots \]

\[
\begin{align*}
\{ & u = aa \\
& v = ab \\
& w = ba \\
\}
\end{align*}
\]
Motivation

\[ x = v \ u \ w \ w \ v \ u \ w \ w \cdots \]
Interval exchange transformations

Let $A$ be a finite set ordered by $<_1$ and $<_2$. An interval exchange transformation (IET) is a map $T : [0, 1] \rightarrow [0, 1]$ defined by

$$T(z) = z + \alpha_z \quad \text{if } z \in I_a.$$
A IET $T$ is said to be *minimal* if for any $z \in [0, 1]$ the orbit $O(z) = \{ T^n(z) | n \in \mathbb{Z} \}$ is dense in $[0, 1]$.

$T$ is said *regular* if the orbits of the separation points $\neq 0$ are infinite and disjoint.

**Theorem [Keane, 1975]**

A regular interval exchange transformation is minimal.
Natural coding

Let $T$ be an IET relative to $(l_a)_{a \in A}$. The natural coding of $T$ relative to $z \in [0, 1]$ is the infinite word $\Sigma_T(z) = a_0a_1 \cdots \in A^\omega$ defined by

$$a_n = a \quad \text{si} \quad T^n(z) \in l_a.$$ 

Example

The Fibonacci word is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$. 

\[\text{Diagram}\]
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$\alpha$

$T$

$\Sigma_T(\alpha) = a$
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\[ a_n = a \quad \text{if} \quad T^n(z) \in I_a. \]

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The *Fibonacci word* is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$.

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Example

The Fibonacci word is the natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$ relative to the point $\alpha$, i.e. $T(z) = z + \alpha \mod 1$. 

$$T^2(\alpha) \quad T^5(\alpha) \quad \alpha \quad T^3(\alpha) \quad T(\alpha) \quad T^4(\alpha)$$

$$\Sigma_T(\alpha) = abaaaba\cdots$$
**Proposition**

If $T$ is minimal, $F(\Sigma_T(z))$ does not depend on $z$.

When $T$ is regular (minimal), $F(T) = F(\Sigma_T(z))$ is said a regular (minimal) interval exchange set.

**Example**

The **Fibonacci set** is the set of factors of a natural coding of the rotation of angle $\alpha = (3 - \sqrt{5})/2$.

\[
F(T) = \{ \varepsilon, a, b, aa, ab, ba, aab, aba, baa, \ldots \}
\]
Theorem

Let $T$ be a regular IET defined over a quadratic field. Then the interval exchange set $F(T)$ is primitive morphic.

Example

\[ |l_a|, |l_b| \in \mathbb{Q}[\sqrt{5}] \]

\[ F(T) = F(x) \quad \text{with} \quad x = id \circ f^\omega(a) \]

\[ f : \begin{cases} 
  a \mapsto ab \\
  b \mapsto a 
\end{cases} \]
Cylinders

For a word $w = b_0 b_1 \cdots b_{m-1}$, let's define

$$I_w = I_{b_0} \cap T^{-1}(I_{b_1}) \cap \cdots \cap T^{-m+1}(I_{b_{m-1}})$$

and $J_w = T^m(I_w)$.

Example

$$I_{aa} = I_a \cap T^{-1}(I_a), \quad I_{ab} = I_a \cap T^{-1}(I_b), \quad I_{ba} = I_b \cap T^{-1}(I_a), \quad I_{bb} = I_b \cap T^{-1}(I_b);$$

$$J_{aa} = T^2(I_a) \cap T(I_a), \quad J_{ab} = T^2(I_a) \cap T(I_b), \quad J_{ba} = T^2(I_b) \cap T(I_a), \quad J_{bb} = T^2(I_b) \cap T(I_b).$$
Cylinders

We denote by $<_1$ the lexicographic order on $A^*$ induced by $<_1$ and by $<_2$ the lexicographic order on the reversal of the words induced by $<_2$.

**Proposition**

- $I_u < I_v$ if and only if $u <_1 v$ and $u$ is not a prefix of $v$.
- $J_u < J_v$ if and only if $u <_2 v$ and $u$ is not a suffix of $v$.

**Example**

\[
\begin{align*}
  &I_{aab} \quad I_{ab} \quad I_b \\
  &J_{ab} \quad J_{ba} \quad J_{aa}
\end{align*}
\]

\[
aab <_1 ab <_1 b \quad \text{while} \quad ab <_2 ba <_2 aa.
\]
A set $X \subset A^+$ of nonempty words over an alphabet $A$ is a code if for every $m, n \geq 1$ and $x_1, \ldots, x_n, y_1, \ldots, y_m$,

$$x_1 \cdots x_n = y_1 \cdots y_m \implies n = m \quad \text{and} \quad x_i = y_i \quad \text{for} \quad i = 1, \ldots, n$$

A prefix code is a set of nonempty words which does not contain any proper prefix of its elements. A suffix code is defined symmetrically. A bifix code is a set which is both a prefix code and a suffix code.

**Example**

- $\{a, ab, ba\}$ is not a code.
- $\{aabb, ababb, abb\}$ is a prefix code but it’s not a suffix code.
- $\{aa, ab, ba\}$ is a bifix code.
**Codes**

A set $X \subset A^+$ of nonempty words over an alphabet $A$ is a *code* if for every $m, n \geq 1$ and $x_1, \ldots, x_n, y_1, \ldots, y_m$,

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A bifix code $X \subset S$ is *$S$-maximal* if it is not properly contained in a bifix code $Y \subset S$. 
Proposition

Let $T$ a minimal IET and $S = F(T)$. If $X$ is a finite $S$-maximal bifix code, the families $(I_w)_{w \in X}$ and $(J_w)_{w \in X}$ are ordered partitions of $[0, 1]$, relatively to the orders $<_1$ and $<_2$.

Example

Let $S$ be the Fibonacci set. The set $X = \{a, baab, bab\}$ is an $S$-maximal bifix code.

\[ a <_1 baab <_1 bab \quad \text{and} \quad bab <_2 baab <_2 a. \]
Transformation associated with a bifix code

Let $T$ be a regular IET and $S = F(T)$. Let $X$ be a finite $S$-maximal bifix code on the alphabet $A$. Let’s define the transformation

$$T_X(z) = T_{|u|}(z) \quad \text{if } z \in I_u.$$
Transformation associated with a bifix code

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$$T_X(z) = T^{|u|}(z) \quad \text{if} \quad z \in I_u.$$

Example
Let $T$ be a regular IET and $S = F(T)$. Let $X$ be a finite $S$-maximal bifix code on the alphabet $A$. Let’s define the transformation $T_X(z) = T^{|u|}(z)$ if $z \in I_u$.

**Example**

![Diagram showing transformations and intervals]

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$$T_X(z) = T^{|u|}(z) \quad \text{if } z \in I_u.$$
A *coding morphism* for a prefix code $X \subset A^+$ is a morphism $f : B^* \to A^*$ which maps bijectively $B$ onto $X$.

**Example**

Let’s consider the bifix code $X = \{aa, ab, ba\}$ on $A = \{a, b\}$ and let $B = \{u, v, w\}$. The map

$$f : \begin{cases} 
  u \mapsto aa \\
  v \mapsto ab \\
  w \mapsto ba 
\end{cases}$$

is a coding morphism for $X$. 

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Transformation associated with a coding morphism

Let \( f : B^* \to A^* \) be a coding morphism for \( X \). Let \((K_b)_{b \in B}\), with \( K_b = I_{f(b)} \). Let \( T_f \) be the IET relative to \((K_b)_{b \in B}\).

**Proposition**

If \( X \) is a finite \( S \)-maximal bifix code, one has \( T_f = T_X \).

**Example**

Let \( X = \{a, baab, bab\} \), \( B = \{u, v, w\} \) and

\[
\begin{align*}
f : u &\mapsto a, \\
v &\mapsto baab, \\
w &\mapsto bab.
\end{align*}
\]
Theorem [2014]

Let $T$ a regular IET and $S = F(T)$. For any finite $S$-maximal bifix code $X$ with coding morphism $f$, the transformation $T_f$ is regular.

Example

$X = \{aa, ab, ba\}$ and $f : u \mapsto aa, \ v \mapsto ab, \ w \mapsto ba$. 

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IET on a stack

Let $T$ a IET and $G$ a transitive permutation group on a finite set $Q$. Let $\varphi : A^* \to G$ be a morphism and let $\psi : I \to G$ definend by $\psi(z) = \varphi(a)$ if $z \in I_a$. The skew product of $T$ and $G$ is the transformation $U$ on $I \times Q$ definend by

$$U(z, q) = (T(z), q \psi(z))$$

$G = S_2$

$\varphi : a \mapsto (1), \ b \mapsto (12)$

$X = \{a, baab, bab\}$

Theorem [2014]

A regular interval exchange set has the finite index basis property $^a$.

$^a$ A finite bifix code $X \subset S$ is an $S$-maximal bifix code of $S$-degree $d$ if and only if it is a basis of a subgroup of index $d$ of $F_A$. 
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**IET on a stack**

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Decoding

Let $f$ be a coding morphism for a $S$-maximal prefix code. The decoding of $x$ is the infinite word $y$ s.t. $x = f(y)$.

**Proposition**

Let $T$ be a minimal IET, $S = F(T)$, $X$ a finite $S$-maximal prefix code and $f : B^* \rightarrow A^*$ a coding morphism. Then, for all $z \in [0, 1[$, one has $\Sigma_T(z) = f(\Sigma_{T^f}(z))$. 
Example

\[ T(\alpha) = a b a a b a b a b a b a a \cdots \]

\[ \Sigma_T(\alpha) = \{ aa, ab, ba \} \text{ and } f : u \mapsto aa, v \mapsto ab, w \mapsto ba. \]

\[ T_f(\alpha) = v u w w v u \cdots \]

\[ \Sigma_{T_f}(\alpha) = v u w w v u \cdots \]

\[ f (v u w w v u \cdots) = ab a a b a b a b a b a b a b a \cdots \]
Maximal bifix decoding

Let $f$ be a coding morphism for a finite $S$-maximal bifix code $X \subset S$. The set $f^{-1}(S)$ is called a *maximal bifix decoding* of $S$.

**Theorem [2014]**

The family of regular interval exchange sets is closed under maximal bifix decoding.

**Proof.** $f^{-1}(S) = F(T_f)$.

Actually, this property is true for a larger class of sets...
**Extension graphs**

Let $S$ be a biextendable set of words. For $w \in S$, we denote

$$L(w) = \{ a \in A \mid aw \in S \}, \quad R(w) = \{ a \in A \mid wa \in S \}$$

and

$$E(w) = \{ (a, b) \in A \times A \mid awb \in S \}.$$ 

The *extension graph* of $w$ is the undirected bipartite graph $G(w)$ with vertices $L(w) \sqcup R(w)$ and edges $E(w)$.

**Example**

Let $S$ be the Fibonacci set.

- $G(\varepsilon)$
  - $a$ connected to $a$
  - $b$ connected to $b$

- $G(a)$
  - $a$ connected to $a$
  - $b$ connected to $b$

- $G(b)$
  - $a$ connected to $a$
Tree sets

We say that a biextendable set $S$ is a *tree set* if the graph $G(w)$ is a tree (connected and acyclic) for all $w \in S$.

**Example**

Let $A = \{a, b, c\}$. The set $S$ of factors of $a^*\{bc, bcba\}a^*$ is not a tree set.
Planar tree sets

Let $<_1$ and $<_2$ be two orders on $A$. For a set $S$ and a word $w \in S$, the graph $G(w)$ is compatible with $<_1$ and $<_2$ if for any $(a, b), (c, d) \in E(w)$, one has

$$a <_1 b \implies b \leq_2 d$$

Example

Let $S$ be the Fibonacci set. Set $a <_1 b$ and $b <_2 a$.

We say that a biextendable set $S$ is a planar tree set w.r.t. $<_1$ and $<_2$ on $A$ if for any $w \in S$, the graph $G(w)$ is a tree compatible with $<_1$ and $<_2$. 
Example

Let $A = \{a, b, c\}$. The Tribonacci set is the set of factors of the Tribonacci word, i.e. is the fixpoint $x = f^\omega(a) = abacaba \cdots$ of the morphism

$$f : a \mapsto ab, \quad b \mapsto ac, \quad c \mapsto a.$$

It is not possible to find two orders on $A$ making the three graphs planar.
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\[
\begin{align*}
G(\varepsilon) & \quad G(a) & \quad G(aba) \\
\begin{array}{c}
\text{a} \\
\text{\quad b} \\
\text{\quad c}
\end{array} & \quad \begin{array}{c}
\text{b} \\
\text{\quad a} \\
\text{\quad c}
\end{array} & \quad \begin{array}{c}
\text{c} \\
\text{\quad a} \\
\text{\quad b}
\end{array}
\end{align*}
\]

It is not possible to find two orders on $A$ making the three graphs planar.

Theorem [Ferenczi, Zamboni, 2008]

A set $S$ is a regular interval exchange set on $A$ if and only if it is a uniformly recurrent planar tree set containing $A$. 
Theorem [2014]

The family of uniformly recurrent tree set is closed under maximal bifix decoding.
Merci!