Return words and palindromes in specular sets

Francesco Dolce

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based on a joint work with

V. Berthé, C. De Felice, V. Delecroix,
J. Leroy, D. Perrin, C. Reutenauer, G. Rindone
return words

HOME SWEET HOME

palindromes

TACO CAT
Outline

Introduction

1. Specular sets
2. Return words
3. Palindromes

Conclusions
The *extension graph* of a word \( w \in S \) is the undirected bipartite graph \( \mathcal{E}(w) \) with vertices \( L(w) \sqcup R(w) \) and edges \( B(w) \), where

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\begin{align*}
L(w) &= \{ a \in A \mid aw \in S \}, \\
R(w) &= \{ a \in A \mid wa \in S \}, \\
B(w) &= \{ (a, b) \in A \times A \mid awb \in S \}.
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\textbf{Example (Fibonacci)}

\( S = \{ \varepsilon, a, b, aa, ab, ba, aab, aba, baa, bab, \ldots \} \).
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![Diagram of extension graphs for \( \epsilon \), \( a \), and \( b \)]
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![Diagram](image-url)
A factorial set $S$ is called a *tree set of characteristic* $c$ if $E(w)$ is a tree for any nonempty $w \in S$, and $E(\varepsilon)$ is a union of $c$ trees.
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**Theorem**

Families of (uniformly) recurrent tree sets of characteristic 1:

- Factors of Arnoux-Rauzy (*Sturmian*) words;
  
  [Berthé, De Felice, D., Leroy, Perrin, Reutenauer, Rindone (2014)]

- Natural coding of regular interval exchanges.
  
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**Example (Tribonacci)**

![Diagram of Tribonacci tree set](image)
Let $\theta : A \to A$ be an involution (possibly with some fixed point).
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A word is $\theta$-reduced if it has no factor of the form $a\theta(a)$ for $a \in A$.

**Example**

Let $\theta : a \mapsto a$, $b \mapsto d$, $c \mapsto c$, $d \mapsto b$.

The $\theta$-reduction of the word $\text{daaacad}b$ is $\text{dac}$.
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The $\theta$-reduction of the word $d a a a c d b$ is $d a c$.

A set is called \textit{$\theta$-symmetric} if it is closed under taking inverses (under $\theta$).

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The set $X = \{a, adc, b, cba, d\}$ is symmetric for $\theta : b \leftrightarrow d$ fixing $a, c$. 
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A *specular set* on an alphabet $A$ (w.r.t. an involution $\theta$) is a set

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- tree set of characteristic 2.
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**Example**

Let $A = \{a, b\}$ and $\theta$ be the identity on $A$. The set of factors of $(ab)^\omega$ is a specular set.

$$
\begin{align*}
E(\varepsilon) & \\
\begin{array}{c}
a \\
b
\end{array} & \begin{array}{c}
b \\
a
\end{array}
\end{align*}
$$

$$
\begin{align*}
E(baba) & \\
\begin{array}{c}
a \\
b
\end{array} & \begin{array}{c}
b \\
a
\end{array}
\end{align*}
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**Proposition [using J. Cassaigne (1997)]**

The factor complexity of a specular set is given by $p_n = n(\text{Card}(A) - 2) + 2$ for all $n \geq 1$. 
Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

The natural coding of a linear involution without connections is a specular set.

\[ T = \sigma_2 \circ \sigma_1 \]
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The natural coding of a linear involution without connections is a specular set.

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A *doubling transducer* is a transducer with set of states \{0, 1\} such that:

1. the input automaton is a group automaton,
2. the output labels of the edges are all distinct.

**Example**

\[
\begin{align*}
\Sigma &= \{\alpha\} \\
A &= \{a, b\} \\
\end{align*}
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![Diagram](attachment:image.png)
A **doubling transducer** is a transducer with set of states \( \{0, 1\} \) such that:

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A **doubling map** is a pair \( \delta = (\delta_0, \delta_1) \), where \( \delta_i(u) = v \) for a path starting at the state \( i \) with input label \( u \) and output label \( v \).

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\begin{align*}
\delta_0 (\alpha^\omega) &= (ab)^\omega \\
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The **image** of a set \( T \) is \( \delta(T) = \delta_0(T) \cup \delta_1(T) \).

### Example

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A = \{a, b\}
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\[
\begin{align*}
\alpha & | a \\
0 \quad \rightarrow \quad 1 \quad \alpha & | b \\
\delta_0(\alpha^\omega) &= (ab)^\omega \\
\delta_1(\alpha^\omega) &= (ba)^\omega \\
\delta(\text{Fac}(\alpha^\omega)) &= \text{Fac}((ab)^\omega)
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Proposition [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2015)]

The image of a tree set of characteristic $1$ closed under reversal is a specular set with respect to $\theta_A$. 
Example (two doublings of Fibonacci on $\Sigma = \{\alpha, \beta\}$)

$\text{Fac}(abaababa\ldots) \cup \text{Fac}(cdccdcdc\ldots)$

$\theta_A : \begin{cases} 
\alpha \mapsto c \\
\beta \mapsto d \\
c \mapsto a \\
d \mapsto b 
\end{cases}$
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\[ \alpha | a \quad \beta | b \quad 0 \quad \alpha | c \quad \beta | d \quad 1 \]

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$\theta_A : \begin{cases} 
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\end{cases}$
A *right return word* to $w$ in $S$ is a nonempty word $u$ such that $wu \in S$, starts and ends with $w$ but has no $w$ as an internal factor. Formally,

$$\mathcal{R}(w) = \{ u \in A^+ | \ wu \in (A^+w \setminus A^+wA^+) \cap S \}.$$ 

**Example (Fibonacci)**

$$\mathcal{R}(aa) = \{ baa, babaa \}.$$ 

$$\varphi(a)^\omega = ababaabaabaababaababaababaabaabaabaababaabab.$$
A **right return word** to \( w \) in \( S \) is a nonempty word \( u \) such that \( wu \in S \), starts and ends with \( w \) but has no \( w \) as an internal factor. Formally,

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\]

**Cardinality Theorem for Right Return Words [BDDDLPRR (2015)]**

For any \( w \) in a recurrent specular set, one has

\[
\text{Card } (\mathcal{R}(w)) = \text{Card } (A) - 1.
\]
A complete return word to a set $X \subseteq S$ is a word starting and ending with a word of $X$ but having no internal factor in $X$. Formally,

$$\mathcal{CR}(X) = S \cap (XA^+ \cap A^+X) \setminus A^+XA^+.$$ 

**Example (Fibonacci)**

$$\mathcal{CR}\{aa, bab\} = \{aabaa, aabab, babaa\}.$$ 

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Cardinality Theorem for Complete Return Words [BDDDLPRR (2015)]

Let $S$ be a recurrent specular set and $X \subset S$ be a finite bifix code\(^1\) with empty kernel\(^2\). Then,

$$\text{Card}(\mathcal{CR}(X)) = \text{Card}(X) + \text{Card}(A) - 2.$$ 

1. **bifix code**: set that does not contain any proper prefix or suffix of its elements.
2. **kernel**: set of words of $X$ which are also internal factors of $X$. 

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Numeration 2016  
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13 / 23
Two words $u, v$ overlap if a nonempty suffix of one of them is a prefix of the other.
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Consider a word $w$ not overlapping with $w^{-1}$.
A mixed return word to $w$ is the word $N(u)$ obtained from $u \in CR(\{w, w^{-1}\})$ erasing the prefix if it is $w$ and the suffix if it is $w^{-1}$.
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**Cardinality Theorem for Mixed Return Words [BDDDLPRRR (2015)]**

Let $S$ be a recurrent specular set and $w \in S$ such that $w, w^{-1}$ do not overlap. Then,

$$\text{Card}(\mathcal{MR}(w)) = \text{Card}(A).$$
A palindrome is a word $w = \tilde{w}$ as, for instance:
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Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]
A word of length $n$ has at most $n + 1$ palindrome factors.

A word with maximal number of palindromes is rich.
A factorial set is rich if all its elements are rich.

Example (Fibonacci)
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$\text{Pal}(\text{abaab}) = \{\varepsilon, a, b, aa, aba, baab\}$. 

**Theorem [A. Glen, J. Justin, S. Widmer, L.Q. Zamboni (2009)]**

Let $S$ be a recurrent set closed under reversal. $S$ is rich $\iff$ every complete return word to a palindrome is a palindrome.
Families of rich sets:

- Factors of Arnoux-Rauzy (∗Sturmian∗) words.
  
  [X. Droubay, J. Justin, G. Pirillo (2001)]

- Natural coding of regular interval exchanges defined by a symmetric permutation.
  
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Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

Recurrent tree sets of characteristic 1 closed under reversal are rich.
Let $\sigma$ be an antimorphism.
A word $w$ is a $\sigma$-palindrome if $w = \sigma(w)$.
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Let $\sigma : A \leftrightarrow T, \; C \leftrightarrow G$.
The word $CTTAAG$ is a $\sigma$-palindrome.
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**Example**

Let $\sigma : A \leftrightarrow T, \ C \leftrightarrow G$.
The word CTTAAG is a $\sigma$-palindrome.

**Theorem [Š. Starosta (2011)]**

Let $\gamma_{\sigma}(w)$ be the number of transpositions of $\sigma$ affecting $w$. Then,

$$\text{Card (Pal}_{\sigma}(w)) \leq |w| + 1 - \gamma_{\sigma}(w).$$

A word (set) is $\sigma$-rich if the equality holds (for all its elements).
Let $G$ be a group of morphisms and antimorphisms, containing at least an antimorphism. A word $w$ is a $G$-palindrome if there exists a nontrivial $g \in G$ s.t. $w = g(w)$. 
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Let $G = \langle \sigma, \tau \rangle$, with

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The following words are $G$-palindromes:

- **NUMERATION**, fixed by $\sigma$,
- **PRAGUE**, fixed by $\tau$,
- **PÍT**, fixed by $\sigma \tau \sigma$.

A word (set) is $G$-rich* if... “the number of $G$-palindromes if maximal”.
Theorem [E. Pelantová, Š. Starosta (2014)]

A set $S$ closed under $G$ is $G$-rich if for every $w \in S$, every complete return word to the $G$-orbit of $w$ is fixed by a nontrivial element of $G$. 
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Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

The specular set obtained as image under a doubling transducer \( \mathcal{A} \) is \( G_{\mathcal{A}} \)-rich.

\[
G_{\mathcal{A}} = \{ \text{id}, \sigma, \tau, \sigma \tau \} \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})
\]

with \( \sigma \) an antimorphism and \( \tau \) a morphism.
Conclusions

Summing up

- Tree and specular sets.
  
  Linear involutions and doubling maps.
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\[
\begin{align*}
\text{Card } (\mathcal{R}(w)) &= \text{Card } (A) - 1 \\
\text{Card } (\mathcal{CR}(X)) &= \text{Card } (X) + \text{Card } (A) - 2 \\
\text{Card } (\mathcal{MR}(w)) &= \text{Card } (A)
\end{align*}
\]
Conclusions
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- Tree and specular sets.
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  \[ \text{Card} \left( R(w) \right) = \text{Card} \left( A \right) - 1 \]
  \[ \text{Card} \left( CR(X) \right) = \text{Card} \left( X \right) + \text{Card} \left( A \right) - 2 \]
  \[ \text{Card} \left( MR(w) \right) = \text{Card} \left( A \right) \]

- New family of $G$-rich sets.
  Specular sets obtained by doubling maps are $G_A$-rich.
Decidability of the tree (and specular) condition.

[work in progress with Julien Leroy and Revekka Kyriakoglou]
Further Research Directions
and other works in progress

- Decidability of the tree (and specular) condition.
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- Tree set and free groups.
  Tree set of $\chi = 1 \implies R(w)$ is a basis of the free group for every $w$
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- New classes of $G$-rich sets (or new groups $G$).
Děkuji