Palindromes and Tree Sets

Francesco DOLCE

Atelier
“Combinatoire des mots et pavages”
“Combinatorics on Words and Tilings”
Workshop

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«You can summon him by trying to take on his characteristics - relaxing, fantasising that you’re ‘cool’, and letting go of your frustration momentarily. Visualise him zipping along on his skateboard, accompanied by a slight breeze and his Mantra: ‘Neeeeooow’».
GoFlowolFoG

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«We decided that the 'name' of the Spirit would [...] be Go Flow. This was mirrored to give the name GoFlowolFoG - which sounds suitably ‘magical’»
A palindrome is a word $w = \tilde{w}$ as, for instance:

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- ici, été, coloc, kayak, …
- saippuakivikauppias, …
- ojo, somos, reconocer, …
- Krk, potop, ici, …
- топот, довод, кабак,
- وَلَوْ، وَدُ، مُهِم، آبَا، …
- À Laval elle l’avalà, …
Palindromes

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- وَلَوْ , وَذُ , مُهِمْ , آبَا ,
- À Laval elle l’avalala, ...
Conway's Criterion: $B$, $C$, $D$, $E$ palindromes.

\[ B = \downarrow\rightarrow\downarrow, \quad C = \leftarrow\downarrow\rightarrow\rightarrow\rightarrow\leftarrow, \]
\[ D = \uparrow\rightarrow\rightarrow\leftarrow\leftarrow\leftarrow\rightarrow\rightarrow\rightarrow\uparrow, \quad E = \uparrow\uparrow. \]
Conway's Criterion: $B, C, D, E$ palindromes.

\[
\begin{align*}
B &= 303, & C &= 23033032, \\
D &= 1001333331001, & E &= 11.
\end{align*}
\]
Theorem [A. Blondin-Massé, A. Garon, S. Labbé (2013)]

If $AB\hat{A}\hat{B}$ is a $BN$-factorisation of a Fibonacci tile, then $A$ and $B$ are palindromes.

$A = 0103032303010, \quad B = 3032321232303,$
Theorem [A. Blondin-Massé, S. Brlek, A. Garon, S. Labbé (2009)]

If $AB\hat{A}\hat{B}$ and $CD\hat{C}\hat{D}$ are the $BN$-factorisation of a prime double square, then $A, B, C, D$ are palindromes.
A word of length $n$ has at most $n + 1$ palindrome factors.

A word with maximal number of palindromes is full (or rich).
**Full words**

**Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]**

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**Example**

- **Trump**, **Putin**, **Le Pen**, **Fillon** are rich.
- **Trudeau**, **Merkel**, **Gentiloni**, **Mélenchon** are not rich.
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- **Trump, Putin, Le Pen, Fillon** are rich.
- **Trudeau, Merkel, Gentiloni, Mélenchon** are not rich.

<table>
<thead>
<tr>
<th>FRANÇOIS</th>
<th>= 8</th>
<th>and</th>
<th>Card ({$\varepsilon, F, R, A, N, \zeta, O, I, S$}) = 9 = 8 + 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>PENELope</td>
<td>= 8</td>
<td>and</td>
<td>Card ({$\varepsilon, P, E, N, L, O, ENЕ$}) = 7 &lt; 8 + 1</td>
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Full words

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A word of length \( n \) has at most \( n + 1 \) palindrome factors

A word with maximal number of palindromes is \textit{full} (or \textit{rich}).
A factorial set is \textit{full} if all its elements are full.

Example (Fibonacci)

Let \( S \) be the set of factors of the fixed-point \( \varphi^\omega(0) \) of

\[
\varphi : 0 \mapsto 01, \quad 1 \mapsto 0.
\]

Every word \( w \in S \) is full. For instance,

\[
\text{Pal}(01001) = \{\varepsilon, 0, 1, 00, 010, 1001\}.
\]
Arnoux-Rauzy sets

Definition

An Arnoux-Rauzy set is a factorial set closed under reversal with $p_n = (\text{Card}(A) - 1)n + 1$ having a unique right special factor for each length.

Examples

- **Fibonacci**: factors of the fixed-point $\varphi^\omega(0)$, where
  \[ \varphi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases} \]

- **Tribonacci**: factors of the fixed-point $\psi^\omega(0)$, where
  \[ \psi : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 02 \\ 2 \mapsto 0 \end{cases} \]
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Theorem [X. Droubay, J. Justin, G. Pirillo (2001)]
Arnoux-Rauzy sets are full.
Interval exchanges

Let \((I_\alpha)_{\alpha \in A}\) and \((J_\alpha)_{\alpha \in A}\) be two partitions of a semi-interval \(I\).
An interval exchange transformation (IET) is a map \(T : I \rightarrow I\) defined by

\[ T(z) = z + y_\alpha \quad \text{if } z \in I_\alpha. \]
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$T$ is \textit{minimal} if for any point $z \in I$ the orbit $O(z) = \{T^n(z) \mid n \in \mathbb{Z}\}$ is dense in $I$.

$T$ is \textit{regular} if the orbits of the separation points are infinite and disjoint.

\textbf{Theorem [M. Keane (1975)]}

A regular interval exchange transformation is minimal.
**Interval exchanges**

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$T$ is *regular* if the orbits of the separation points are infinite and disjoint.

**Theorem** [M. Keane (1975)]

A regular interval exchange transformation is minimal.

**Example (the converse is not true)**

Diagram showing the transformation $T$ and the orbits of separation points.
Interval exchanges

The *natural coding* of $T$ relative to $z \in I$ is the infinite word $\Sigma_T(z) = a_0a_1 \cdots \in A^\omega$ defined by

$$a_n = \alpha \quad \text{if} \quad T^n(z) \in I_\alpha.$$
The **natural coding** of \( T \) relative to \( z \in I \) is the infinite word \( \Sigma_T(z) = a_0a_1 \cdots \in A^\omega \) defined by

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a_n = \alpha \quad \text{if} \quad T^n(z) \in l_\alpha.
\]

**Example (Fibonacci, \( z = (3 - \sqrt{5})/2 \))**

\[
\Sigma_T(z) = 0
\]
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Interval exchanges

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The set $\mathcal{L}(T) = \bigcup_{z \in I} \text{Fac}(\Sigma_T(z))$ is said a (minimal, regular) interval exchange set.

**Remark.** If $T$ is minimal, $\text{Fac}(\Sigma_T(z))$ does not depend on the point $z$.

**Example (Fibonacci)**

$$\mathcal{L}(T) = \{ \varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, \ldots \}$$
Interval exchanges

The set \( \mathcal{L}(T) = \bigcup_{z \in I} \text{Fac}(\Sigma_T(z)) \) is said a (minimal, regular) interval exchange set.

**Remark.** If \( T \) is minimal, \( \text{Fac}(\Sigma_T(z)) \) does not depend on the point \( z \).

**Example (Fibonacci)**

\[
\mathcal{L}(T) = \{ \varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, \ldots \}
\]

**Proposition**

Regular interval exchange sets have factor complexity \( p_n = (\text{Card}(A) - 1)n + 1 \).
Interval exchanges

Theorem [P. Baláži, Z. Masáková, E. Pelantová (2007)]

Regular interval exchange sets closed under reverse are full.

\[ T \text{ closed under reverse} \iff \pi = (n \ n-1 \cdots 2 \ 1) \]
The *extension graph* of a word $w \in S$ is the undirected bipartite graph $\mathcal{E}(w)$ with vertices $L(w) \sqcup R(w)$ and edges $B(w)$, where

$$
L(w) = \{ a \in A \mid aw \in S \}, \quad R(w) = \{ a \in A \mid wa \in S \}, \quad B(w) = \{ (a,b) \in A \mid awb \in S \}.
$$

**Example (Fibonacci, $S = \{ \varepsilon, 0, 1, 00, 01, 10, 001, 010, 100, 101, \ldots \}$)**

\[
\begin{align*}
\mathcal{E}(\varepsilon) & \quad \mathcal{E}(0) & \quad \mathcal{E}(1) \\
0 & \quad 0 & \quad 0 \\
1 & \quad 1 & \quad 1 \\
\end{align*}
\]
Definition

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**Tree sets**

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\[ n(\text{Card}(A) - 1) + 1 \]

Arnoux - Rauzy (unif.) rec.

Sturm

regular interval exchange (unif.) rec.

[D., Perrin (2016)]
Tree sets

Definition

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**Theorem** [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

A (uniformly) recurrent tree set closed under reversal is full.
Let $\sigma$ be an antimorphism.
A word $w$ is a $\sigma$-palindrome if $w = \sigma(w)$.

Example
Let $\sigma : A \leftrightarrow T$, $C \leftrightarrow G$.
The word $CTTAAG$ is a $\sigma$-palindrome.
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Theorem [Š. Starosta (2011)]

$$\text{Card}(\text{Pal}_\sigma(w)) \leq |w| + 1 - \gamma_\sigma(w)$$

with $\gamma_\sigma(w) = \# \text{ transposition acting on } w$.

A word (resp. set) is $\sigma$-full if the equality holds (resp. for all its elements).
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**Example**

Let $\sigma : I \leftrightarrow M, \ O \leftrightarrow T$ and $\tau = J \leftrightarrow O, \ K \leftrightarrow R$, fixing all other letters.
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**Example**

Let $\sigma : I \leftrightarrow M, O \leftrightarrow T$ and $\tau = J \leftrightarrow O, K \leftrightarrow R$, fixing all other letters.

\[
\text{Card}(\text{Pal}_\sigma(\text{TIMO})) = \text{Card}(\{\varepsilon, IM, TIMO\}) = 3 = 4 + 1 - 2
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\( \sigma \)-palindromes

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Let \( \sigma : I \leftrightarrow M, \ 0 \leftrightarrow T \) and \( \tau = J \leftrightarrow O, \ K \leftrightarrow R \), fixing all other letters.

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\text{Card}(\text{Pal}_\sigma(\text{TIMO})) = \text{Card}(\{\varepsilon, \text{IM}, \text{TIMO}\}) = 3 = 4 + 1 - 2
\]

\[
\text{Card}(\text{Pal}_\tau(\text{JARKKO})) = \text{Card}(\{\varepsilon, \text{A}, \text{RK}\}) = 3 < 5 = 6 + 1 - 2
\]
Let $G$ be a group containing at least one antimorphism. A word $w$ is a \textit{$G$-palindrome} if there exists a nontrivial $g \in G$ s.t. $w = g(w)$. 

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**Example**

Let $G = \langle \sigma, \tau \rangle$ with

\[
\begin{align*}
\sigma &: A \leftrightarrow E, \ I \leftrightarrow V, \ R \leftrightarrow X, \ O \leftrightarrow L \\
\tau &: A \leftrightarrow J, \ L \leftrightarrow S
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and fixing the other letters.

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- **XAVIER**, fixed by $\sigma$,
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while **NADIA** is fixed only by $\text{id}$. 

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**Example**

Let $G = \langle \sigma, \tau \rangle$ with $\sigma : A \leftrightarrow E$, $I \leftrightarrow V$, $R \leftrightarrow X$, $O \leftrightarrow L$ and $\tau : A \leftrightarrow J$, $L \leftrightarrow S$ fixing the other letters.

The following are $G$-palindromes:

- **XAVIER**, fixed by $\sigma$,
- **ÉLISE**, fixed by $\tau$,
- **JOSÉ**, fixed by $\sigma \tau \sigma$,

while **NADIA** is fixed only by $id$.

A word (set) is $G$-full if “the number of $G$-palindromes is maximal”.
Doubling transducer

A *doubling transducer* is a transducer with set of states \( \{q_0, q_1\} \) such that:

1. the input automata is a group automaton,
2. the output labels of the edges are all distinct.

**Example**

\[
\begin{align*}
\Sigma &= \{\alpha\} \\
A &= \{a, b\}
\end{align*}
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Doubling transducer

A *doubling transducer* is a transducer with set of states \( \{q_0, q_1\} \) such that:

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\( \delta_0, \delta_1 : \Sigma^* \to A^* \) are defined by \( \delta_i(u) = v \) for a path starting at \( q_i \) with input label \( u \) and output label \( v \).

**Example**

\[
\Sigma = \{\alpha\} \\
A = \{a, b\}
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\[
\begin{array}{c}
\delta_0(\alpha^3) = aba \\
\delta_1(\alpha^3) = bab
\end{array}
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The **image** of a set \( T \) is \( \delta_0(T) \cup \delta_1(T) \).

**Example**

\[
\begin{align*}
\Sigma &= \{ \alpha \} \\
A &= \{ a, b \} \\
\delta_0(\alpha^3) &= aba \\
\delta_1(\alpha^3) &= bab \\
\delta(\alpha^*) &= (\varepsilon + a)(ba)^*(\varepsilon + b)
\end{align*}
\]
Theorem [Berthé, De Felice, Delecroix, D., Leroy, Perrin, Reutenauer, Rindone (2016)]

Let \( S \) be a recurrent tree set \textbf{closed under reversal}.
The image of \( S \) by a doubling transducer is \( G \)-full, with \( G \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \).

Example (doubling of Fibonacci)

\[
\begin{align*}
q_0 & \quad 0|0 \quad \{010\} \\
0|2 \quad q_1
\end{align*}
\]

\[
\begin{align*}
1|3 \quad 0|0 \quad \{010\} \rightarrow \{012, 230\} \\
1|1 \quad 0|2 \quad q_1
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**G-palindromes**

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**Example (doubling of Fibonacci)**

Diagram:

- States: $q_0$, $q_1$.
- Transitions:
  - $0|0$ from $q_0$ to $q_1$.
  - $0|2$ from $q_0$ to $q_0$.
  - $1|1$ from $q_1$ to $q_1$.
  - $1|3$ from $q_0$ to $q_0$.

- Productions:
  - $\{010\} \rightarrow \{012, 230\}$

- Transducers:
  - $\sigma: 0 \leftrightarrow 2, \ 1 \leftrightarrow 3$
  - $\tau: 0, 2 \leftrightarrow, \ 1 \leftrightarrow 3$

- $G = \{\text{id}, \sigma, \tau, \sigma\tau\}$
MERCI CREM

THANK YOU YOYO KNIGHT