Least and greatest fixed points in ludics

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Least and greatest fixed points in Ludics

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- A logic modelling *inductive and coinductive* reasoning:
  - not only to express statements, but also a proof system in *sequent calculus*: in the paper, *MALL with fixed points*.
  - extends the *proof-program correspondence* to recursive and co-recusive programming, with coinductive datatypes.

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- A logic modelling *inductive and coinductive* reasoning:
  - not only to express statements, but also a proof system in *sequent calculus*: in the paper, *MALL with fixed points*.
  - extends the *proof-program correspondence* to recursive and co-recusive programming, with coinductive datatypes.
- Semantics of proofs:
  - Interprets not only formulas, but also *inteprets proofs*.
  - *Interactive semantics* (ludics, geometry of interaction, Hyland-Ong game semantics, ...)
  - *completeness properties* (not only at the level of *provability*, but also at the level of *proofs*).

On the completeness of an interactive semantics for a logic with least and greatest fixed points.

Logics with fixed points

Formulas

$$\begin{array}{rcl} F & ::= & F \otimes F \mid F \parr F \mid \ldots \\ & \mid & \mu X.F \\ & \mid & \nu X.F \end{array}$$

Propositional logic with least fixed point and greatest fixed point.

- μ and ν are dual.
- Examples:

$$\begin{array}{rcl} \mathrm{Nat} & := & \mu X.1 \parr X \\ \mathrm{List}(A) & := & \mu X.1 \parr (A \otimes X) \\ \mathrm{Stream}(A) & := & \nu X.1 \parr (A \otimes X) \end{array}$$

Sequent calculus

- Usual logical rules

$$\frac{\Delta \vdash \Gamma, F_1, F_2}{\Delta \vdash \Gamma, F_1 \parr F_2}\ (\parr) \qquad \frac{\Delta_1 \vdash \Gamma_1, F_1 \qquad \Delta_2 \vdash \Gamma_2, F_2}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2, F_1 \otimes F_2}\ (\otimes) \qquad \ldots$$

- Identity rules

$$\frac{}{F \vdash F}\ (\mathtt{ax}) \qquad \frac{\Delta_1 \vdash \Gamma_1, F \qquad F, \Delta_2 \vdash \Gamma_2}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2}\ (\mathtt{cut})$$

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- Rules for μ and ν

See next couple of slides

Knaster-Tarski fixed point theorem

Theorem

Let C be a complete lattice and F a monotonic operator on C.

- F has a **least** fixed point $\mu X.F$.
- $\mu X.F$ is the **least pre-fixed** point, i.e.:

$$F(\mu X.F) \subseteq \mu X.F \quad \text{and} \quad \forall S \quad F(S) \subseteq S \quad \Rightarrow \quad \mu X.F \subseteq S$$

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This gives right and left rules for μ:

$$\frac{H \vdash F[\mu X.F/X]}{H \vdash \mu X.F}\ (\mu_r) \qquad \frac{F[S/X] \vdash S}{\mu X.F \vdash S}\ (\mu_l)$$

Knaster-Tarski fixed point theorem

Theorem

Let C be a complete lattice and F a monotonic operator on C.

- F has a **greatest** fixed point $\nu X.F$.
- $\nu X.F$ is the **greatest post-fixed** point, i.e.:

$$\nu X.F \subseteq F(\nu X.F) \quad \text{and} \quad \forall S \quad S \subseteq F(S) \quad \Rightarrow \quad S \subseteq \nu X.F$$

This gives right and left rules for ν:

$$\frac{F[\nu X.F/X] \vdash H}{\nu X.F \vdash H}\ (\nu_l) \qquad \frac{S \vdash F[S/X]}{S \vdash \nu X.F}\ (\nu_r)$$

Ludics

Semantics of proofs

- In matematics, more interest for theorems and their truth than for their proofs.
  $\Rightarrow$ In Logic one traditionaly interprets formulas only.
- The proof-programs correspondence changed this perspective:

Curry-Howard correspondence

Proof theory	Programming languages
Formulas	Types
Proofs	Programs
Cut elimination	Evaluation

- Consequently, one aims at understanding the meaning of programs and proofs.

Semantics of proofs and programs

- From *extentional semantics*: interpret programs (or proofs) by the functions they compute.

What?

- To *intentional semantics*: the semantics keeps information about how computation is achieved, e.g. Game semantics.

How?

Proof theory	Programming languages	Game semantics
Proofs	Programs	Strategies
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How?

Proof theory	Programming languages	Ludics
Proofs	Programs	Designs
Formulas	Types	Behaviours
Cut elimination	Evaluation	Interaction

Ludics

Designs

- Signature: A set of names

Designs

The set of positive designs p and negative designs n are coinductively generated by:

p	$:=$	$\maltese$	Daimon	*Success*
	$\mid$	Ω	Omega	*Failure*
	$\mid$	$n_0 \mid \overline{a}\langle n_1, \ldots, n_k \rangle$	Named application	
n	$:=$	x	Variables	
	$\mid$	$\sum a(\vec{x}_a).p_a$	Sum of named abstractions	

- Reduction rule:

$$(\sum a(\vec{x}_a).p_a) \mid \overline{b}\langle \vec{n} \rangle \to p_b[\vec{n}/\vec{x}_b].$$

Ludics

Behaviours

- Orthogonality between two designs p and n:

$$p \perp n \quad \Longleftrightarrow \quad p[n/x] \to^{\star} \maltese$$

- Orthogonal of a set of designs $\mathbf{X}$:

$$\mathbf{X}^{\perp} = \{d \quad | \quad \forall e \in \mathbf{X}, \quad e \perp d\}.$$

- Behaviours are set of designs such that:

$$\mathbf{X} = \mathbf{X}^{\perp\perp}$$

Interpretation of propositional logic

Interpretation of formulas

$$
\begin{aligned}
[\![X]\!]_{\mathscr{E}} &= \mathscr{E}(X) \\
[\![F_1 \otimes F_2]\!]_{\mathscr{E}} &= \{(x \mid \otimes\langle r_1, r_2\rangle) : r_1 \in [\![F_1]\!]_{\mathscr{E}}, r_2 \in [\![F_2]\!]_{\mathscr{E}}\}^{\perp\perp} \\
[\![F_1 \parr F_2]\!]_{\mathscr{E}} &= [\![\neg F_1 \otimes \neg F_2]\!]_{\mathscr{E}}{}^{\perp}
\end{aligned}
$$

Interpretation of proofs

By induction on the last applied rule:

$$
\frac{p \vdash \Gamma, x : F_1, y : F_2}{\parr(x,y).p \vdash \Gamma, F_1 \parr F_2}\ (\parr)
\qquad
\frac{n_1 \vdash \Delta, F_1 \qquad n_2 \vdash \Gamma, F_2}{x \mid \otimes\langle n_1, n_2\rangle \vdash \Gamma, \Delta, x : F_1 \wedge F_2}\ (\otimes)
$$

Properties of the interpretation

Theorem (Soundness)

If π is a proof of F, then $[\![\pi]\!] \in [\![F]\!]$.

Theorem

The interpretation is invariant under cut elimination.

Theorem (Completeness)

If $s \in [\![F]\!]$ and $\maltese \notin s$, then there is a proof π of F such that $s = [\![\pi]\!]$.

Interpretation of fixed points

Interpretation of fixed point

Interpretation of fixed point formulas

- $[\![\mu X.F]\!]_{\mathscr{E}} = \mathrm{lfp}(\Phi)$ and $[\![\nu X.F]\!]_{\mathscr{E}} = \mathrm{gfp}(\Phi)$
- where $\Phi : \mathbf{C} \longmapsto [\![F]\!]_{\mathscr{E} \cup (X \mapsto \mathbf{C})}$.

Interpretation of fixed point rules

Should behave well w.r.t cut elimination!

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Interpretation of fixed point rules

- Rule μ: $$\frac{p \vdash \Gamma, x : P[\mu X.P/X]}{p \vdash \Gamma, x : \mu X.P} \ (\mu)$$
- Rule ν: $$\frac{d \vdash x : S, N[S^{\perp}/X]}{\mathbb{G}_{N,d} \vdash S, \nu X.N} \ (\nu)$$

Should behave well w.r.t cut elimination!

Interpretation of ν rule

The key $(\mu)-(\nu)$ step:

$$\dfrac{\dfrac{\dfrac{\Pi}{\vdash \Gamma, N^{\perp}[(\mu X.N^{\perp})/X]}}{\vdash \Gamma, \mu X.N^{\perp}}\,(\mu) \qquad \dfrac{\dfrac{\Theta}{\vdash S, N[S^{\perp}/X]}}{\vdash S, \nu X.N}\,(\nu)}{\vdash \Gamma, S}\,(cut)$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi}{\vdash \Gamma, N^{\perp}[(\mu X.N^{\perp})/X]} \qquad \dfrac{\dfrac{\Theta}{\vdash S, N[S^{\perp}/X]} \qquad \dfrac{\dfrac{\dfrac{\Theta}{\vdash S, N[S^{\perp}/X]}}{\vdash S, \nu X.N}\,(\nu)}{\vdash N^{\perp}[S/X], N[(\nu X.N)/X]}\,(N)}{\vdash S, N[(\nu X.N)/X]}\,(cut)}{\vdash \Gamma, S}\,(cut)$$

When we annotate these two proofs, we obtain an equation which we take as the definition of the ν rule.

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What about completeness?

On completeness

Completeness for Essentially Finite Designs

Completeness is at the level of proof, not at the level of provability, that means a class of elements of the models which are all interpretations of proofs.

Definition (Essentially finite designs – EFD)

Designs with a finite prefix, followed by a *copycat*.

Theorem (Completeness for EFD)

Let d be an EFD.
If $d \in [\![F]\!]$ then there is a proof π of F such that $d = [\![\pi]\!]$.

Idea of the proof

The completeness for EFD reduces to:

Theorem (Completeness for semantic inclusion)

If $[\![Q]\!] \subseteq [\![P]\!]$ then there is a proof π of $P \vdash Q$.

- Introducing an infinitary proof system S_∞ with a validity condition, inspired by Santocanale's proof systems.
- Every valid proof in S_∞ can be translated into a proof in our proof system.
- If $[\![Q]\!] \subseteq [\![P]\!]$ then there is a valid proof of $Q \vdash P$ in S_∞ (and conversely)

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As a bonus

- Validity of S_∞ proofs is a decidable property. This gives us decidability of semantic inclusion.

Conclusion

- A correct semantics for a fixed point logic in Ludics.
- Completeness for essentially finite designs.
- Decidability of semantic inclusion.

Future work

- Completeness for regular strategies.
- Full abstraction.

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Thank you for your attention!