Least and greatest fixed points in ludics

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Least and greatest fixed points in Ludics

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A logic modelling inductive and coinductive reasoning: not only to express statements, but also a proof system in sequent calculus: in the paper, MALL with fixed points extends the proof-program correspondence to recursive and co-recursive programming, with coinductive datatypes.

Semantics of proofs: interprets not only formulas, but also interprets proofs. Interactive semantics (ludics, geometry of interaction, Hyland-Ong game semantics, ...) completeness properties (not only at the level of provability, but also at the level of proofs).

On the completeness of an interactive semantics for a logic with least and greatest fixed points.
A logic modelling *inductive and coinductive* reasoning:

- not only to express statements, but also a proof system in *sequent calculus*: in the paper, *MALL with fixed points*.
- extends the *proof-program correspondence* to recursive and co-recursive programming, with coinductive datatypes.
Least and greatest fixed points in Ludics

A logic modelling *inductive and coinductive* reasoning:
- not only to express statements, but also a proof system in *sequent calculus*: in the paper, *MALL with fixed points*.
- extends the *proof-program correspondence* to recursive and co-recursive programming, with coinductive datatypes.

Semantics of proofs:
- Interprets not only formulas, but also *interprets proofs*.
- *Interactive semantics* (ludics, geometry of interaction, Hyland-Ong game semantics, ...)
- *completeness properties* (not only at the level of *provability*, but also at the level of *proofs*).

*On the completeness of an interactive semantics for a logic with least and greatest fixed points.*
Logics with fixed points
Formulas

\[ F ::= \ F \otimes F \mid F \otimes F \mid \ldots \]

Propositional logic with

\[ \mu X. F \mid \nu X. F \]

greatest fixed point.

\( \mu \) and \( \nu \) are dual.

Examples:

\[
\begin{align*}
\text{Nat} & \ ::= \ \mu X. 1 \otimes X \\
\text{List}(A) & \ ::= \ \mu X. 1 \otimes (A \otimes X) \\
\text{Stream}(A) & \ ::= \ \nu X. 1 \otimes (A \otimes X)
\end{align*}
\]
Sequent calculus

- **Usual logical rules**
  
  \[
  \frac{\Delta \vdash \Gamma, F_1, F_2}{\Delta \vdash \Gamma, F_1 \& F_2} \quad \text{(\&)} \quad \frac{\Delta_1 \vdash \Gamma_1, F_1 \quad \Delta_2 \vdash \Gamma_2, F_2}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2, F_1 \otimes F_2} \quad \text{(...)}
  \]

- **Identity rules**
  
  \[
  \frac{F \vdash F}{(ax)} \quad \frac{\Delta_1 \vdash \Gamma_1, F \quad F, \Delta_2 \vdash \Gamma_2}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2} \quad \text{(cut)}
  \]
Sequent calculus

- Usual logical rules

\[
\frac{\Delta \vdash \Gamma, F_1, F_2}{\Delta \vdash \Gamma, F_1 \otimes F_2} \quad (\otimes) \\
\frac{\Delta_1 \vdash \Gamma_1, F_1 \quad \Delta_2 \vdash \Gamma_2, F_2}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2, F_1 \otimes F_2} \quad (\otimes) \quad \ldots
\]

- Identity rules

\[
\frac{F}{F \vdash F} \quad (\text{ax}) \\
\frac{\Delta_1 \vdash \Gamma_1, F \quad F, \Delta_2 \vdash \Gamma_2}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2} \quad (\text{cut})
\]

- Rules for $\mu$ and $\nu$

See next couple of slides
Knaster-Tarski fixed point theorem

Theorem
Let $C$ be a complete lattice and $F$ a monotonic operator on $C$.

- $F$ has a **least** fixed point $\mu X.F$.
- $\mu X.F$ is the **least pre-fixed** point, i.e.:

$$F(\mu X.F) \subseteq \mu X.F \quad \text{and} \quad \forall S \quad F(S) \subseteq S \quad \Rightarrow \quad \mu X.F \subseteq S$$
Knaster-Tarski fixed point theorem

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This gives right and left rules for $\mu$:

$$
\frac{H \vdash F[\mu X. F/X]}{H \vdash \mu X. F} (\mu_r) \quad \frac{F[S/X] \vdash S}{\mu X. F \vdash S} (\mu_l)
$$
Knaster-Tarski fixed point theorem

**Theorem**

Let $C$ be a complete lattice and $F$ a monotonic operator on $C$.

- $F$ has a **greatest** fixed point $\nu X.F$.
- $\nu X.F$ is the **greatest post-fixed** point, i.e.:
  \[ \nu X.F \subseteq F(\nu X.F) \quad \text{and} \quad \forall S \quad S \subseteq F(S) \quad \Rightarrow \quad S \subseteq \nu X.F \]

This gives right and left rules for $\nu$:

\[
\frac{F[\nu X.F/X] \vdash H}{\nu X.F \vdash H} \quad (\nu_l) \quad \frac{S \vdash F[S/X]}{S \vdash \nu X.F} \quad (\nu_r)
\]
Ludics
Semantics of proofs

- In matematics, more interest for theorems and their truth than for their proofs.
  ⇒ In Logic one tradiotionaly interprets formulas only.
- The proof-programs correspondence changed this perspective:

  Curry-Howard correspondence

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- Consequently, one aims at understanding the meaning of programs and proofs.
Semantics of proofs and programs

- From *extensional semantics*: interpret programs (or proofs) by the functions they compute.

- To *intentional semantics*: the semantics keeps information about how computation is achieved, e.g. Game semantics.

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Semantics of proofs and programs

- From *extentional semantics*: interpret programs (or proofs) by the functions they compute.  
  **What?**

- To *intentional semantics*: the semantics keeps information about how computation is achieved, e.g. Game semantics.  
  **How?**

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Ludics
Designs

- Signature: A set of names

Designs

The set of positive designs $p$ and negative designs $n$ are coinductively generated by:

$$
p ::= \top \quad \text{Daimon} \quad \text{Success} \\
| \quad \Omega \quad \text{Omega} \quad \text{Failure} \\
| \quad n_0 \mid \overline{a}\langle n_1, \ldots, n_k \rangle \quad \text{Named application} \\

n ::= x \quad \text{Variables} \\
| \quad \sum a(x_a).p_a \quad \text{Sum of named abstractions}
$$

- Reduction rule:

$$
(\sum a(x_a).p_a) \mid \overline{b}\langle \overline{n} \rangle \rightarrow p_b[\overline{n}/x_b].
$$
Ludics
Behaviours

- Orthogonality between two designs $p$ and $n$:

$$p \perp n \iff p[n/x] \rightarrow^* \not\!	imes$$

- Orthogonal of a set of designs $X$:

$$X^\perp = \{ d \mid \forall e \in X, \ e \perp d \}.$$ 

- Behaviours are set of designs such that:

$$X = X^\perp\perp$$
Interpretation of propositional logic

Interpretation of formulas

\[
\begin{align*}
[X]_\epsilon &= \epsilon(X) \\
[F_1 \otimes F_2]_\epsilon &= \{(x \mid \otimes(r_1, r_2)) : r_1 \in [F_1]_\epsilon, r_2 \in [F_2]_\epsilon\}^\perp \\
[F_1 \otimes F_2]_\epsilon &= [\neg F_1 \otimes \neg F_2]_\epsilon^\perp
\end{align*}
\]

Interpretation of proofs

By induction on the last applied rule:

\[
\frac{p \vdash \Gamma, x : F_1, y : F_2}{\otimes(x, y).p \vdash \Gamma, F_1 \otimes F_2} \quad (\otimes)
\]

\[
\frac{n_1 \vdash \Delta, F_1 \quad n_2 \vdash \Gamma, F_2}{x \mid \otimes\langle n_1, n_2 \rangle \vdash \Gamma, \Delta, x : F_1 \land F_2} \quad (\otimes)
\]
Properties of the interpretation

Theorem (Soundness)

If $\pi$ is a proof of $F$, then $[\pi] \in [F]$.

Theorem

The interpretation is invariant under cut elimination.

Theorem (Completeness)

If $s \in [F]$ and $\Box \notin s$, then there is a proof $\pi$ of $F$ such that $s = [\pi]$. 
Interpretation of fixed points
Interpretation of fixed point

Interpretation of fixed point formulas

- $[[\mu X. F]]_\mathcal{E} = \text{lfp}(\Phi)$ and $[[\nu X. F]]_\mathcal{E} = \text{gfp}(\Phi)$

- where $\Phi : C \mapsto \mathcal{E}[F] \cup (X \mapsto C)$.

Interpretation of fixed point rules

Should behave well w.r.t cut elimination!
Interpretation of fixed point

Interpretation of fixed point formulas

- \([\mu X. F]_{\mathcal{E}} = \text{lf}(\Phi)\) and \([\nu X. F]_{\mathcal{E}} = \text{gfp}(\Phi)\)
- where \(\Phi : \mathcal{C} \rightarrow \mu F K E \cup (X \rightarrow \mathcal{C})\).

Interpretation of fixed point rules

- Rule \(\mu\):
  \[ p \vdash \Gamma, x : P[\mu X.P/X] \]
  \[ \frac{p \vdash \Gamma, x : \mu X.P}{p \vdash \Gamma, x : \mu X.P} \quad (\mu) \]
- Rule \(\nu\):
  \[ d \vdash x : S, N[S⊥/X] \]
  \[ \frac{G_{N,d} \vdash S, \nu X.N}{G_{N,d} \vdash S, \nu X.N} \quad (\nu) \]

Should behave well w.r.t cut elimination!
Interpretation of $\nu$ rule

The key $(\mu) - (\nu)$ step:

$$\begin{align*}
\Pi & \vdash \Gamma, N^\perp[(\mu X. N^\perp)/X] \\
\vdash & \Gamma,\mu X.N^\perp \\
\vdash & \Gamma, S \\
\downarrow & \\
\Theta & \vdash S, N[S^\perp/X] \\
\vdash & S, \nu X. N
\end{align*}$$

$(\mu)$

$(\nu)$

$(\text{cut})$

When we annotate these two proofs, we obtain an equation which we take as the definition of the $\nu$ rule.
Properties of the interpretation

Theorem (Soundness)

*If* $\pi$ *is a proof of* $F$, *then* $[\pi] \in [F]$.

Theorem

*The interpretation is invariant under cut elimination.*
Properties of the interpretation

Theorem (Soundness)

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Theorem

The interpretation is invariant under cut elimination.

What about completeness?
On completeness
Completeness for Essentially Finite Designs

Completeness is at the level of proof, not at the level of provability, that means a class of elements of the models which are all interpretations of proofs.

Definition (Essentially finite designs – EFD)
Designs with a finite prefix, followed by a copycat.

Theorem (Completeness for EFD)
Let $d$ be an EFD.
If $d \in \left[ F \right]$ then there is a proof $\pi$ of $F$ such that $d = \left[ \pi \right]$. 
Idea of the proof

The completeness for EFD reduces to:

Theorem (Completeness for semantic inclusion)

If $\llbracket Q \rrbracket \subseteq \llbracket P \rrbracket$ then there is a proof $\pi$ of $P \vdash Q$.

- Introducing an infinitary proof system $S_\infty$ with a validity condition, inspired by Santocanale’s proof systems.
- Every valid proof in $S_\infty$ can be translated into a proof in our proof system.
- If $\llbracket Q \rrbracket \subseteq \llbracket P \rrbracket$ then there is a valid proof of $Q \vdash P$ in $S_\infty$ (and conversely)
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As a bonus

- Validity of $S_\infty$ proofs is a decidable property. This gives us decidability of semantic inclusion.
Conclusion

- A correct semantics for a fixed point logic in Ludics.
- Completeness for essentially finite designs.
- Decidability of semantic inclusion.

Future work

- Completeness for regular strategies.
- Full abstraction.
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Thank you for your attention!