Cut elimination for infinitary proofs

Amina Doumane
IRIF-Université Paris Diderot

August 2016 - CSL

Joint work with:
David Baelde & Alexis Saurin
LSV-ENS Cachan & IRIF-Université Paris 7
Introduction
Introduction

• Inductive and coinductive definitions

A natural number is either 0 or the successor of a natural number.
Introduction

- Inductive and coinductive definitions

\[ N = 1 \oplus N \]
Introduction

- Inductive and coinductive definitions

\[ N = \mu X.1\oplus X \]
Introduction

- Inductive and coinductive definitions

\[ N = \mu X.1 \oplus X \]

A stream is made of a natural number (head) and a stream (tail).
Introduction

- Inductive and coinductive definitions

\[ N = \mu X.1 \oplus X \]
\[ S = N \otimes S \]
Introduction

Inductive and coinductive definitions

\[ N = \mu X.1 \oplus X \]
\[ S = \nu X.N \otimes X \]
Introduction

- Inductive and coinductive definitions

\[ N = \mu X.1 \oplus X \]
\[ S = \nu X.N \otimes X \]
Introduction

- Inductive and coinductive definitions

\[
N = \mu X.1 \oplus X \\
S = \nu X.N \otimes X
\]

- Proofs-programs over these data types

\[
double(n) = \begin{cases} 
0 & \text{if } n = 0 \\
\text{succ}(\text{succ}(\text{double}(m))) & \text{if } n = \text{succ}(m)
\end{cases}
\]
Introduction

- Inductive and coinductive definitions

\[ N = \mu X. 1 \oplus X \]
\[ S = \nu X. N \otimes X \]

- Proofs-programs over these data types

\[ \text{double}(n) = \begin{cases} 0 & \text{if } n = 0 \\ \text{succ}(\text{succ}(\text{double}(m))) & \text{if } n = \text{succ}(m) \end{cases} \]

\[ \Pi_{\text{double}} = \]
\[ \begin{array}{c}
\frac{1 \vdash 1}{1 \vdash 1 \oplus N} (\ominus_1) \\
\frac{1 \vdash 1 \oplus N}{1 \vdash N} (\mu_l) \\
\frac{1 \vdash N}{1 \oplus N \vdash N} (\mu_i) \\
\frac{N \vdash N}{N \vdash 1 \oplus N} (\oplus_2) \\
\frac{N \vdash 1 \oplus N}{N \vdash N} (\mu_r) \\
\frac{N \vdash N}{1 \oplus N \vdash N} (\ominus_2) \\
\frac{N \vdash N}{N \vdash N} (\mu_r) \\
\end{array} \]
Infinitary (circular) proofs in the literature

- **Verification device**: Complete deduction system giving algorithms for checking validity (Tableaux, sequent calculi)

  \[ \text{Success} \rightarrow \text{Validity} \]

  \[ \text{Failure} \rightarrow \text{Invalidity} \]

  \[ \mu\text{-calculus formula} \rightarrow \text{Proof search} \]

  Completeness arguments: Intermediate objects between syntax and semantics (Kozen, Kaivola, Walukiewicz)

But rarely as proof/program objects in themselves
Infinitary (circular) proofs in the litterature

- **Verification device**: Complete deduction system giving algorithms for checking validity (Tableaux, sequent calculi)

  \[
  \text{Success} \rightarrow \text{Validity} \\
  \mu\text{-calculus formula} \rightarrow \text{Proof search} \uparrow \downarrow \\
  \text{Failure} \rightarrow \text{Invalidity}
  \]

- **Completeness arguments**: Intermediate objects between syntax and semantics (Kozen, Kaivola, Walukiewicz)

  \[
  \mu\text{-calculus formula} \rightarrow \text{Circular proof} \rightarrow \text{Finite axiomatization}
  \]
Infinitary (circular) proofs in the literature

- **Verification device**: Complete deduction system giving algorithms for checking validity (Tableaux, sequent calculi)

- **Completeness arguments**: Intermediate objects between syntax and semantics (Kozen, Kaivola, Walukiewicz)

- But rarely as proof/programm objects in themselves
Structural proof theory

Two main properties:

- Syntactic cut-elimination
Structural proof theory

Two main properties:

- **Syntactic cut-elimination**
  - **Motivation:** At the heart of proofs-as-programmes viewpoint

- **Focalization**
  - **Motivation:** Proof search strategy based on the notion of polarity

State of art de la focalization: le nothing peut être provoquant.
Structural proof theory

Two main properties:

- **Syntactic cut-elimination**
  - **Motivation:** At the heart of proofs-as-programmes viewpoint
  - **State of art:** Semantical cut elimination (Brotherstone), Additive fragment (Fortier-Santocanale)

- **Focalization**
  - **Motivation:** Proof search strategy based on the notion of polarity
  - **State of art:** Nothing

State of art de la focalization: le nothing peut être provoquant.
Structural proof theory

Two main properties:

- **Syntactic cut-elimination**
  - **Motivation:** At the heart of proofs-as-programms viewpoint
  - **State of art:** Semantical cut elimination (Brotherstone), Additive fragment (Fortier-Santocanale)

- **Focalization**
  - **Motivation:** Proof search strategy based on the notion of polarity
  - **State of art:** Nothing

State of art de la focalization: le nothing peut être provoquant.
Infinitary proof system $\mu \text{MALL}^\infty$
Formulas

$\mu MALL^\infty$ formulas

\[
F ::= X \mid \top \mid \bot \mid 0 \mid 1 \mid F \otimes F \mid F \otimes F \mid F \land F \mid F \oplus F \quad \text{MALL}
\]

| $\mu X.F$ |
| $\nu X.F$ |

- $\mu$ and $\nu$ are dual.

**Example:** $\neg (\nu X. X \otimes X) = \mu X. X \otimes X$.

- Data types encoding

\[
\text{Nat} ::= \mu X.1 \oplus X
\]

\[
\text{Stream}(A) ::= \nu X. A \otimes X
\]
Sequent calculus

$\mu MALL^{\infty}$ pre-proofs are the trees coinductively generated by:

**Usual logical rules**

$$\Rightarrow \Gamma, F, \Gamma, \Delta, G \quad \Rightarrow \Gamma, F, G \quad \Rightarrow \Gamma, F \quad \Rightarrow \Gamma, G \quad \Rightarrow \Gamma, F_i \quad \Rightarrow \Gamma, F_1 \oplus F_2$$

- **$\otimes$**
  $$\Rightarrow \Gamma, \Delta, F \otimes G$$

- **$\otimes$**
  $$\Rightarrow \Gamma, F \otimes G$$

- **$\&$**
  $$\Rightarrow \Gamma, F \& G$$

- **$\oplus$**
  $$\Rightarrow \Gamma, F_1 \oplus F_2$$

**Identity rules**

$$\Rightarrow \Gamma, F, \neg F$$

$$\Rightarrow \Gamma, F, \Delta, \neg F$$

**Rules for $\mu$ and $\nu$**

$$\Rightarrow \Gamma, F[\mu X.F/X]$$

$$\Rightarrow \Gamma, \mu X.F$$

$$\Rightarrow \Gamma, F[\nu X.F/X]$$

$$\Rightarrow \Gamma, \nu X.F$$
Pre-proofs are unsound, hence the need for a validity condition.

Validity condition

A proof is a pre-proof such that every infinite branch must unfold a $\nu$ formula infinitely often.
Sequent calculus

\[
\vdash \mu X. X \quad (\mu) \\
\vdash \nu X. X, F \quad (v)
\]

\[
\vdash \nu X. X, F \quad (v) \\
\vdash F \quad (cut)
\]

Pre-proofs are unsound, hence the need for a validity condition.
Pre-proofs are unsound, hence the need for a validity condition.

Validity condition

A proof is a pre-proof such that every infinite branch must unfold a $\nu$ formula infinitely often.
Focalization
Focalization in MALL

Idea: classify the connectives into 2 categories

- **Negative connectives:** Invertible connectives ie. we don’t lose provability by applying these rules ($\otimes, \&$).

  If $\vdash \Gamma, A \otimes B$ is provable then $\vdash \Gamma, A, B$ is also provable.

- **Positive connectives:** Non Invertible connectives ie. there is a choice to make, a bad choice may lead to a loss of provability ($\oplus, \otimes$).

  $\vdash \bot$  
  $\vdash T \oplus \bot$  
  $\vdash X \vdash 1, X^\perp$  
  $\vdash X \otimes 1, X^\perp$
Focalization in MALL

To prove a sequent $\Gamma$, apply the following:

<table>
<thead>
<tr>
<th>$\Gamma$ contains a negative formula</th>
<th>$\Gamma$ contains no negative formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>choose a negative formula and apply the unique negative rule available.</td>
<td>choose some positive formula and decompose it hereditarily until negative subformulas are reached.</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\Gamma \vdash B, B^{\perp} & \quad \text{(ax)} \\
\Gamma \vdash B, D \oplus B^{\perp} & \quad \text{($\oplus$)} \\
\Gamma \vdash B \otimes C, D \oplus B^{\perp}, D \oplus C^{\perp} & \quad \text{($\otimes$)} \\
\Gamma \vdash A \oplus (B \otimes C), D \oplus B^{\perp}, D \oplus C^{\perp} & \quad \text{($\oplus$)} \\
\Gamma \vdash A \oplus (B \otimes C), (D \oplus B^{\perp}) \otimes (D \oplus C^{\perp}) & \quad \text{(\otimes)}
\end{align*}
\]
Focalization in MALL

To prove a sequent $\Gamma$, apply the following:

<table>
<thead>
<tr>
<th>$\Gamma$ contains a negative formula</th>
<th>$\Gamma$ contains no negative formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>choose a negative formula and apply the unique negative rule available.</td>
<td>choose some positive formula and decompose it hereditarily until negative subformulas are reached.</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
  \vdash B, B^{\bot} & \quad \text{(ax)} \\
  \vdash B, D \oplus B^{\bot} & \quad \text{(⊕)} \\
  \vdash B \otimes C, D \oplus B^{\bot}, D \oplus C^{\bot} & \quad \text{(⊗)} \\
  \vdash B \otimes C, (D \oplus B^{\bot}) \otimes (D \oplus C^{\bot}) & \quad \text{(⊕)} \\
  \vdash A \oplus (B \otimes C), (D \oplus B^{\bot}) \otimes (D \oplus C^{\bot}) & \quad \text{(⊕)} \\
\end{align*}
\]
Focalization in MALL

To prove a sequent $\Gamma$, apply the following:

<table>
<thead>
<tr>
<th>$\Gamma$ contains a negative formula</th>
<th>$\Gamma$ contains no negative formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>choose a negative formula and apply the unique negative rule available.</td>
<td>choose some positive formula and decompose it hereditarily until negative subformulas are reached.</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\vdash B, B^\bot & \quad (ax) \\
\vdash C, C^\bot & \quad (ax) \\
\vdash B \otimes C, B^\bot, D \oplus C^\bot & \quad (\oplus) \\
\vdash B \otimes C, C^\bot, D \oplus C^\bot & \quad (\otimes) \\
\vdash A \oplus (B \otimes C), D \oplus B^\bot, D \oplus C^\bot & \quad (\oplus) \\
\vdash A \oplus (B \otimes C), (D \oplus B^\bot) \otimes (D \oplus C^\bot) & \quad (\otimes)
\end{align*}
\]
Classification of connectives

\( \nu \) is classified **negative** and \( \mu \) is classified **positive**, even though both are invertible.

If \( \mu \) is classified negative, we would have

\[
\Gamma \vdash T \otimes T, \mu X.X (\mu) \\
\Gamma \vdash T \otimes T, \mu X.X (\mu) \\
\Gamma \vdash T \otimes T, \mu X.X (\mu)
\]

... which is not a valid proof.
Proof of completeness of Focalization for MALL

Transforms a MALL proof into a focused proof by using:

- **Strong commutation of Negatives:** negative connectives commute down with all other connectives.

  Exemple: \((\otimes/\&\))

  \[
  \begin{align*}
  \vdash F, P, Q & \quad \vdash G, P, Q \\
  \vdash F, P \otimes Q & \quad \vdash G, P \otimes Q \\
  \vdash F \& G, P \otimes Q & \quad \vdash F \& G, P, Q \\
  \vdash F \& G, P \otimes Q & \quad \vdash F \& G, P, Q
  \end{align*}
  \]

- **Weak commutation of positives:** positive connectives commute with each others only.

  Exemple: \((\oplus/\oplus\))

  \[
  \begin{align*}
  \vdash G, P, \Gamma & \quad \vdash G, P, \Gamma \\
  \vdash G, P \oplus Q, \Gamma & \quad \vdash G, P \oplus Q, \Gamma \\
  \vdash F \oplus G, P \oplus Q, \Gamma & \quad \vdash F \oplus G, P, \Gamma \\
  \vdash F \oplus G, P \oplus Q, \Gamma & \quad \vdash F \oplus G, P, \Gamma
  \end{align*}
  \]
Proof of completeness of Focalization for $\mu MALL$

Works in the same way, under some adaptations.

- Rules commutations cannot be performed locally:

\[
\begin{align*}
\frac{}{(\ast)} & F, P \quad Q \\
\frac{}{F, P \& Q} & F \quad P \quad Q \\
\pi & F, P, Q \\
\frac{}{F, P \quad Q} & (\&) \\
\frac{}{F \& F, P \quad Q} & F \quad F \quad P \quad Q \\
\pi & F, P, Q \\
\frac{}{F, P \quad Q} & (\&) \\
\frac{}{F \& F, P \quad Q} & (v)
\end{align*}
\]

- The commutation process is productive.
- The commutation process preserves validity.
Cut elimination
**Cut elimination procedure**

- **Strategy:** “push” the cuts away from the root.

- **Cut-Cut:**

\[
\begin{align*}
\vdash \Gamma, F & \quad \vdash \neg F, \Delta, G \\
\vdash \Gamma, \Delta, G & \quad \text{(cut)} \quad \vdash \neg G, \Sigma \\
\vdash \Gamma, \Delta, \Sigma & \quad \text{(cut)}
\end{align*}
\]

\[
\uparrow
\]

\[
\begin{align*}
\vdash \neg F, \Delta, G & \quad \vdash \neg G, \Sigma \\
\vdash \Gamma, F & \quad \text{(cut)} \quad \vdash \neg F, \Delta, \Sigma \\
\vdash \Gamma, \Delta, \Sigma & \quad \text{(cut)}
\end{align*}
\]
Cut elimination procedure

- **Strategy:** “push” the cuts away from the root.

- **Cut-Cut:**

  \[
  \frac{
  \Gamma, F \quad \Gamma, \neg F, \Delta, G
  }{
  \Gamma, \Delta, G \quad \text{(cut)}
  \frac{
  \Gamma, \Delta, G \quad \Gamma, \neg G, \Sigma
  }{
  \Gamma, \Delta, \Sigma \quad \text{(cut)}
  }\]

  \[
  \Gamma, F \quad \Gamma, \neg F, \Delta, G \quad \Gamma, \neg G, \Sigma \quad \text{(m-cut)}
  \]

  \[
  \Gamma, \Delta, \Sigma
  \]
Cut elimination procedure - External operations

\[
\begin{align*}
\Gamma &\vdash \Delta, F, G & (\otimes) \\
\Gamma &\vdash \Delta, F \otimes G & \Rightarrow \\
\Gamma &\vdash \Sigma, F \otimes G & (m\text{-cut}) \\
\end{align*}
\]

\[
\begin{align*}
\Gamma &\vdash \Delta, F & (\&) \\
\Gamma &\vdash \Delta, G & \Rightarrow \\
\Gamma &\vdash \Delta, F \& G & (m\text{-cut}) \\
\Gamma &\vdash \Sigma, F \& G & (m\text{-cut}) \\
\end{align*}
\]

\[
\begin{align*}
\Gamma &\vdash \Delta, F[\mu X.F/X] & (\mu) \\
\Gamma &\vdash \Delta, \mu X.F & \Rightarrow \\
\Gamma &\vdash \Sigma, \mu X.F & (m\text{-cut}) \\
\end{align*}
\]

External operations are productive
Cut elimination procedure - Internal operations

Internal operations are not productive
Cut elimination algorithm

- **Internal phase**: Perform internal transformations while you can’t do anything else.

- **External phase**: Build a part of the output tree whenever you can.
Cut elimination algorithm

- **Internal phase:** Perform internal transformations while you can’t do anything else.

- **External phase:** Build a part of the output tree whenever you can.

- Repeat.
Cut elimination algorithm

- **Internal phase:** Perform internal transformations while you can’t do anything else.

- **External phase:** Build a part of the output tree whenever you can.

- Repeat.
Cut elimination is productive

Theorem

Internal phase always halts.

Proof sketch:
Suppose that the internal phase diverges for a proof $\pi$ of $\vdash \Delta$. Let $\theta$ be the sub-derivation of $\pi$ explored by the reduction. Extract from $\theta$ a proof of the empty sequent. We define a truth semantics for $\mu$MALL$_\infty$ formulas and show that the proof system is sound with respect to it. Contradiction.
Cut elimination is productive

Theorem
Internal phase always halts.

Proof sketch: Suppose that the internal phase diverges for a proof $\pi$ of $\vdash \Delta$.

- Let $\theta$ be the sub-derivation of $\pi$ explored by the reduction.
- Extract from $\theta$ a proof of the empty sequent.
- We define a truth semantics for $\mu MALL^\infty$ formulas and show that the proof system is sound with respect to it. Contradiction.
Theorem
The pre-proof obtained by the cut elimination algorithm is valid.

Follows the same proof idea.
Conclusion
Conclusion

Contributions:

- Proper foundations for infinitary proof theory
- Syntactic cut elimination and Focalization

Future work:

- Go beyond Linear Logic and handle structural rules
- Translate infinitary proofs to finitary ones
- Same question by preserving the computational content

Thank you for your attention!
Conclusion

- Contributions:
  - Proper foundations for infinitary proof theory
  - Syntactic cut elimination and Focalization

- Future work:
  - Go beyond Linear Logic and handle structural rules
  - Translate infinitary proofs to finitary ones
  - Same question by preserving the computational content

Thank you for your attention!