Cut elimination for infinitary proofs

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Introduction

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- **Inductive and coinductive definitions**

A natural number is either 0 or the successor of a natural number.

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$$\mathsf{N} = 1 \oplus \mathsf{N}$$

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A stream is made of a natural number (head) and a stream (tail).

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$$\mathsf{N} = \mu X.1 \oplus X$$
$$\mathsf{S} = N \otimes \mathsf{S}$$

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- **Proofs-programs over these data types**

$$\begin{array}{rcll} double(n) & = & 0 & \text{if } n = 0 \\ & = & succ(succ(double(m))) & \text{if } n = succ(m) \end{array}$$

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$$\Pi_{double} = \dfrac{\dfrac{\dfrac{\dfrac{}{1 \vdash 1}\,(1)}{1 \vdash 1 \oplus N}\,(\oplus_1)}{1 \vdash N}\,(\mu_l) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Pi_{double}}{N \vdash N}}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu_r)}{N \vdash 1 \oplus N}\,(\oplus_2)}{N \vdash N}\,(\mu_r)}{1 \oplus N \vdash N}\,(\oplus_l)}{N \vdash N}\,(\mu_l)$$

Infinitary (circular) proofs in the litterature

- **Verification device**: Complete deduction sytem giving algorithms for checking validity (Tableaux, sequent calculi)

$$\begin{array}{ccccccc} & & & & \text{Success} & \rightarrow & \text{Validity} \\ \mu\text{-calulus formula} & \rightarrow & \text{Proof search} & \begin{array}{c}\nearrow \\ \searrow\end{array} & & & \\ & & & & \text{Failure} & \rightarrow & \text{Invalidity} \end{array}$$

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- **Completeness arguments**: Intermediate objects between syntax and semantics (Kozen, Kaivola, Walukiewicz)

$$\mu\text{-calulus formula} \quad \rightarrow \quad \text{Circular proof} \quad \rightarrow \quad \text{Finite axiomatization}$$

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- **Completeness arguments**: Intermediate objects between syntax and semantics (Kozen, Kaivola, Walukiewicz)

μ-calulus formula $\rightarrow$ Circular proof $\rightarrow$ Finite axiomatization

- **But rarely as proof/programm objects in themselves**

Structural proof theory

Two main properties:

- Syntactic cut-elimination

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  - **Motivation:** At the heart of proofs-as-programms viewpoint
- Focalization
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State of art de la focalization: le nothing peut être provoquant.

Structural proof theory

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Infinitary proof system $\mu MALL^{\infty}$

Formulas

$\mu MALL^\infty$ formulas

$$
\begin{array}{rcll}
F & ::= & X \mid \top \mid \bot \mid 0 \mid 1 \mid F \otimes F \mid F \parr F \mid F \& F \mid F \oplus F & \textit{MALL} \\
 & \mid & \mu X.F & \textbf{least fixed point} \\
 & \mid & \nu X.F & \textbf{greatest fixed point}
\end{array}
$$

- μ and ν are dual.

  <u>Example:</u> $\neg(\nu X.X \otimes X) = \mu X.X \parr X$.

- Data types encoding

$$
\begin{array}{rcl}
\mathrm{Nat} & := & \mu X.1 \oplus X \\
\mathrm{Stream}(A) & := & \nu X.A \otimes X
\end{array}
$$

Sequent calculus

$\mu MALL^\infty$ pre-proofs are the trees coinductively generated by:

Usual logical rules

$$\frac{\vdash \Gamma, F \qquad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}\ (\otimes) \qquad \frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\ (\parr) \qquad \frac{\vdash \Gamma, F \qquad \vdash \Gamma, G}{\vdash \Gamma, F \& G}\ (\&) \qquad \frac{\vdash \Gamma, F_i}{\vdash \Gamma, F_1 \oplus F_2}\ (\oplus_i)$$

Identity rules

$$\frac{}{\vdash F, \neg F}\ (\mathtt{ax}) \qquad \frac{\vdash \Gamma, F \qquad \vdash \Delta, \neg F}{\vdash \Gamma, \Delta}\ (\mathtt{cut})$$

Rules for μ and ν

$$\frac{\vdash \Gamma, F[\mu X.F/X]}{\vdash \Gamma, \mu X.F}\ (\mu) \qquad \frac{\vdash \Gamma, F[\nu X.F/X]}{\vdash \Gamma, \nu X.F}\ (\nu)$$

Sequent calculus

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\,(\mu)}{\vdash \mu X.X}\,(\mu) \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, F}\,(\nu)}{\vdash \nu X.X, F}\,(\nu)}{\vdash F}\,(\mathtt{cut})$$

Sequent calculus

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Pre-proofs are unsound, hence the need for a validity condition.

Sequent calculus

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Validity condition

A proof is a pre-proof such that every infinite branch must unfold a ν formula infinitly often.

Focalization

Focalization in MALL

Idea: classify the connectives into 2 categories

- **Negative connectives:** Invertible connectives ie. we don't lose provability by applying these rules ($\parr, \&$).

  If $\vdash \Gamma, A \parr B$ is provable then $\vdash \Gamma, A, B$ is also provable.

- **Positive connectives:** Non Invertible connectives ie. there is a choice to make, a bad choice may lead to a loss of provability ($\oplus, \otimes$).

$$\frac{\vdash \bot}{\vdash \top \oplus \bot}\ (\oplus) \qquad \frac{\vdash X \quad \vdash 1, X^{\bot}}{\vdash X \otimes 1, X^{\bot}}\ (\otimes)$$

Focalization in MALL

To prove a sequent Γ, apply the following:

Γ **contains a negative formula**	Γ **contains no negative formula**
choose a negative formula and apply the unique negative rule available.	choose some positive formula and decompose it hereditarily until negative subformulas are reached.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^{\perp}}\,(ax)}{\vdash B, D \oplus B^{\perp}}\,(\oplus) \qquad \dfrac{\dfrac{}{\vdash C, C^{\perp}}\,(ax)}{\vdash C, D \oplus C^{\perp}}\,(\oplus)}{\vdash B \otimes C, D \oplus B^{\perp}, D \oplus C^{\perp}}\,(\otimes)}{\vdash A \oplus (B \otimes C), D \oplus B^{\perp}, D \oplus C^{\perp}}\,(\oplus)}{\vdash A \oplus (B \otimes C), (D \oplus B^{\perp}) \parr (D \oplus C^{\perp})}\,(\parr)
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$$

Focalization for $\mu MALL$

Classification of connectives

ν is classified **negative** and μ is classified **positive**, even though both are invertible.

If μ is classified negative, we would have

$$\dfrac{\dfrac{\vdash \quad \vdots}{\vdash \top \otimes \top, \mu X.X}\,(\mu)}{\vdash \top \otimes \top, \mu X.X}\,(\mu)$$

... which is not a valid proof.

Proof of completeness of Focalization for MALL

Transforms a MALL proof into a focused proof by using:

- **Strong commutation of Negatives:** negative connectives commute down with <u>all other connectives</u>.

  <u>Exemple:</u> $(\parr/\&)$

$$\dfrac{\dfrac{\vdash F,P,Q}{\vdash F,P\parr Q}\,(\parr) \qquad \dfrac{\vdash G,P,Q}{\vdash G,P\parr Q}\,(\parr)}{\vdash F\&G,P\parr Q}\,(\&) \rightarrow \dfrac{\dfrac{\vdash F,P,Q \qquad \vdash G,P,Q}{\vdash F\&G,P,Q}\,(\&)}{\vdash F\&G,P\parr Q}\,(\parr)$$

- **Weak commutation of positives:** positive connectives commute with <u>each others only</u>.

  <u>Exemple:</u> $(\oplus/\oplus)$

$$\dfrac{\dfrac{\vdash G,P,\Gamma}{\vdash G,P\oplus Q,\Gamma}\,(\oplus)}{\vdash F\oplus G,P\oplus Q,\Gamma}\,(\oplus) \rightarrow \dfrac{\dfrac{\vdash G,P,\Gamma}{\vdash F\oplus G,P,\Gamma}\,(\oplus)}{\vdash F\oplus G,P\oplus Q,\Gamma}\,(\oplus)$$

Proof of completeness of Focalization for $\mu MALL$

Works in the same way, under some adaptations.

- **Rules commutations cannot be performed locally:**

$$
\dfrac{\dfrac{\dfrac{\dfrac{(\star)}{\vdash F, P \parr Q} \quad \dfrac{\dfrac{\pi}{\vdash F, P, Q}}{\vdash F, P \parr Q}\,(\parr)}{\vdash F \& F, P \parr Q}\,(\&)}{\vdash F, P \parr Q} \quad \dfrac{\dfrac{\pi}{\vdash F, P, Q}}{\vdash F, P \parr Q}\,(\parr)}{\vdash F \& F, P \parr Q}\,(\&)}{(\star)\ \vdash F, P \parr Q}\,(v)
$$

- **The commutation process is productive.**
- **The commutation process preserves validity.**

Cut elimination

Cut elimination procedure

- **Strategy:** "push" the cuts away from the root.
- **Cut-Cut:**

$$\dfrac{\dfrac{\vdash \Gamma, F \qquad \vdash \neg F, \Delta, G}{\vdash \Gamma, \Delta, G}\,(\mathtt{cut}) \qquad \vdash \neg G, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(\mathtt{cut})$$

$$\updownarrow$$

$$\dfrac{\vdash \Gamma, F \qquad \dfrac{\vdash \neg F, \Delta, G \qquad \vdash \neg G, \Sigma}{\vdash \neg F, \Delta, \Sigma}\,(\mathtt{cut})}{\vdash \Gamma, \Delta, \Sigma}\,(\mathtt{cut})$$

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Cut elimination procedure - External operations

$$\dfrac{\dfrac{\vdash \Delta, F, G}{\vdash \Delta, F \parr G}\,(\parr) \qquad \dots}{\vdash \Sigma, F \parr G}\,(\texttt{m-cut}) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F, G \qquad \dots}{\vdash \Sigma, F, G}\,(\texttt{m-cut})}{\vdash \Sigma, F \parr G}\,(\parr)$$

$$\dfrac{\dfrac{\vdash \Delta, F \qquad \vdash \Delta, G}{\vdash \Delta, F \& G}\,(\&) \qquad \dots}{\vdash \Sigma, F \& G}\,(\texttt{m-cut}) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F \qquad \dots}{\vdash \Sigma, F}\,(\texttt{m-cut}) \qquad \dfrac{\vdash \Delta, G \qquad \dots}{\vdash \Sigma, G}\,(\texttt{m-cut})}{\vdash \Sigma, F \& G}\,(\&)$$

$$\dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}\,(\mu) \qquad \dots}{\vdash \Sigma, \mu X.F}\,(m-cut) \quad \Rightarrow \quad \dfrac{\dfrac{\vdash \Delta, F[\mu X.F/X] \qquad \dots}{\vdash \Sigma, F[\mu X.F/X]}\,(m-cut)}{\vdash \Sigma, \mu X.F}\,(\mu)$$

External operations are productive

Cut elimination procedure - Internal operations

$$\dfrac{\ldots \qquad \dfrac{\vdash \Delta, F_2 \qquad \vdash \Delta, F_1}{\vdash \Delta, F_2 \& F_1}\,(\&) \qquad \dfrac{\vdash \Gamma, F_i^{\perp}}{\vdash \Gamma, F_1^{\perp} \oplus F_2^{\perp}}\,(\oplus_i)}{\vdash \Sigma}\,(\texttt{m-cut})$$

$$\Rightarrow \quad \dfrac{\ldots \qquad \vdash \Delta, F_i \qquad \vdash \Gamma, F_i^{\perp}}{\vdash \Sigma}\,(\texttt{m-cut})$$

$$\dfrac{\ldots \qquad \dfrac{\vdash \Delta, F[\mu X.F/X]}{\vdash \Delta, \mu X.F}\,(\mu) \qquad \dfrac{\vdash \Gamma, F^{\perp}[\nu X.F^{\perp}/X]}{\vdash \Gamma, \nu X.F^{\perp}}\,(\nu)}{\vdash \Sigma}\,(\texttt{m-cut})$$

$$\Rightarrow \quad \dfrac{\ldots \qquad \vdash \Delta, F[\mu X.F/X] \qquad \vdash \Gamma, F^{\perp}[\nu X.F^{\perp}/X]}{\vdash \Sigma}\,(\texttt{m-cut})$$

Internal operations are not productive

Cut elimination algorithm

- Internal phase: Perform internal transformations while you can't do anything else.
- External phase: Build a part of the output tree whenever you can.

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Cut elimination is productive

Theorem

Internal phase always halts.

Cut elimination is productive

Theorem

Internal phase always halts.

Proof sketch: Suppose that the internal phase diverges for a proof π of $\vdash \Delta$.

- Let θ be the sub-derivation of π explored by the reduction.
- Extract from θ a proof of the empty sequent.
- We define a truth semantics for $\mu MALL^{\infty}$ formulas and show that the proof system is sound with respect to it. Contradiction.

Cut elimination produces a proof

Theorem

The pre-proof obtained by the cut elimination algorithm is valid.

Follows the same proof idea.

Conclusion

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- **Contributions:**
  - Proper foundations for infinitary proof theory
  - Syntactic cut elimination and Focalization
- **Future work:**
  - Go beyond Linear Logic and handle structural rules
  - Translate infinitrary proofs to finitary ones
  - Same question by preserving the computational content

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Thank you for your attention!