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Coxeter factorizations and the Matrix Tree theorem with generalized Jucys-Murphy weights (extended abstract)

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Abstract. We prove a universal (case-free) formula for the *weighted* enumeration of factorizations of Coxeter elements into products of reflections valid in any well-generated reflection group, in terms of the spectrum of an associated Laplacian matrix that we introduce. This covers in particular all finite Coxeter groups. For symmetric groups and for minimal length, our statement is an instance of the Matrix Tree theorem.

The formula is relative to the choice of a weighting system, that corresponds to the choice of n free scalar parameters and of a parabolic tower of subgroups. This leads us to introduce (a class of) variants of the Jucys-Murphy elements for every group, from which we define a new notion of 'tower equivalence' of virtual characters. The main technical point is to prove the tower equivalence between virtual characters naturally appearing in the problem, and exterior products of the reflection representation.

Keywords: Coxeter groups, factorisations, matrix tree theorem, higher genus, Laplacian.

1 Introduction

More than a century after its discovery and despite its simplicity, Kirchoff's "Matrix Tree" theorem is still one of the most beautiful and remarkable enumeration formulas in mathematics. It expresses the weighted number of labelled trees on the vertex-set $\{1, 2, ..., n\}$, where each edge $\{i, j\}$ receives an arbitrary complex weight ω_{ij}, as the determinant of the so-called *Laplacian matrix* constructed form these weights. Kirchoff's theorem expresses the solution to an enumerative problem in terms of the spectrum of a related operator, a paradigm that has now been widely used in enumeration. In this work, we will solve an enumeration problem related to complex reflection groups by relating it to the spectrum of a natural Laplacian operator that we introduce.

When all weights are equal to one, Kirchoff's determinant evaluates to Cayley's famous n^{n-2} formula that counts (unweighted) labelled trees of size n. Via the monodromy

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correspondence and the interpretation of trees as branched coverings of the sphere by itself, n^{n-2} is also the number of factorisations $t_1...t_{n-1} = (1,2,...,n)$ of the increasing long cycle in the symmetric group $\mathfrak{S}_n$ into a product of $n-1$ transpositions. Thus Kirchoff's theorem can be viewed as a far-reaching weighted generalization of Cayley's formula.

But Cayley's formula has at least two other sorts of generalizations. The first one, motivated by the study of Hurwitz numbers, considers coverings of the sphere by a surface of higher genus g, which amounts to enumerating factorizations $t_1...t_\ell = (1,2,\ldots,n)$ with $\ell = n-1+2g$. This was done by Jackson [9] (see also Shapiro, Shapiro, Vainstein [14]) who obtained a striking product form for the generating function of these numbers. Jackson's formula was unified with the matrix tree theorem in a beautiful paper of Burman and Zvonkine [4] (independently, Alon and Kozma [1]), which was one of the inspirations for this work. They show that the generating function of factorisations of arbitrary length of long cycles into transpositions in the group $\mathfrak{S}_n$, where the transposition (i,j) receives the weight ω_{ij}, has the remarkable form

$$\frac{1}{n!} e^{t\sum_i \lambda_i} \prod_i (1 - e^{-t\lambda_i}), \tag{1.1}$$

where the λ_i are the non zero eigenvalues of the Laplacian. When $t \to 0$, one recovers the matrix tree theorem, while Jackson's unweighted formula corresponds to $\lambda_i = n$.

Another generalization of Cayley's formula is to replace the symmetric group $\mathfrak{S}_n$ by some other finite group W of matrices, for example a real reflection group with transpositions and the long cycle replaced respectively by reflections and the so-called Coxeter element. Such a generalization was conjectured by Looijenga and proved by Deligne ([7], crediting discussions with Tits and Zagier) and later rediscovered by Chapoton [5]. Here again, Deligne's formula, as well as Bessis' version [2] for *complex* reflection groups, takes a remarkable product form as $h^n n!/|W|$ where h and n are parameters of the group that we will encounter later in this paper. This formula was later generalized by the first author and Stump [6], who show that factorizations of arbitrary length have a fully factored generating function given by (here $\mathcal{R}$ denotes the set of reflections of W):

$$\frac{1}{|W|} e^{t|\mathcal{R}|} (1 - e^{-th})^n. \tag{1.2}$$

This gives a common generalisation of Jackson's and Deligne's formulas.

The remarkable product form of all these formulas, and the similarity between (1.2) and (1.1), raise several natural questions: can these results be put under a common roof, i.e. are they shadows of a more general universal result? What would be the "Laplacian" for other groups? And, is there a conceptual *explanation* for this product form?

In this paper we will see that the answers to these questions are very much related. We will give an explanation of the product form by the existence of a correspondence between some virtual characters of reflection groups and the exterior powers of their reflection representations. This also leads us to the good framework for the level of generality at which a general statement can be made, which turns out to involve towers of subgroups and generalizations of the Jucys-Murphy elements.

2 Main results

Given a complex vector space $V \cong \mathbb{C}^n$, we call a *finite* subgroup $W \leq \mathrm{GL}(V)$ a *complex reflection group of rank n* if it is generated by unitary reflections, which are $\mathbb{C}$-linear maps whose fixed spaces are hyperplanes. We further say that W is *irreducible* if it has no stable linear subspaces apart from V and $\{0\}$. A *parabolic* subgroup W_X of W is the fixator of some flat $X := \bigcap_{i\in I} H_i$. We say that a complex reflection group W is *well-generated* if it can be generated by n reflections. Well-generated groups are precisely the ones for which a class of *Coxeter elements* can be defined, see [8, §2.3]. In particular they include the symmetric group with the long cycles, and more generally real reflection groups with the product of all simple generators. The *Coxeter number* of the group, denoted by h, is the order of its Coxeter element.

For such a group W with set of reflections $\mathcal{R}$ and a tower of parabolic subgroups

$$T := \big(\{\mathbf{1}\} = W_0 < W_1 < \cdots < W_n = W\big), \tag{2.1}$$

we consider the weight function $\mathbf{w}_T : \mathcal{R} \to \boldsymbol{\omega} := (\omega_i)_{i=1}^n$ defined by the filtration of $\mathcal{R}$ by T, namely $\mathbf{w}_T(\tau) := \omega_i$ if $\tau \in W_i \setminus W_{i-1}$. We study the exponential generating function of weighted reflection factorizations of *any* element c of the Coxeter class $\mathcal{C}$:

$$\mathrm{FAC}_W^T(t,\boldsymbol{\omega}) := \sum_{\ell\geq 0} \frac{t^\ell}{\ell!} \sum_{\substack{(\tau_1,\cdots,\tau_\ell,c)\in\mathcal{R}^\ell\times\mathcal{C}\\ \tau_1\cdots\tau_\ell=c}} \mathbf{w}_T(\tau_1)\cdots\mathbf{w}_T(\tau_\ell). \tag{2.2}$$

Thm. 1 gives a product formula for (2.2) which is a simultaneous extension of both the Chapuy-Stump formula (1.2) and (partially) the Burman-Zvonkine formula (1.1). For an arbitrary complex reflection group W, we introduce first the *W-Laplacian*

$$L_W^T(\boldsymbol{\omega}) := \sum_{\tau\in\mathcal{R}} \mathbf{w}_T(\tau)\big(\,\mathrm{id} - \tau\big) \in \mathrm{GL}(V). \tag{2.3}$$

Theorem 1 (Main combinatorial result)**.** *For a well-generated reflection group W of rank n and Coxeter number h, and a parabolic tower T, the weighted enumeration (2.2) is given by*

$$\mathrm{FAC}_W^T(t,\boldsymbol{\omega}) = \frac{e^{t\mathbf{w}_T(\mathcal{R})}}{h} \cdot \prod_{i=1}^n \big(1 - e^{-t\lambda_i^T(\boldsymbol{\omega})}\big),$$

where $\mathbf{w}_T(\mathcal{R}) := \sum_{\tau\in\mathcal{R}} \mathbf{w}_T(\tau)$, and the $\lambda_i^T(\boldsymbol{\omega})$ are the eigenvalues of the W-Laplacian $L_W^T(\boldsymbol{\omega})$.

The weighting systems of (2.2) are closely related to the notion of *generalized Jucys Murphy elements* that we propose in this paper. Given a tower as in (2.1), we introduce the elements $J_1, \ldots, J_n$ in the group algebra $\mathbb{C}[W]$, where J_i is the sum of all reflections of W that belong to $W_i \setminus W_{i-1}$:

$$J_i := \sum_{\tau \in \mathcal{R} \cap W_i \setminus W_{i-1}} \tau \quad \in \mathbb{C}[W]. \tag{2.4}$$

The cornerstone of the Okounkov-Vershik approach is that for the standard tower of $\mathfrak{S}_n$, these elements generate the Gelfand-Tsetlin algebra, in particular they generate the center of $\mathfrak{S}_n$ and separate all conjugacy classes, hence all characters. For a general tower as (2.1) this is no longer the case, which leads us to the following question: which characters does this algebra separate? And what does it *mean* for characters to be separated, or not? We will not answer these questions here, but our work shows that they deserve further interest. We say that two (virtual) characters of the group W are *tower equivalent* if they are equal on the subalgebra $\mathbb{C}[\boldsymbol{J}^T]$ of $\mathbb{C}[W]$ generated by the generalized Jucys-Murphy elements (2.4), for *any* choice of the parabolic tower T. As we will see, Theorem 1 is a direct consequence of the following result, which is the main representation-theoretic result of this paper and really the *de facto* explanation for the nice factored form in Theorem 1.

Recall that for any character χ of W, we may define the Coxeter number of χ as the normalized trace $c_\chi := (\dim \chi)^{-1} \chi\big(\sum_{\tau \in \mathcal{R}} (1-\tau)\big)$ (which is an integer, see [8, Cor. 4.16]). Grouping characters with respect to this statistic was a key ingredient of the uniform proof of the Chapuy-Stump formula in [8]. Here too, consider the virtual character

$$P_m := \sum_{\chi \in \widehat{W},\ c_\chi = m} \chi(c^{-1}) \cdot \chi,$$

for any integer m. The main theorem in [8] implies that the (virtual) dimension of P_m is $(-1)^k \cdot \binom{n}{k}$ when $m = kh$ and 0 otherwise. The following is a vast generalization.

Theorem 2 (Tower equivalence, main result)**.** *For W and T as in Theorem 1 we have that*

$$P_m \equiv (-1)^k \chi_{\Lambda^k V_{ref}} \tag{2.5}$$

if $m = hk$ and $P_m \equiv 0$ otherwise. Here $\equiv$ denotes tower equivalence, and $\chi_{\Lambda^k V_{ref}}$ is the character of the k-th exterior power $\Lambda^k V_{ref}$ of the reflection representation V_{ref} of W.

Tower equivalence sheds a new light on the existence of product formulas, even if (for now) our proof of Theorem 2 is not case-free. In fact, all formulas in the field including the ones in [7, 2, 6] first received non-case-free proofs, using computer verification. Only very recently were case-free proofs of the Deligne formula obtained (for Weyl groups by Michel [11] and for arbitrary real groups by the second author [8]), and both prove in fact the more general Chapuy-Stump formula, whose additional structure played an important role in their discovery. We believe that our results, in addition to their intrinsic interest, will play a similar role. In particular, the study of the generalized Jucys-Murphy algebras can lead to interesting progress, and the links between Theorem 2 and the unipotent characters U_{χ_k} indexed by $\chi_k := \chi_{\Lambda^k(V_{ref})}$ which appear in [11] could be further investigated. This may lead in the future to a case-free proof of the Tower equivalence.

In fact, our paper already contains an example of this phenomenon. In Section 5, we will give a generalization of the *Matrix forest* theorem to reflection groups. This relies on our main theorem, and on other developments (Thm. 3) which are completely case-free but were actually motivated by this generalization. As an unexpected bonus, we obtain from these a new recursion for Coxeter numbers of parabolic subgroups (Cor. 4). In a separate project we deduce from this a new uniform derivation of Deligne's formula from the Deligne-Reading recursion [13] (which so far had led only to case-by-case proofs).

Corollaries and further comments.

Looking at the first non-zero coefficient in $\mathrm{FAC}_W^T(t,\boldsymbol{\omega})$ in Theorem 1, we obtain:

Corollary 1 (Matrix-tree with generalized Jucys-Murphy weights, for reflection groups)**.** *Let W be a well-generated complex reflection group, T a parabolic tower, and assume the notation and hypotheses of Theorem 1. Then the weighted number of reduced factorisations of a Coxeter element into a product of transpositions is given by*

$$\sum_{\substack{(\tau_1,\tau_2,\dots,\tau_n,c)\in\mathcal{R}^n\times\mathcal{C}\\ \tau_1\tau_2\dots\tau_n=c}} \mathbf{w}_T(\tau_1)\mathbf{w}_T(\tau_2)\dots\mathbf{w}_T(\tau_n) = \frac{n!}{h}\det L_W^T(\boldsymbol{\omega}), \tag{2.6}$$

where $L_W^T(\boldsymbol{\omega})$ is the Laplacian matrix.

In the case of the symmetric group, $W = A_n$, it is well known that the equivalence $\equiv$ in Theorem 2 is an equality, which underlies the fact the the results of [4, 1] work with arbitrary weights and no reference to a tower structure. Note that the usual practice is to define the Laplacian matrix in type $A_n = \mathfrak{S}_{n+1}$ as an $(n+1)\times(n+1)$ matrix having a zero eigenvalue. Here our matrix is defined directly as a generically invertible $n\times n$ matrix but it is essentially the same object. For other groups, it seems hopeless to expect a general factored formula under general weights as it already fails for dihedral groups – for which computations can be done explicitely. Some isolated facts will be discussed in the final long version, for example for the group B_n (signed permutations), it is indeed possible to work under general weights at the price of defining the Laplacian in a different, reducible, representation.

Plan of the paper. In Section 3 we discuss towers, generalized Jucys-Murphy elements, and generalizations of the Gelfand-Tsetlin basis. In Section 4 we show the equivalence between Theorems 1 and 2 via the concept of Lie-like elements due to Burman and Zvonkine [4]. In Section 5 we give a version of the *matrix-forest* theorem for reflection groups and some case-free developments concerning the Coxeter numbers of parabolics. Section 6 contains a *very* partial sketch of the proofs of our main results.

3 Frobenius lemmas and Gelfand-Tsetlin decomposition

A classical technique in the enumeration of factorizations in groups is via a lemma of Frobenius that relates the enumeration to character evaluations (see e.g. [6]). We give here a variant of it (see also [4]) as it applies for the weighted case (2.2). To simplify our formulas, we extend characters $\chi \in \widehat{W}$ to the power series $\mathbb{C}[W][[t]]$ by defining $\chi\big(\sum a_i \cdot t^i\big) := \sum \chi(a_i) \cdot t^i$, for any coefficients $a_i \in \mathbb{C}[W]$.

Lemma 1 (weighted Frobenius)**.** *The enumeration of weighted factorizations* (2.2) *is given by*

$$FAC_W^T(t,\boldsymbol{\omega}) = (1/h) \cdot \sum_{\chi\in\widehat{W}} \chi(c^{-1}) \cdot \chi\big(e^{t\mathbf{A}(\boldsymbol{\omega})}\big),$$

where $\mathbf{A}(\boldsymbol{\omega}) := \sum_{\tau\in\mathcal{R}} \mathbf{w}_T(\tau)\cdot\tau$ *belongs to the group algebra* $\mathbb{C}[W]$ *and* c *is a Coxeter element.*

Proof. Notice that if $[w](\alpha)$ denotes the coefficient of w in an element α of the group algebra $\mathbb{C}[W]$, we can express the weighted enumeration of (2.2) as

$$\sum_{\substack{(\tau_1,\tau_2,\dots,\tau_\ell,c)\in\mathcal{R}^\ell\times\mathcal{C}\\ \tau_1\tau_2\dots\tau_\ell=c}} \mathbf{w}(\tau_1)\mathbf{w}(\tau_2)\dots\mathbf{w}(\tau_\ell) = \sum_{c\in\mathcal{C}}[\mathbf{1}]\big(\mathbf{A}(\boldsymbol{\omega})^\ell\cdot c^{-1}\big) = [\mathbf{1}]\big(\mathbf{A}(\boldsymbol{\omega})^\ell\cdot\mathbf{C}^{-1}\big),$$

where $\mathbf{1}$ is the identity in W and $\mathbf{C}^{-1} := \sum_{c\in\mathcal{C}} c^{-1}$ is *central* in $\mathbb{C}[W]$ (recall that $\mathcal{C}$ is the Coxeter conjugacy class). One can detect the coefficient $[\mathbf{1}](\alpha)$ as the trace of α, normalized by $1/|W|$, under the regular representation $\mathbb{C}[W]$. This is because in $\mathbb{C}[W]$ all group elements apart from $\mathbf{1}$ act as fixed-point-free permutation matrices. We have

$$\begin{aligned}[\mathbf{1}]\big(\mathbf{A}(\boldsymbol{\omega})^\ell\cdot\mathbf{C}^{-1}\big) &= \frac{1}{|W|}\cdot\mathrm{Tr}_{\mathbb{C}[W]}\big(\mathbf{A}(\boldsymbol{\omega})^\ell\cdot\mathbf{C}^{-1}\big)\\ &= \frac{1}{|W|}\sum_{\chi\in\widehat{W}}\chi(1)\cdot\chi\big(\mathbf{A}(\boldsymbol{\omega})^\ell\cdot\mathbf{C}^{-1}\big) = \frac{|\mathcal{C}|}{|W|}\sum_{\chi\in\widehat{W}}\chi(c^{-1})\cdot\chi\big(\mathbf{A}(\boldsymbol{\omega})^\ell\big),\end{aligned}$$

where for the second line we decompose the group algebra as $\mathbb{C}[W]\cong\bigoplus_{\chi\in\widehat{G}}\chi(1)\cdot U_\chi$ with U_χ the representation afforded by χ, and use that the central element $\mathbf{C}^{-1}$ acts on U_χ as multiplication by the scalar $|\mathcal{C}|\cdot\chi(c^{-1})/\chi(1)$ (for any $c\in\mathcal{C}$). Applying this to all terms in the series (2.2) and noticing that $\chi\big(e^{t\mathbf{A}(\boldsymbol{\omega})}\big) = \sum_{\ell\geq 0}\chi\big(\mathbf{A}(\boldsymbol{\omega})^l\big)\cdot\dfrac{t^l}{l!}$ and that $h = |W|/|\mathcal{C}|$ for the Coxeter class $\mathcal{C}$, the statement is proven. □

Jucys-Murphy elements, Gelfand-Tsetlin decomposition

The proof of Lemma 1 does not use the tower structure of the weight system $\mathbf{w}_T$ and actually works for arbitrary weight assignments. However it is often impractical as evaluating the traces $\chi(e^{t\mathbf{A}(\boldsymbol{\omega})})$ amounts to calculating the multiset $\mathrm{Spec}_\chi(\mathbf{A}(\boldsymbol{\omega}))$ of eigenvalues of $\mathbf{A}(\boldsymbol{\omega})$ over the representation U_χ of W. Indeed, we will have

$$\chi(e^{t\mathbf{A}(\boldsymbol{\omega})}) = \sum_{\ell\geq 0}\frac{t^\ell}{\ell!}\cdot\sum_{\lambda_j(\boldsymbol{\omega})\in\mathrm{Spec}_\chi(\mathbf{A}(\boldsymbol{\omega}))}\lambda_j(\boldsymbol{\omega})^\ell = \sum_{\lambda_j(\boldsymbol{\omega})\in\mathrm{Spec}_\chi(\mathbf{A}(\boldsymbol{\omega}))} e^{t\lambda_j(\boldsymbol{\omega})}, \tag{3.1}$$

where the $\lambda_j(\boldsymbol{\omega})$ are a priori algebraic functions on the parameters ω_i. We will show in this section (Lem. 2), that for systems $\mathbf{w}_T$ they are in fact *linear* in the ω_i.

The element $\mathbf{A}(\boldsymbol{\omega})$ belongs to the algebra $\mathbb{C}[\boldsymbol{J}] := \mathbb{C}[J_1,\ldots,J_n]$ generated by the generalized Jucys Murphy elements (2.4); it is equal to $\sum_{i=1}^n \omega_i J_i$. It is easy to see that $\mathbb{C}[\boldsymbol{J}]$ is commutative and in fact the J_i are simultaneously diagonalizable and hence their eigenvalues determine those of $\mathbf{A}(\boldsymbol{\omega})$. In what follows we study the spectra of the J_i with a technique inspired from the Okounkov-Vershik approach [16].

The notation $\psi \nearrow \chi$ indicates that ψ and χ are characters of *consecutive* groups W_{i-1} and W_i in the tower T, and that ψ appears with positive multiplicity in the restriction of χ on W_{i-1}. If $\underline{\chi}$ denotes a tuple of characters $(\chi_0,\chi_1,\cdots,\chi_n)$ with $\chi_i\in\widehat{W_i}$, we write

$$\mathrm{Res}_T(\chi) := \{\underline{\chi}:\ \chi_i\nearrow\chi_{i+1} \text{ and } \chi_n=\chi\} \tag{3.2}$$

for the set of possible chains $\underline{\chi}$ that may appear as we restrict χ down the tower T.

We can use the canonical projection into isotypic components to construct a decomposition of V_χ (for any $\chi\in\widehat{W}$) into spaces $V_{\underline{\chi}}$ indexed by the chains $\underline{\chi}\in\mathrm{Res}_T(\chi)$ as defined above. Start by writing $V_{\underline{\chi}}(n) := V_\chi$ and inductively define $V_{\underline{\chi}}(i)$ as the χ_i-isotypic component of $V_{\underline{\chi}}(i+1)$. At the last level, the spaces $V_{\underline{\chi}} := V_{\underline{\chi}}(0)$ satisfy

$$V_\chi = \bigoplus_{\underline{\chi}\in\mathrm{Res}_T(\chi)} V_{\underline{\chi}}. \tag{3.3}$$

We call the expression above the *Gelfand-Tsetlin decomposition* of V_χ with respect to the tower T. At each step i of the inductive construction, the character χ_{i-1} appears with multiplicity m_{χ_{i-1},χ_i} in the restriction of χ_i on W_{i-1}. Then, if we define

$$\mathrm{mult}(\underline{\chi}) := \prod_{i=1}^n m_{\chi_{i-1},\chi_i} = \prod_{i=1}^n \left(\chi_{i-1},\downarrow^{W_i}_{W_{i-1}}\chi_i\right)_{W_{i-1}}, \tag{3.4}$$

and since $\dim(V_{\chi_0})=1$ ($W_0=\{\mathbf{1}\}$), we get that $\dim(V_{\underline{\chi}})=\mathrm{mult}(\underline{\chi})$. In combinatorial terms, (3.4) counts the number of chains of type $\underline{\chi}$ in the branching graph for T.

The significance of the Gelfand-Tsetlin decomposition (3.3) in our setting is that it provides simultaneous eigenspaces for the elements J_i. Indeed, consider in the group algebra $\mathbb{C}[W]$ the partial sums $R_i := J_1 + \cdots + J_i$ so that $J_i = R_i - R_{i-1}$. Each R_i is a central element of $\mathbb{C}[W_i]$ (it is the sum of all reflections of W_i) and therefore acts as scalar multiplication by $\widetilde{\chi}_i(R_i) := \chi_i(R_i)/\chi_i(1)$ on each space $V_{\underline{\chi}}(i)$.

Finally, we have by construction $V_{\underline{\chi}} \subset V_{\underline{\chi}}(i)$ for all i, which means that (3.3) gives a decomposition of V_χ in simultaneous eigenspaces of the R_i and thus by definition, also of the J_i. Recalling that $\mathbf{A}(\boldsymbol{\omega}) = \sum_{i=1}^n \omega_i J_i$ and using (3.1), Lemma 1 becomes:

Lemma 2. *For a tower T in W and with the R_i as above, the enumeration* (2.2) *is given by:*

$$FAC_W^T(t,\boldsymbol{\omega}) = \frac{1}{h} \cdot \sum_{\chi \in \widehat{W}} \chi(c^{-1}) \sum_{\underline{\chi} \in \mathrm{Res}_T(\chi)} \mathrm{mult}(\underline{\chi}) \cdot \exp\left(t \cdot \sum_{i=1}^n \left(\widetilde{\chi}_i(R_i) - \widetilde{\chi}_{i-1}(R_{i-1})\right) \cdot \omega_i\right)$$

Remark 1. All the arguments in this section, and in particular Lemma 2 above, work for an arbitrary group G and tower of subgroups T. We hope that this method we introduced here may be used successfully in other instances where similar weighted enumeration questions are natural. An obvious first candidate would be the groups $\mathrm{GL}_n(F_q)$ [10].

4 Lie-like elements and proof of Theorem 1

Theorem 2 relates the virtual characters appearing in the Frobenius formula to the exterior powers $\Lambda^k(V_{ref})$ of the reflection representation V_{ref}. The following lemma asserts that reflections act as "Lie-like elements" on those exterior powers. It was first stated by Burman and Zvonkine [4] in the context of type A_n (their proof trivially extends from reflections to pseudo-reflections).

Lemma 3 ([4, Proposition 2.2])**.** *Let $W \leq \mathrm{GL}(V)$ be a complex reflection group and $\tau \in \mathbb{C}[W]$ be a pseudo-reflection. Then the action of $1-\tau$ on $\Lambda^k V = V \wedge V \wedge \cdots \wedge V$ is given by*

$$\sum_{i=1}^k id \wedge \cdots \wedge id \wedge (1-\tau) \wedge id \wedge \cdots \wedge id, \tag{4.1}$$

where in the sum, $(1-\tau)$ appears at position i (and only the identity appear at other positions).

Corollary 2 ([4, Proposition 2.4])**.** *Consider the "group algebra version" of the Laplacian, namely the element $\mathcal{L} := \sum_{\tau \in \mathcal{T}} x_\tau \cdot (1-\tau)$ in $\mathbb{C}[x_\tau][W]$. Then the $\binom{n}{k}$ eigenvalues of $\mathcal{L}$ on the representation $\Lambda^k V$ are given by the sums*

$$\sigma_{i_1} + \sigma_{i_2} + \cdots + \sigma_{i_k}$$

for all $1 \leq i_1 < i_2 < \cdots < i_k \leq N$, where $\sigma_1, \sigma_2, \ldots, \sigma_N$ are the eigenvalues of $\mathcal{L}$ on V.

We are now ready to show how Theorem 2 implies Theorem 1. From Lemma 1, and from the fact that $\mathcal{L} = \mathbf{w}_T(\mathcal{R}) - \mathbf{A}(\boldsymbol{\omega})$ in previous notation, we have

$$FAC_W^T(t, \boldsymbol{\omega}) = \frac{1}{h} e^{t\mathbf{w}_T(\mathcal{R})} \sum_{\chi \in \widehat{W}} \chi(c^{-1}) \cdot \chi(e^{-t\mathcal{L}}) \tag{4.2}$$

$$= \frac{1}{h} e^{t\mathbf{w}_T(\mathcal{R})} \sum_{m \geq 0} P_m(e^{-t\mathcal{L}}), \tag{4.3}$$

where we have grouped the characters $\chi \in \widehat{W}$ according to their Coxeter number c_χ using the notation P_m of Theorem 2. Since all the coefficients in the t-expansion of $e^{t\mathcal{L}}$ are elements of $\mathbb{C}[\mathbf{J}]$, we can use the tower equivalence in Theorem 2 and we get

$$FAC_W^T(t, \mathbf{w}) = \frac{1}{h} e^{t\mathbf{w}_T(\mathcal{R})} \sum_{k=0}^{n} (-1)^k \chi_{\Lambda^k V_{ref}}(e^{-t\mathcal{L}}) = \frac{1}{h} e^{t\mathbf{w}_T(\mathcal{R})} \sum_{k=0}^{n} (-1)^k \sum_{\lambda} e^{-t\lambda},$$

where the last sum is taken over the eigenvalues λ of $\mathcal{L}$ on the representation $\Lambda^k V_{ref}$. But Corollary 2 gives us these eigenvalues explicitly! We get

$$FAC_W^T(t, \mathbf{w}) = \frac{1}{h} e^{t\mathbf{w}_T(\mathcal{R})} \sum_{k=0}^{n} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} e^{-t\sigma_{i_1} - \cdots - t\sigma_{i_k}} = \frac{1}{h} e^{t\mathbf{w}_T(\mathcal{R})} \prod_{i=1}^{n} (1 - e^{-t\sigma_i}),$$

which is precisely Theorem 1.

Remark 2. A bit more work shows that for a given W, Theorem 1 is in fact *equivalent* to Theorem 2. This relies on showing that the powers $A(\boldsymbol{\omega})^\ell$ linearly generate $\mathbb{C}[\boldsymbol{J}]$.

5 A Matrix-forest theorem for reflection groups

After Cor. 1, it is natural to ask for an analog of the whole Matrix-Forest theorem for reflection groups; namely a combinatorial description of all the coefficients of the characteristic polynomial of the W-Laplacian $L_W^T(\boldsymbol{\omega})$. The answer, Corollary 3 below, was initially guessed via computer calculations and suggested a relation with determinants of smaller Laplacians. This led to the following much broader theorem.

For an arbitrary hyperplane arrangement $\mathcal{A}$ in a space V, we define the $\mathcal{A}$-Laplacian as a sum of rank 1 operators $L_{\mathcal{A}}(\boldsymbol{\omega}) := \sum_{H \in \mathcal{A}} \omega_H(\mathrm{Id} - s_H) \in \mathrm{GL}(V)$ with a family of weights $\boldsymbol{\omega} = (\omega_H)_{H \in \mathcal{A}}$ and where s_H denotes the reflection across H. We write qdet for the *quasi-determinant* of an operator, i.e. the product of its nonzero eigenvalues.

Theorem 3. *For any arrangement $\mathcal{A}$, the characteristic polynomial of $L_{\mathcal{A}}(\boldsymbol{\omega})$ is given by*

$$\det\left(t \cdot \mathrm{Id} + L_{\mathcal{A}}(\boldsymbol{\omega})\right) = \sum_{X \in \mathcal{L}_{\mathcal{A}}} \mathrm{qdet}\left(L_{\mathcal{A}_X}(\boldsymbol{\omega}_X)\right) \cdot t^{\dim(X)},$$

where $\mathcal{A}_X := \{H \in \mathcal{A} : \ H \supset X\}$ denotes the localization on a flat X.

Sketch. The proof is an almost immediate application of [3, Cor. 2.4] where a formula is given for the characteristic polynomial of *any* sum of rank 1 operators. Instead of summing over all linearly independent subcollections of hyperplanes we group them together with respect to their intersection. □

For reflection arrangements $\mathcal{A}_W$, any localization $(\mathcal{A}_W)_X$ is also a reflection arrangement (for the group W_X). Then combining Cor. 1 with Thm. 3 immediately gives the following Corollary. One thing to notice is that any parabolic tower T in W defines a parabolic tower $W_i' := W_i \bigcap W_X$ for any flat X.

Corollary 3. *The characteristic polynomial of the W-Laplacian $L_T^W(\omega)$ is given by*

$$\det(t\cdot\mathbf{1}+L_T^W(\boldsymbol{\omega})) = \sum_{\substack{\tau_1\cdots\tau_{n-k}=c_X\\ \dim(X)=k}} [W_X : C_{W_X}(c_X)]\cdot\mathbf{w}_T(\tau_1)\cdots\mathbf{w}_T(\tau_{n-k})\cdot\frac{t^k}{(n-k)!},$$

where the sum is over all Coxeter elements c_X, over any parabolic subgroup W_X.

Notice how Cor. 3 and Thm. 1 together imply a remarkable connection between arbitrary length and reduced reflection factorizations for the parabolic and Coxeter classes.

Even more than this, Thm. 3 produces some very interesting numerology for reflection groups W. For instance, identifying all weights to 1, the eigenvalues of the W-Laplacian equal, by definition, the Coxeter number h, so that Thm. 3 implies:

Corollary 4. *If $\{h_i(W_X)\}$ denotes the multiset of Coxeter numbers of the parabolic W_X, then*

$$(t+h)^n = \sum_{X\in\mathcal{L}_W} t^{\dim(X)}\cdot\prod_{i=1}^{\mathrm{rk}(W_X)} h_i(W_X).$$

6 A glimpse at the proof of Theorem 2

In this section we only give an idea of the structure of our proofs (not even a sketch), refering to the long version for complete proofs. Our proof uses the Sheppard-Todd classification of well generated complex reflection groups (see [15]), into a finite set of "exceptional" groups, and the three infinite families $\mathfrak{S}_n$, $G(r,1,n)$, $G(r,r,n)$.

1- Exceptional groups. For any fixed group and a given tower, Theorem 1 (and thus also Theorem 2, see Remark 2) can be proved from Lemma 2 provided one has access to the irreducible characters. We ran this check for all exceptional groups and all conjugacy classes of towers, using GAP3 and Sage.

2- Symmetric groups. For $\mathfrak{S}_n$, everything is already known, see [4]. The only characters not vanishing the Coxeter element (the long cycle) are hooks, which correspond to exterior powers of V_{ref}. Therefore the equivalence $\equiv$ in Theorem 2 is an equality.

3- Other infinite types. The proof for infinite types $G(r,1,n)$ and $G(r,r,n)$ is not as simple and relies on three ingredients, all of which require proof:

(i) The inductive nature of tower equivalence: In order to check that two virtual characters χ_1, χ_2 of W (that take the same value on the sum of reflections) are tower equivalent, it suffices to show that their restriction to any maximal parabolic subgroup $W_{sub} \subset W$ are tower equivalent (for the smaller group W_{sub}). Now, the maximal parabolics are well understood, and up to conjugation they are either symmetric groups, or of the form:

$$G(r,1,a) \times S_{n-a} \hookrightarrow G(r,1,n) \text{ and } G(r,r,a) \times S_{n-a} \hookrightarrow G(r,r,n).$$

(ii) The irreducible characters of groups $G(r,d,n)$ are understood. They are generically indexed by r-tuples of partitions of total size n, considered modulo the action of the cyclic group $\mathbb{Z}_d$, see [6]. For example, the values of the (virtual) character P_{hk} of Theorem 2 for the group $G(r,1,n)$ is given (with $h = rn$) by

$$P_{(rn)k} = {}_0\mathfrak{h}_k^n - \sum_{0<q<r} \xi^{-q} \cdot {}_q\mathfrak{h}_{k-1}^n,$$

where ξ is a primitive r-th root of unity, and ${}_q\mathfrak{h}_k^n := (0, ..., 0, \mathfrak{h}_k^n, 0, ...0)$ where the hook partition $\mathfrak{h}_k^n := (n-k, 1^k)$ appears at the q-th position. Using either the Littlewood-Richardson rule and its generalisation to $G(r,1,n)$ (e.g. [12]), or direct exterior product calculations, the restriction of this character to a subgroup $G(r,1,a) \times S_{n-a}$ can explicitely be written.

(iii) It turns out, and this is a remarkable fact, that after cancellation of many terms, this restriction of the character P_{hk} to a parabolic subgroup can be put in an "recursive" form. In the previous example of $G(r,1,n)$, we get after calculation

$$P_{(rn)k}\Big\downarrow_{G(r,1,a)\times S_{n-a}} = \sum_{\substack{i,j\geq 0, \epsilon\in\{0,1\} \\ i+j+\epsilon=k}} P_{(ra)i} \otimes \mathfrak{h}_j^{n-a}, \tag{6.1}$$

where we see the quantity $P_{(ra)i}$ corresponding to the smaller group $G(r,1,a)$ appear. From this formula, and from the easy fact that exterior products obey a similar recursion, one can show by induction[1] that P_{nk} is ideed equal to $\Lambda^k \mathbb{C}^n$ for all n and k.

The case of $G(r,r,n)$ is similar in structure but much more complicated, as the characters involved are more numerous and do not involve only hooks but also quasi-hooks, as already noticed in [6]. The computation of the restriction to a parabolic subgroup is more involved and is only possible via a cautious use of the Littlewood-Richardson rule, see the appendix in the long version. The same phenomenon as for $G(r,1,n)$ arises, namely that after tedious calculations and simplifications, the restriction of the virtual characters appearing in Theorem 2 have a "recursive" form w.r.t. to parabolic subgroups, from which we can use induction. We refer again to the long version for these calculations, that form a large part of the full version of this paper.

[1] The present sketch omits many details including initial conditions, and parabolic subgroups that are single symmetric groups.

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