Cyclic Sieving for reduced reflection factorizations of the Coxeter element

Theo Douvropoulos

Paris VII, IRIF (ERC CombiTop)

July 18, 2018
The number of reduced reflection factorizations of $c$

**Theorem (Hurwitz, 1892)**

There are $n^{n-2}$ (minimal length) factorizations $t_1 \cdots t_{n-1} = (12 \cdots n) \in S_n$ where the $t_i$'s are transpositions.

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(12)(23) = (123) \quad (13)(12) = (123) \quad (23)(13) = (123).
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$$(12)(23) = (123) \quad (13)(12) = (123) \quad (23)(13) = (123).$$

**Theorem (Deligne-Arnol’d-Bessis)**

For a well-generated, complex reflection group $W$, with Coxeter number $h$, there are $\frac{h^n n!}{|W|}$ (minimal length) reflection factorizations $t_1 \cdots t_n = c$ of the Coxeter element $c$. 

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Ingredients:

1. A set $X$.
2. A polynomial $X(q)$.
3. A cyclic group $C = \langle c \rangle$ of some order $n$, acting on $X$.

Definition (Reiner-Stanton-White)

We say that the triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon if for all $d$, 

$$\{x \in X : c^d \cdot x = x\} = X(\zeta_d),$$

where $\zeta$ is a(n) primitive $n$th root of unity.

The polynomial $X(q)$ is sometimes a statistic on a combinatorial object, a Poincare polynomial, a Hilbert Series, a formal character. (for more, have a look at What is... Cyclic Sieving?)
What is a Cyclic Sieving Phenomenon (CSP)?

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The polynomial $X(q)$ is sometimes a statistic on a combinatorial object, a Poincare polynomial, a Hilbert Series, a formal character.. (for more, have a look at What is... Cyclic Sieving?).
$W$ is a well-generated complex reflection group of rank $n$, $c$ is a Coxeter element of $W$ and the $t_i$’s are reflections.

1. $X = \text{Red}_W(c)$ ( := \{factorizations $t_1 \cdots t_n = c$ \}) enumerated by $\frac{h^n n!}{|W|}$.
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   1. \(\Psi_{\text{cyc}} : (t_1, \cdots, t_n) \rightarrow (c t_n, t_1, \cdots, t_{n-1}),\) of order \(nh\).
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   2. \( \Phi_{\text{cyc}}: (t_1, \cdots t_n) \rightarrow (c^{t_n}) t_1, c t_n, t_2, \cdots, t_{n-1} \), of order \((n-1)h\),
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   - $\Phi_{\text{cyc}} : (t_1, \cdots, t_n) \rightarrow (c t_n, c t_n, t_2, \cdots, t_{n-1})$, of order $(n-1)h$,
   - $\text{Twist} : (t_1, \cdots, t_n) \rightarrow \left( (t_1 \cdots t_{n-1}) t_n, (t_1 \cdots t_{n-2}) t_{n-1}, \cdots, t_1, t_2, t_1 \right)$, $2h$,
   - ...

where $w t := wt w^{-1}$ stands for conjugation.
A CSP for reduced reflection factorizations of $c$

Theorem (D. ’17, Conjectured in Williams’ Cataland)

For an irreducible, well-generated complex reflection group $W$, with degrees $d_1, \cdots, d_n$, Coxeter element $c$ and Coxeter number $h = d_n$, the triple

$$\left( \text{Red}_W(c), \prod_{i=1}^n \frac{[hi]_q}{[di]_q}, C \right),$$

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For $W = S_4$, we have:

$$X(q) = [4]_q^2 \cdot [4]_q^3 = \frac{1 - q^8}{1 - q^2} \cdot \frac{1 - q^{12}}{1 - q^4}.$$

$$\zeta = e^{2\pi i/12}, \quad \zeta^4 = e^{2\pi i/3}$$

$$X(\zeta^4) = 1 \cdot (1 + \zeta^{12} + \zeta^{24} + \zeta^{32}) = 4.$$

$$X(\zeta) = X(\zeta^2) = X(\zeta^3) = X(\zeta^6) = 0.$$

The other orbit is free and contains $12 \cdot 23 \cdot 34$. The other orbit is free and contains $12 \cdot 34 \cdot 24$. 
Coxeter elements via Springer

Theorem (For us definition)

Coxeter elements are characterized by having an eigenvector $\vec{v}$, which lies on no reflection hyperplane, with eigenvalue $\zeta = e^{2\pi i/h}$, where $h = \frac{|R| + |R^*|}{n}$. 
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For the symmetric group \( S_n \), the Coxeter element is the (any) long cycle \((12 \cdots n)\); its eigenvectors are of the form \((\zeta^{n-1}, \zeta^{n-2}, \cdots, \zeta, 1)\).
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For real reflection groups:
Towards a geometric construction of the Coxeter element

Theorem (Steinberg)

$W$ acts freely on the complement of the hyperplane arrangement $V^{\text{reg}} := V \setminus \bigcup H$. That is, $\rho : V^{\text{reg}} \to W \setminus V^{\text{reg}}$ is a covering map.
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1 \hookrightarrow \pi_1(V^\text{reg}) \xrightarrow{\rho^*} \pi_1(W \setminus V^\text{reg}) \xrightarrow{\pi} W \to 1
\]

\[
\begin{array}{c}
\| & \| \\
\pi_1(V^\text{reg}) & \pi_1(W \setminus V^\text{reg}) \\
P(W) & B(W)
\end{array}
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Towards a geometric construction of the Coxeter element

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\[ P(W) \hspace{1cm} B(W) \]

Theorem (Shephard-Todd-Chevalley, GIT)

\[ W \text{ is realized as the group of deck transformations of a covering map } \rho \text{ which is explicitly given via the fundamental invariants } f_i. \]
The loop $\delta$ maps to the Coxeter element $c$. 

As we vary $y$, the slice $L_y \sim = C$ intersects the discriminant hypersurface $H := W \setminus \bigcup H$ in $n$-many points (with multiplicity). Loops around these points (that are prescribed by a choice of a base star) map to factorizations of $c$. We call this construction a labeling map and we write $\text{rlbl}(y, c_1, \ldots, c_k)$, to indicate the dependence on the base star.
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As we vary $y$, the slice $L_y \cong \mathbb{C}$ intersects the discriminant hypersurface $\mathcal{H} := W \setminus \bigcup H$ in $n$-many points (with multiplicity).
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We call this construction a *labeling map* and we write

$$\text{rlbl} \left( y, \begin{array}{c} \delta \\ \end{array} \right) = (c_1, \cdots, c_k),$$

to indicate the dependence on the base star.
We define the LL map:

$$LL : Y \rightarrow \{ \text{centered configurations} \}$$

$$y \rightarrow \text{multiset } L_y \cap \mathcal{H}.$$
The Lyashko-Looijenga (LL) morphism

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$$LL : Y \rightarrow \{ \text{centered configurations} \} \quad \{ \text{of } n \text{ points in } \mathbb{C} \}$$

$$y \rightarrow \text{multiset } L_y \cap H.$$  

Algebraically, it is given as:

$$(f_1, \cdots, f_{n-1}) \xrightarrow{LL} (\alpha_2(f), \cdots, \alpha_n(f)),$$

where $\alpha_i$ is weighted homogeneous of degree $h_i$. 

Our polynomial $X(q)$ is precisely the Hilbert series:

$$Hilb(LL^{-1}(0), q) = \prod_{i=1}^{n} [h_i] q^{d_i}.$$
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It is a finite morphism whose degree is:

\[ \deg(LL) = \frac{h^n n!}{|\mathcal{W}|} \quad (= |\text{Red}_W(c)|). \]
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Our polynomial \( X(q) \) is precisely the Hilbert series:

\[ \text{Hilb} \left( LL^{-1}(0), q \right) = \prod_{i=1}^{n} \frac{[h_i]_q}{[d_i]_q}. \]
Bessis’ trivialization theorem

Theorem (Bessis)

The points in a generic fiber $LL^{-1}(e)$ of the LL map are in bijection with reduced reflection factorizations of the Coxeter element $c$. The bijection is given by the labeling map and depends non-trivially on a choice of base-star for the configuration $e$.

\[ LL^{-1}(e) \ni y \rightarrow \text{rlbl} \left( y, \begin{array}{c} \vdots \end{array} \right) \in \text{Red}_W(c). \]
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The points in a generic fiber $LL^{-1}(e)$ of the $LL$ map are in bijection with reduced reflection factorizations of the Coxeter element $c$. The bijection is given by the labeling map and depends non-trivially on a choice of base-star for the configuration $e$.

$$LL^{-1}(e) \ni y \rightarrow \text{rlbl} \left( y, \begin{array}{c} \delta \end{array} \right) \in \text{Red}_W(c).$$

This remarkably relies on the numerological coincidence $\deg(LL) = |\text{Red}_W(c)|$. 
The cyclic actions $\Psi, \Phi, Twist, \cdots$ via the rlbl map

$$\text{rlbl} \left( y'', \bigcirc \right) = \text{rlbl} \left( y, \bigcirc \right) = (b_1, b_2, b_3) \quad \text{rlbl} \left( y'', \bigcirc \right) = (b_1, b_2, b_3).$$

$$(b_1, b_2, b_3) = (b_2, b_3, b_1 b_2 b_3) \quad \text{i.e. } \text{rlbl}(y) = \Psi^{-1} \cdot \text{rlbl}(y'')$$
Proof of the CSP (sketch)

\( X(q) = \text{Hilb} (LL^{-1}(0), q) = \prod_{i=1}^{n} \frac{[h_i]_q}{[d_i]_q} \)
Proof of the CSP (sketch)

1. \( X(q) = \text{Hilb} \left( LL^{-1}(0), q \right) = \prod_{i=1}^{n} \frac{[hi]_q}{[d_i]_q} \)

2. For \( k \)-symmetric point configurations \( e \), the fiber \( LL^{-1}(e) \) carries a natural (scalar) action of a cyclic group \( C_{kh} = \langle \xi \rangle \leq \mathbb{C}^* \), \( \xi = e^{2\pi i / kh} \), given by:

\[
    y \in LL^{-1}(e), \quad \xi \star y = \xi \star (f_1, \cdots, f_{n-1}) := (\xi^{d_1} f_1, \cdots, \xi^{d_{n-1}} f_{n-1})
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3. This scalar action on the fiber \( LL^{-1}(e) \) is equivalent to some combinatorial action \( \Phi, \Psi, \text{Twist}, \cdots \):

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\text{rlbl}(\xi \star y) = \Psi \cdot \text{rlbl}(y).
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4. On the other hand, any fiber \( LL^{-1}(e) \) is a flat deformation of the special fiber \( LL^{-1}(0) \) and retains part of its \( \mathbb{C}^* \)-structure:

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LL^{-1}(e) \cong_{C_{kh}} LL^{-1}(0).
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proof of the CSP (sketch)

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5. This allows Hilbert series \( \text{Hilb}(LL^{-1}(0), q) \) to give a CSP for the \( C_{kh} \) (scalar) action on \( LL^{-1}(e) \), and hence the combinatorial actions \( \Psi, \cdots \) as well.
Lemma (See much more generally in Broer-Reiner-Smith-Webb)

Let \( f : \mathbb{C}^n \to \mathbb{C}^n \) be a polynomial morphism, finite and quasi-homogeneous. Consider further a point \( \epsilon \), such that the fiber \( f^{-1}(\epsilon) \) is stable under weighted multiplication by \( c := e^{-2\pi i/N} \) for some number \( N \). Then, if \( C_N = \langle c \rangle \), we have the isomorphism of \( C_N \)-modules:

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1. Springer’s theorem \( \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \cong_{W \times C} \mathbb{C}[W] \), and in particular the CSP for the \( q \)-binomial \( \binom{n}{k}_q \).
CSP’s through finite quasi-homogeneous morphisms

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A C.S.P. for reflection factorizations of c
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2. Primitive factorizations \( w \cdot t_1 \cdots t_k = c \), counted by \( \frac{h^{\dim(X)}(\dim(X))!}{[N(X) : W_X]} \).
3. (Possibly..) many of the factorization enumeration formulas in \( S_n \) that have geometric interpretation. In particular, the Goulden-Jackson formula:

\[
\text{Fact}_{[\lambda_1, \ldots, \lambda_m]}((n)) = n^{m-1} \prod_{i=1}^{m} \frac{(l(\lambda_i) - 1)!}{\text{Aut}(\lambda_i)}.
\]
Thank you!