Coxeter factorizations and the Matrix Tree theorem with generalized Jucys-Murphy weights

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A famous theorem of Cayley after Hurwitz

**Theorem (Hurwitz, 1892)**

*There are \( n^{n-2} \) (minimal length) factorizations \( t_1 \cdots t_{n-1} = (12 \cdots n) \in S_n \) where the \( t_i \)'s are transpositions.***

For example, the \( 3^1 \) factorizations

- \((12)(23) = (123)\)
- \((13)(12) = (123)\)
- \((23)(13) = (123)\)

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**Theorem (Matrix-Tree theorem)**

*The (weighted) enumeration \( t(G) \) of spanning trees of a graph \( G \) is given as

\[
t(G) = \frac{1}{n} \cdot \lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1} = q\text{-det} \left( L(G) \right),
\]

where \( L(G) := \sum_{(ij) \in G} \omega_{ij} (1 - (ij)) \) is the weighted Laplacian of \( G \).*
Arbitrary length factorizations of the long cycle

If $\mathcal{R}$ denotes the set of transpositions of $S_n$, we write

$$\text{Fact}_{S_n}(N) := \#\{(t_1, \cdots, t_N) \in \mathcal{R}^n \mid t_1 \cdots t_N = (12 \cdots n)\}.$$  

Now, consider the exponential generating function:

$$\text{FAC}_{S_n}(t) = \sum_{N \geq 0} \text{Fact}_{S_n,c}(N) \frac{t^N}{N!}.$$  

**Theorem (Jackson, '88)**

*For the symmetric group $S_n$, we have*

$$\text{FAC}_{S_n,c}(t) = \frac{e^{t\binom{n}{2}}}{n!} \left(1 - e^{-tn}\right)^{n-1}.$$  

Notice that

$$\left[\frac{t^{n-1}}{(n-1)!}\right] \text{FAC}_{S_n,c}(t) = \frac{1}{n!} \cdot (n)^{n-1} \cdot (n-1)! = n^{n-2}.$$
We consider \( (\binom{n}{2}) \) parameters \( \omega := (\omega_{ij})_{i<j} \) that form a weight system \( w((ij)) = \omega_{ij} \) for the transpositions \( (ij) \in S_n \). If \( C \) denotes the class of the long cycles, we define:

\[
FAC_{S_n}(t, \omega) := \sum_{\ell \geq 1} \frac{t^\ell}{\ell!} \sum_{(\tau_1, \tau_2, \cdots, \tau_\ell, c) \in R^\ell \times C \atop \tau_1 \cdots \tau_\ell = c} w(\tau_1) \cdot w(\tau_2) \cdots w(\tau_\ell).
\]

**Theorem (Burman-Zvonkine ’08, Alon-Kozma ’10)**

The exponential generating function above factors as

\[
FAC_{S_n}(t, \omega) = e^{tw(R)} \cdot \frac{n^{-1}}{n} \prod_{i=1}^{n-1} \left(1 - e^{-t \lambda_i(\omega)}\right),
\]

where \( w(R) = \sum_{i<j} (\omega_{ij}) \) and \( \{\lambda_i(\omega)\} \) are the non-zero eigenvalues of the (weighted) Laplacian \( L(K_n) \).
A finite subgroup $G \leq GL_n(V)$ is called a complex reflection group if it is generated by pseudo-reflections. There are $\mathbb{C}$-linear maps $t$ that fix a hyperplane (i.e. $\text{codim}(V^t) = 1$). Shephard and Todd have classified (irreducible) complex reflection groups into:

1. an infinite 3-parameter family $G(r, p, n)$ of monomial groups
2. 34 exceptional cases indexed $G_4$ to $G_{37}$.

**Definition**

An element $g \in W$ is called $\zeta$-regular if it has a $\zeta$-eigenvector $\vec{v}$ that lies in no reflection hyperplane.

In particular, a **Coxeter element** is defined as a $e^{2\pi i / h}$-regular element for $h = (|\mathcal{R}| + |\mathcal{A}|)/n$. 
You already know this definition of Coxeter elements

Example

1. In $S_n$, the regular elements are $(12 \cdots n)$, $(12 \cdots n - 1)(n)$, and their powers. Indeed, $(\zeta^{n-1}, \zeta^{n-2}, \cdots, 1)$ with $\zeta = e^{2\pi i / n}$ is an eigenvector for $(12 \cdots n)$.

2. For real reflection groups:
Arbitrary length reflection factorizations of $c$

If $\mathcal{R}$ denotes the set of reflections of $W$ and $c$ its Coxeter element, we write

$$\text{Fact}_{W,c}(N) := \# \{(t_1, \cdots, t_N) \in \mathcal{R}^n \mid t_1 \cdots t_N = c\}.$$ 

Now, consider the exponential generating function:

$$\text{FAC}_{W,c}(t) = \sum_{N \geq 0} \text{Fact}_{W,c}(N) \frac{t^N}{N!}.$$ 

**Theorem (Chapuy-Stump, ’12)**

If $W$ is well-generated, of rank $n$, and $h$ is the order of the Coxeter element $c$, then

$$\text{FAC}_{W,c}(t) = \frac{e^{t|\mathcal{R}|}}{|W|} (1 - e^{-th})^n.$$ 

Notice that

$$\left[\frac{t^n}{n!}\right] \text{FAC}_{W,c}(t) = \frac{1}{|W|} \cdot h^n \cdot n! = \frac{h^n n!}{|W|}.$$
Enumeration for arbitrary $W$ with parabolic weights

Consider a (maximal) tower of parabolic subgroups

$$T := (\{1\} = W_0 \leq W_1 \leq W_2 \leq \cdots \leq W_n = W),$$

and a weight system $w_T$ on reflections $\tau \in R$ with parameters $\omega := (\omega_i)$ given by

$$w_T(\tau) = \omega_i \text{ if and only if } \tau \in W_i \setminus W_{i-1}.$$

If $C$ denotes the class of Coxeter elements, define the exp. gen. function

$$FAC_T^W(t, \omega) := \sum_{\ell \geq 1} \frac{t^\ell}{\ell!} \sum_{(\tau_1, \tau_2, \cdots, \tau_\ell, c) \in R^\ell \times C} \prod_{1 \leq i \leq \ell} w_T(\tau_i) \cdot \prod_{1 \leq i \leq \ell} w_T(\tau_i).$$

Theorem (Chapuy, D. 2019)

The exponential generating function above factors as

$$FAC_T^W(t, \omega) = \frac{e^{tw_T(R)}}{h} \cdot \prod_{i=1}^n (1 - e^{-t\lambda_i^T(\omega)}),$$

where $\{\lambda_i^T(\omega)\}$ are the eigenvalues on the reflection representation $V$ of the $W$-Laplacian $L_T(W) := \sum_{\tau \in R} w_T(\tau) \cdot (1 - \tau) \in \mathbb{C}[\omega][W]$. 

But, what do the eigenvalues look like?

$$A = \begin{bmatrix}
2 & 1 & 0 & 1 & 2 & 4 \\
0 & 3 & 0 & 1 & 2 & 4 \\
0 & 0 & 2 & 0 & 0 & 8 \\
0 & 0 & 0 & 4 & 2 & 4 \\
0 & 0 & 0 & 0 & 6 & 4 \\
0 & 0 & 0 & 0 & 0 & 10 \\
\end{bmatrix} \begin{array}{c}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5 \\
\lambda_6 \\
\end{array}$$

$$\{\lambda_i(\omega)\} = \begin{cases} 
2\omega_1 + \omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\
3\omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\
2\omega_3 + 8\omega_6, \\
4\omega_4 + 2\omega_5 + 4\omega_6, \\
6\omega_5 + 4\omega_6, \\
10\omega_6
\end{cases}$$
Representation theoretic interpretation

The filtration by \( T \) defines natural analogs of the Jucys-Murphy elements:

\[
\mathbb{C}[W] \ni J_i := \sum_{\tau \in R \text{ and } \tau \in W_i \setminus W_{i-1}} \tau,
\]

and we write \( \mathbb{C}[J] := \mathbb{C}[J_1, \cdots, J_n] \) for the (commutative) algebra they generate.

**Definition**

We say that two (virtual) characters \( \psi_1 \) and \( \psi_2 \) are *tower equivalent*, and write \( \psi_1 \equiv \psi_2 \), if they agree on the subalgebra \( \mathbb{C}[J] \) of \( \mathbb{C}[W] \) for *any* choice of \( T \).

For any irreducible character \( \chi \in \hat{W} \) Gordon-Griffeth define a *Coxeter number* \( c_\chi \)

\[
c_\chi := (1/\chi(1)) \cdot \chi\left(\sum_{\tau \in R} (1 - \tau)\right).
\]

**Theorem (Chapuy, D. -version appropriate for induction)**

*Our enumeration theorem can be re-phrased via the tower equivalence relation as:*

\[
\sum_{\chi \in \hat{W} \text{ s.th. } c_\chi = kh} \chi(c^{-1}) \cdot \chi \equiv (-1)^k \cdot \bigwedge(V_{ref}) \quad \text{and} \quad \sum_{c_\chi = m \text{ with } h \nmid m} \chi(c^{-1}) \cdot \chi \equiv 0.
\]
\[(12) + (13) + (23) \cdot [(12) + (13) + (23)] = 3 \cdot \text{Id} + 3 \cdot (123) + 3 \cdot (132)\]

Consider the element \( A(\omega) := \sum_{\tau \in R} w_T(\tau) \cdot \tau \) of the group algebra \( \mathbb{C}[\omega][W] \).

\[
\sum_{\ell \geq 1} \sum_{\substack{\tau_1 \cdots \tau_\ell = c \\ (\tau_1, \tau_2, \cdots, \tau_\ell, c) \in \mathcal{R}^\ell \times C}} w_T(\tau_1) \cdot w_T(\tau_2) \cdots w_T(\tau_\ell) \cdot \frac{t^\ell}{\ell!}
\]

\[
= \sum_{\ell \geq 0} [C] (A(\omega))^\ell \cdot \frac{t^\ell}{\ell!}
\]

\[
= \sum_{\ell \geq 0} [\text{id}] \left( (A(\omega))^\ell \cdot C^{-1} \right) \cdot \frac{t^\ell}{\ell!}
\]

\[
= \sum_{\ell \geq 0} \frac{1}{|W|} \text{Tr}_{\mathbb{C}[W]} \left( (A(\omega))^\ell \cdot C^{-1} \right) \cdot \frac{t^\ell}{\ell!}
\]

\[
= \sum_{N \geq 0} \frac{1}{|W|} \sum_{\chi \in \hat{W}} \dim(\chi) \cdot \chi \left( (A(\omega))^\ell \cdot C^{-1} \right) \cdot \frac{t^\ell}{\ell!}
\]
Consider the element $A(\omega) := \sum_{\tau \in R} w_T(\tau) \cdot \tau$ of the group algebra $\mathbb{C}[\omega][W]$. 

$$= \sum_{N \geq 0} \frac{1}{|W|} \cdot \sum_{\chi \in \widehat{W}} \dim(\chi) \cdot \chi\left(\left(A(\omega)\right)^\ell \cdot c^{-1}\right) \cdot \frac{t^\ell}{\ell!}$$

$$= \sum_{N \geq 0} \frac{1}{h} \cdot \sum_{\chi \in \widehat{W}} \dim(\chi) \cdot \left(\frac{\chi(c^{-1})}{\chi(1)}\right) \cdot \chi\left(\left(A(\omega)\right)^\ell\right) \cdot \frac{t^\ell}{\ell!}$$

$$= \frac{1}{h} \sum_{\chi \in \widehat{W}} \chi(c^{-1}) \cdot \sum_{\lambda(\omega) \in \text{Spec}_\chi(A(\omega))} \exp \left(t \cdot \lambda(\omega)\right)$$
Ingredients of our proof

1. A weighted version of the Frobenius Lemma

\[ \text{FAC}_W(t, \omega) = \frac{1}{h} \sum_{\chi \in \hat{W}} \chi(c^{-1}) \cdot \sum_{\lambda(\omega) \in \text{Spec}_\chi(A(\omega))} \exp \left( t \cdot \lambda(\omega) \right) \]

becomes

\[ \text{FAC}_W^T(t, \omega) = \frac{1}{h} \sum_{\chi \in \hat{W}} \chi(c^{-1}) \cdot \sum_{\chi \in \text{Res}_T(\chi)} \text{mult}(\chi) \cdot \exp \left( t \cdot \sum_{i=1}^{n} (\tilde{\chi}_i(R_i) - \tilde{\chi}_{i-1}(R_{i-1}) \cdot \omega_i) \right) \]

2. A recursion to maximal parabolics to prove tower equivalence in \( G(r, 1, n) \) and \( G(r, r, n) \).

3. The eigenvalues of the \( W \)-Laplacian \( L_T(W) \) in \( \bigwedge^k(V_{\text{ref}}) \) are the \( k \)-sums of its eigenvalues in \( V_{\text{ref}} \) (via Burman’s Lie-like elements).
A remarkable $W$-Matrix forest theorem

**Theorem (Matrix Forest theorem)**

If $L(G) := \sum_{(ij) \in G} \omega_{ij} (1 - (ij))$ is the weighted Laplacian of the graph $G$ and

$$
\det (x \cdot 1 + L(G)) = c_0 + \cdots + c_k \cdot x^k + \cdots c_n \cdot x^n,
$$

then the coefficients $c_k(\omega)$ count (weighted) forests in $G$ with $k$ rooted components.

**Theorem (Chapuy, D. 2019)**

If we write the characteristic polynomial for the $W$-Laplacian as

$$
\chi(L_T(W), x) := \det(x \cdot 1 + L_T(W)) = c_0^T + \cdots + c_k^T \cdot x^k + \cdots + c_n^T \cdot x^n,
$$

then the coefficients $c_k^T$ are given by

$$
c_k^T := \sum_{\substack{\tau_1 \cdots \tau_{n-k} = c_X \\ \dim(X) = k}} \frac{|W_X|}{|C_{W_X}(c_X)|} \cdot w_T(\tau_1) \cdots w_T(\tau_{n-k}) \cdot \frac{1}{(n-k)!}.
$$
A brief history of the $W$-Hurwitz number $\frac{h^n n!}{|W|}$

1. Singularity theory: There exist two subgroups $G_1 \leq G_2 \leq B_n$ of the braid group $B_n$ on $n$ strands, with finite indexes $\nu_1$ and $\nu_2$ such that:

   \[ \nu_1 = \frac{h^n n!}{|W|}, \quad \nu_2 = \#\{ \text{reduced reflection factorizations of } c \} \]

2. Deligne-Tits-Zagier, rediscovered by Reading.
   Enumerate factorizations $t_1 \cdots t_n = c$ with respect to the $c$-orbit of $t_n$:

   \[ \text{Hur}(W) = \frac{h}{2} \sum_{s \in S} \text{Hur}(W_{\langle s \rangle}) \]

3. We rewrite the recursion as

   \[ \text{Hur}(W) = \sum_{L \in NC^{n-1}(W)} \text{Hur}(W_L) \] which is \[ h^n \cdot n! = \sum_{L \in \mathcal{L}_W^1} h \cdot \prod_{i=1}^{n-1} h_i(W_L), \]

   and follows from \[ \chi(L_A(\omega), t) = \sum_{X \in \mathcal{L}_A} \text{q-det}(L_{A_X}(\omega)) \cdot (-1)^{\text{codim}(X)} \cdot t^{\text{dim}(X)} \]

Chapuy, D. (Paris VII, IRIF) slides available at www.irif.fr/~douvr001/
The algebra $\mathbb{C}[J]$ is not worthy of the name Gelfand-Tsetlin algebra; often:

$$\mathbb{C}[J] \nsubseteq \langle Z(\mathbb{C}[W_1]), \cdots, Z(\mathbb{C}[W_n]) \rangle \nsubseteq \bigoplus_{\chi \in \hat{W}} \text{Hom}(U_{\chi}, \mathbb{C}[W]).$$

However, if it were, our theorem could never be true!

In the infinite families $G(r, 1, n)$ and $G(r, r, n)$ we get factoring formulas even if we use finer weight systems (but the $W$-Laplacian does not appear). Is there a combinatorial or representation theoretic significance for such maximal weighting systems?

- Type $B_n$: All possible $n^2$ weights $\omega_{ij}$.
- Type $G(r, 1, n)$: The reflections $(ii^\xi)$ that fix the same hyperplane $x_i = 0$ get the same weight.

Our elaboration on the Frobenius lemma allows experimentation in any group that has natural towers of subgroups. What about $GL_n(F_q)$?

A uniform proof of the tower equivalence version? Relations with work of Griffeth, Gusenbauer, Juteau, Lanini? with work of Jean Michel?
Thank you!