Coxeter factorizations and the Matrix Tree theorem with
generalized Jucys-Murphy weights

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Slides available at my website linked in the abstract (www.irif.fr/~douvr001/)

Paris VII, IRIF and CNRS, ERC CombiTop
Counting trees and counting factorizations
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Example, $n = 3$:

$3^3 - 1 = 3$
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$3^{3-1} = 3$

**Corollary** (Denes, 1959).

Call a permutation $c \in S_n$ a long cycle if it is conjugate to $(12 \cdots n)$. There are $n^{n-2} \cdot (n-1)!$ minimal length factorizations $\tau_1 \cdots \tau_{n-1} = c$ of long cycles $c \in S_n$ in transpositions $\tau_i$. 

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Caution: All long cycles appear here! Some long cycles do not appear here!
What if we add weights?
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$L_{K_4}(\omega) := \begin{bmatrix}
\sum_{j \neq 1} \omega_{1j} & -\omega_{12} & -\omega_{13} & -\omega_{14} \\
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-\omega_{14} & -\omega_{24} & -\omega_{34} & \sum_{j \neq 4} \omega_{4j}
\end{bmatrix}$

Laplacian Matrix

1
\omega_{12}

2
\omega_{13}
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3
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4
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**Theorem** ((weighted) Matrix Tree theorem).

The Laplacian of a graph $G$ on $n$ vertices counts the spanning trees of $G$ via the formula

$$\sum_{T \text{ a sp. tree for } G} w(T) = \frac{1}{n} \prod_{\lambda_i \neq 0} \lambda_i(\omega),$$

where the $\lambda_i(\omega)$ are the eigenvalues of the Laplacian $L_G$ and $\text{wt}$ the natural weight on trees.
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Example for $G = K_4$:

$q\det(L_{K_4}(\omega)) := \prod_{\lambda_i \neq 0} \lambda_i(\omega) = 4 \cdot w_{12} \cdot w_{13} \cdot w_{14} + 4 \cdot w_{12} \cdot w_{23} \cdot w_{14} + 4 \cdot w_{13} \cdot w_{23} \cdot w_{14} + \cdots$
Corollary (the Denes argument).

The weighted count of factorizations of long cycles \( c \in S_n \) in transpositions \( \tau_i \) is given via

\[
\sum_{\tau_1 \cdots \tau_{n-1} = c} w(\tau_1) \cdots w(\tau_{n-1}) = \left( \frac{1}{n} \prod_{\lambda_i \neq 0} \lambda_i(\omega) \right) \cdot (n-1)!,
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where the \( \lambda_i(\omega) \) are the eigenvalues of the Laplacian \( L_{K_n} \) and \( w((i,j)) = \omega_{ij} \).
Same for factorizations!

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\[
(14)(13)(12) = (1234) \quad (12)(23)(14) = (1423) \quad (13)(23)(14) = (1432)
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\cdots \quad \cdots \quad \cdots
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Now, consider the exponential generating function

$$\mathcal{F}_{S_n}(t) = \sum_{N \geq 0} F_{S_n}(N) \cdot \frac{t^N}{N!}.$$
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$$\left[ \frac{t^{n-1}}{(n-1)!} \right] \mathcal{F}_{S_n}(t) = \frac{1}{n} \cdot n^{n-1} \cdot (n-1)! = n^{n-2} \cdot (n-1)!.$$ 

exp. gen. fnc.
Can we do both at the same time?
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We consider \( \binom{n}{2} \) parameters \( \omega := (\omega_{ij})_{i<j} \) that form a weight system \( w((ij)) = \omega_{ij} \) for the transpositions \( (ij) \in S_n \). If \( C \) is the class of the long cycles, define:

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F_{S_n}(t, \omega) := \sum_{N \geq 0} \frac{t^N}{N!} \sum_{(\tau_1, \cdots, \tau_N, c) \in R^N \times C} w(\tau_1) \cdot w(\tau_2) \cdots w(\tau_N).
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**Theorem** (Burman-Zvonkine '08, Alon-Kozma '10).

The exponential generating function above is given via the product formula:

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F_{S_n}(t, \omega) = \frac{e^{tw(\mathcal{R})}}{n} \cdot \prod_{\lambda_i \neq 0} \left( 1 - e^{-t\lambda_i(\omega)} \right),
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Taking the leading term then gives:

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\left[ \frac{t^{n-1}}{(n-1)!} \right] \mathcal{F}_{S_n}(t, \omega) = \left( \frac{1}{n} \cdot \prod_{\lambda_i \neq 0} \lambda_i(\omega) \right) \cdot (n - 1)!,
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which is in fact a new proof of the (weighted) Matrix Tree theorem after Denes’ argument.
A complete poset of formulas?

\[ F_{S_n}(t, \omega) = \frac{e^{tw(R)}}{n} \cdot \prod_{\lambda_i \neq 0} (1 - e^{-t\lambda_i(\omega)}) \]

\[ F_{S_n}(n - 1, \omega) = \left( \frac{1}{n} \cdot \prod_{\lambda_i \neq 0} \lambda_i(\omega) \right) \cdot (n - 1)! \]

\[ F_{S_n}(t) = \frac{e^{t(n/2)}}{n} \cdot (1 - e^{-tn})^{n-1} \]

\[ F_{S_n}(n - 1) = n^{n-2} \cdot (n - 1)! \]
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Forgetting the weights

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Forgetting the weights
Complex reflection groups and Coxeter elements
A finite subgroup $W \leq \text{GL}(V)$, for some $V \cong \mathbb{C}^n$ is called a complex reflection group if it is generated by pseudo-reflections. These are $\mathbb{C}$-linear maps $t$ that fix a hyperplane. If $W$ is generated by $n$ reflections we say that it is well-generated.
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Shephard and Todd have classified (irreducible) complex reflection groups into:
1. an infinite 3-parameter family $G(r, p, n)$ of monomial groups
2. 34 exceptional cases indexed $G_4$ to $G_{37}$. 
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**Definition.** Well-generated groups \( W \) possess Coxeter elements \( c \). Those are:
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4. In the general complex case, $c$ is a Springer $e^{2\pi i / h}$-regular element where $h$ is the Gordon-Griffeth Coxeter number $(|R| + |A|)/n$. 

![](image)
Arbitrary length reflection factorizations of Coxeter elements $c$
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$$F_W(N) := \# \{(\tau_1, \cdots, \tau_N, c) \in \mathcal{R}^N \times \mathcal{C} \mid \tau_1 \cdots \tau_N = c \}.$$
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**Theorem** (Chapuy-Stump, ’12).

If $W$ is well-generated, of rank $n$, and $h$ is the order of the Coxeter element $c$, then

$$F_W(t) = \frac{e^{t|\mathcal{R}|}}{h} (1 - e^{-th})^n.$$
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Notice that

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[t^n] F_W(t) = \frac{1}{h} \cdot h^n \cdot n! = |\mathcal{C}| \cdot \frac{h^n n!}{|W|}.
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Looijenga-Deligne-Arnol’d-Chapoton-Reading-Bessis formula for the chain number of the noncrossing lattice $NC'(W)$
A bigger poset of formulas!
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\[ F_{S_n}(t, \omega) = \frac{e^{tw(R)}}{n} \cdot \prod_{\lambda_i \neq 0} \left(1 - e^{-t\lambda_i(\omega)}\right) \]

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Factorizations of Coxeter elements with Jucys-Murphy weights
Consider a (maximal) tower of parabolic subgroups

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and a weight system \( w_T \) on reflections \( \tau \in R \) with parameters \( \omega := (\omega_i) \) assigned by:

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If \( C \) denotes the class of Coxeter elements, define the exponential generating function

\[ \mathcal{F}_W^T(t, \omega) := \sum_{N \geq 0} \frac{t^N}{N!} \sum_{(\tau_1, \cdots, \tau_N, c) \in R^N \times C} w_T(\tau_1) \cdot w_T(\tau_2) \cdots w_T(\tau_N). \]
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**Theorem 1** (Chapuy, D. '19). For any parabolic tower \( T \), the function \( F^T_W(t, \omega) \) is given as
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F^T_W(t, \omega) = e^{tw_T(R)} \cdot \frac{1}{h} \cdot \prod_{i=1}^{n} \left( 1 - e^{-t\lambda^T_i(\omega)} \right),
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**Theorem 1** (Chapuy, D. ’19). For any parabolic tower \( T \), the function \( F^T_W(t, \omega) \) is given as

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where \( \{\lambda_i^T(\omega)\} \) are the eigenvalues of the \( W \)-Laplacian:

\[ L^T_W(\omega) := \sum_{\tau \in \mathcal{R}} w_T(\tau) \cdot (1 - \rho_V(\tau)) \in \text{GL}(V). \]

(\( \rho_V \) is the reflection representation of \( W \))
Why call it the $W$-Laplacian?
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$L_{K_4}(\omega) := \begin{bmatrix}
\sum_{j \neq 1} \omega_{1j} & -\omega_{12} & -\omega_{13} & -\omega_{14} \\
-\omega_{12} & \sum_{j \neq 2} \omega_{2j} & -\omega_{23} & -\omega_{24} \\
-\omega_{13} & -\omega_{23} & \sum_{j \neq 3} \omega_{3j} & -\omega_{34} \\
-\omega_{14} & -\omega_{24} & -\omega_{34} & \sum_{j \neq 4} \omega_{4j}
\end{bmatrix}$

Laplacian Matrix

\begin{tikzpicture}
  \node (1) at (0,0) [circle,fill,inner sep=2pt] {} node[above]{1};
  \node (2) at (2,0) [circle,fill,inner sep=2pt] {} node[above]{2};
  \node (3) at (2,2) [circle,fill,inner sep=2pt] {} node[above]{3};
  \node (4) at (0,2) [circle,fill,inner sep=2pt] {} node[above]{4};
  \draw (1) -- (2) node[midway,above right]{$\omega_{12}$};
  \draw (1) -- (3) node[midway,above]{$\omega_{13}$};
  \draw (1) -- (4) node[midway,above left]{$\omega_{14}$};
  \draw (2) -- (3) node[midway,above left]{$\omega_{23}$};
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Laplacian Matrix

1 \quad \omega_{12} \\
\omega_{14} \quad 2 \\
\omega_{13} \quad \omega_{23} \\
\omega_{34} \quad 3 \\
\omega_{41} \\

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Why call it the $W$-Laplacian?

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Laplacian Matrix

So that the definition

$L_T^W(\omega) := \sum_{\tau \in R} w_T(\tau) \cdot \left(1 - \rho_V(\tau)\right) \in GL(V)$

is a direct generalization of the graph Laplacian.
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\[ F_{W}^{T}(t, \omega) = \frac{e^{tw_{T}(R)}}{h} \cdot \prod_{i=1}^{n} (1 - e^{-t\lambda_{i}^{T}(\omega)}) \]

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Leading term

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\[ \omega_{i} = 1 \]

Leading term

\[ F_{S_{n}}(t) = \frac{e^{t(\frac{n}{2})}}{n} \cdot (1 - e^{-tn})^{n-1} \]

\[ F_{S_{n}}^{T}(t, \omega) = \frac{e^{tw(TR)}}{n^{2}} \cdot \prod_{i \neq 0} (1 - e^{-t\lambda_{i}(\omega)}) \]

Leading term

\[ F_{S_{n}}(n - 1, \omega) = \left( \frac{1}{n} \cdot \prod_{i \neq 0} \lambda_{i}(\omega) \right) \cdot (n - 1)! \]

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Representation theoretic interpretation
The filtration of $\mathcal{R}$ by the tower $T$ defines natural analogs of the Jucys-Murphy elements:

$$C[W] \ni J_i := \sum_{\tau \in \mathcal{R} \text{ and } \tau \in W_i \setminus W_{i-1}} \tau,$$

and we write $C[J] := C[J_1, \ldots, J_n]$ for the (commutative) algebra they generate.
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Definition. We say that two virtual characters $\psi_1$ and $\psi_2$ of $W$ are tower equivalent, and write $\psi_1 \equiv \psi_2$, if they agree on the subalgebra $\mathbb{C}[J]$ of $\mathbb{C}[W]$ for any choice of $T$. 

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**Theorem 2 (Chapuy, D. '19).** Our Thm. 1 can be rephrased as the tower equivalence:

$$\sum_{\chi \in \hat{W}} \chi(c^{-1}) \cdot \chi \equiv \sum_{k=1}^{n} (-1)^k \cdot \bigwedge (V_{\text{ref}}).$$
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That the virtual characters agree on the identity $\text{id} \in W$ and the element of the group algebra $\mathbb{R} := \sum_{i=1}^{n} J_i = \sum_{\tau \in \mathcal{R}} \tau$ is in fact equivalent with the Chapuy-Stump formula.

It has relatively difficult uniform proofs.
The product form is forced by the tower equivalence (Thm. 2)
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\[ F_T^W(t, \omega) = \frac{e^{tw(R)}}{h} \sum_{\chi \in \hat{W}} \chi(c^{-1}) \cdot \chi(-L_T^W(\omega)^N) \cdot \frac{t^N}{N!}, \]

where we write \( L_T^W(\omega) \) for the Laplacian element \( \sum_{\tau \in \mathbb{R}} w_T(\tau)(\text{id} - \tau) \in \mathbb{C}[W] \).
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By Theorem 2 we can rewrite this as

\[ \mathcal{F}_W^T(t, \omega) = \frac{e^{tw(R)}}{h} \sum_{k=0}^{n} (-1)^k \left( \bigwedge_k (V_{\text{ref}}) \right) (-L_W^T(\omega)^N) \cdot \frac{t^N}{N!} \]
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Burman’s theory of Lie-like elements completely determines the eigenvalues of \( \mathcal{L}^T_W(\omega) \) on \( \bigwedge^k (V_{\text{ref}}) \). They are precisely the \( k \)-sums of the eigenvalues of the \( W \)-Laplacian \( \mathcal{L}^T_W(\omega) \).
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So now, we have

\[
\mathcal{F}_W^T(t, \omega) = \frac{e^{tw(R)}}{h} \sum_{k=0}^{n} (-1)^k \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} e^{-t\lambda_{i_1}(\omega) - \ldots - t\lambda_{i_k}(\omega)} = \frac{e^{tw(R)}}{h} \cdot \prod_{i=1}^{n} (1 - e^{-t\lambda_{i}(\omega)}).
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2) A non-trivial recursion in the infinite families \( G(r, 1, n) \) and \( G(r, r, n) \).
Their characters and parabolic subgroups are indexed by combinatorial objects and restriction to (parabolic) subgroups can be described via a variant of the Littlewood-Richardson’s rules (John Stembridge’s notes were very helpful).
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3) Burman’s theory of Lie-like elements and our ability to experiment in Sage-Gap-Chevie were key. Also a love for the ”Okounkov-Vershik approach” (thanks Vic!).
A $W$-matrix forest theorem!
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The whole characteristic polynomial of the Laplacian of a graph $G$ has a combinatorial interpretation. This is usually referred to as the (weighted) Matrix-forest theorem:

$$\det (x + L_G(\omega)) = \sum_{\mathcal{F}} w(\mathcal{F}) \cdot x^{c(\mathcal{F})},$$

where the sum is over all forests $\mathcal{F}$ of rooted trees in $G$, and where $c(\mathcal{F})$ counts the number of trees in the forest (and hence also roots).
A $W$-matrix forest theorem!

The whole characteristic polynomial of the Laplacian of a graph $G$ has a combinatorial interpretation. This is usually referred to as the (weighted) Matrix-forest theorem:

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**Theorem 3** (Chapuy, D. '19). The characteristic polynomial of the $W$-Laplacian is given as

$$\det (x + L_W^T(\omega)) = \sum_{\tau_1 \cdots \tau_{n-k} = c_X} |C_{W_X}(c_X)| \cdot \mathbf{w}_T(\tau_1) \cdots \mathbf{w}_T(\tau_{n-k}) \cdot \frac{x^k}{(n-k)!},$$

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**Corollary** (Chapuy, D. '19). *If we set all weights equal to 1 we get a generalization of the Deligne-Arnol’d-Bessis formula $\text{Hur}(W) = \frac{h^n n!}{|W|}$:*

$$(x + h)^n = \sum_{X \in \mathcal{L}_A} |W_X| \cdot \text{Hur}(W_X) \cdot \frac{x^{\dim(X)}}{(\text{codim}(X))!}.$$
A Laplacian $L_A(\omega)$ for general hyperplane arrangements
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Let $\mathcal{A}$ a hyperplane arrangement in some $V \cong \mathbb{C}^n$ and $\omega := (\omega_i)_{i=1}^N$ a weight system for each of its $N$-many hyperplanes $H_i$. 
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**Definition.** We give an $\mathcal{A}$-Laplacian matrix as

$$GL(V) \ni L_{\mathcal{A}}(\omega) := \sum_{i=1}^N \omega_i \cdot (\text{Id}(n) - S_{H_i}),$$

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**Lemma** (Burman et al. ’15). *(Abstract Matrix-forest theorem)*

For each hyperplane $H_i \in A$ choose an orthogonal vector $r_i$ of unit norm. Then

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where the sum is over all linearly independent sets of vectors $r_i$ (and $k = 0 \ldots n$).
A recursion for the $\mathcal{A}$-Laplacian
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**Proposition** (Chapuy, D. '19). The characteristic polynomial of the $\mathcal{A}$-Laplacian matrix satisfies

$$\det(x + L_{\mathcal{A}}(\omega)) = \sum_{X \in \mathcal{L}_{\mathcal{A}}} \text{qdet}(L_{\mathcal{A}_X}(\omega)) \cdot x^{\dim(X)},$$

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**Proposition** (Chapuy, D. '19). *The characteristic polynomial of the $A$-Laplacian matrix satisfies*

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**Corollary.** *Our $W$-Matrix-forest theorem.*

The recursion looks very similar to Brieskorn’s lemma:

$$\text{Poin}(V \setminus A, t) = \sum_{X \in \mathcal{L}_A} \text{rank} \left( H^{\text{top}}(V \setminus A_X) \right) \cdot t^{\dim(X)},$$

which in fact shows furthermore a natural decomposition of the corresponding cohomology spaces. Could the previous proposition be interpreted in a similar way?
A uniform proof of the chain number $\frac{h^n n!}{|W|}$ of $NC(W)$. 
A uniform proof of the chain number \( \frac{h^n n!}{|W|} \) of \( NC(W) \).

Write \( \text{Hur}(W) \) for the number of reduced reflection factorizations of a fixed Coxeter element \( c \):

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\text{Hur}(W) = \# \{(\tau_1, \cdots, \tau_n) \in R^n : \tau_1 \cdots \tau_n = c \} = \frac{h^n n!}{|W|}.
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$$\text{Hur}(W) = \frac{h}{2} \sum_{s \in S} \text{Hur}(W_{\langle s \rangle}).$$

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indeed this is equivalent as enumerating factorizations with respect to just the last reflection:

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The point is that we know:

$$\text{Krew}(X) = \frac{\prod_{i=1}^{\dim(X)}(h + 1 - b_i^X)}{[N(X) : W_X]}$$

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The point is that we know:

$$Krew(X) = \prod_{i=1}^{\dim(X)} \left( h + 1 - b_i^X \right) / \left[ N(X) : W_X \right]$$ and

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Uniformely!

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which after plugging in the formula to be proven demands that

$$h^{n-1} n! = \sum_{L \in L_{AW}^1/W} \frac{|W|}{|N(L)|} \cdot (\prod_{i=1}^{n-1} h_i(W_L)) \cdot (n - 1)!,$$
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and now summing over all flats (instead of orbits of flats):

$$n \cdot h^{n-1} = \sum_{L \in L_{AW}^1} \prod_{i=1}^{n-1} h_i(W_L).$$
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But in fact the recursion for the characteristic polynomial of the $W$-Laplacian gives us more:

$$(h + x)^n = \sum_{X \in \mathcal{L}_{AW}} \left( \prod_{i=1}^{\text{codim}(X)} h_i(X) \right) \cdot x^{\text{dim}(X)}.$$
The end!
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Thank you very much!
A combinatorial description of the eigenvalues of the $W$-Laplacian
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\[
A = \begin{bmatrix}
\cdot & \cdot & & & & \\
2 & 1 & 0 & 1 & 2 & 4 \\
0 & 3 & 0 & 1 & 2 & 4 \\
0 & 0 & 2 & 0 & 0 & 8 \\
0 & 0 & 0 & 4 & 2 & 4 \\
0 & 0 & 0 & 0 & 6 & 4 \\
0 & 0 & 0 & 0 & 0 & 10 \\
\end{bmatrix}
\]

\[
\{\lambda_i(\omega)\} = \left\{ \begin{array}{l}
2\omega_1 + \omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\
3\omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\
2\omega_3 + 8\omega_6, \\
4\omega_4 + 2\omega_5 + 4\omega_6, \\
6\omega_5 + 4\omega_6, \\
10\omega_6 \end{array} \right\}
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