Slides available at my website linked in the abstract (www.irif.fr/∼douvr001/)

Coxeter factorizations and the Matrix Tree theorem with generalized Jucys-Murphy weights

Guillaume Chapuy, Theo Douvropoulos

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Paris VII, IRIF and CNRS, ERC CombiTop

Counting trees and counting factorizations

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Theorem (Cayley 1889, +other people)**.** *There are n^{n-2} trees on n labeled vertices.*

Example, $n = 3$:

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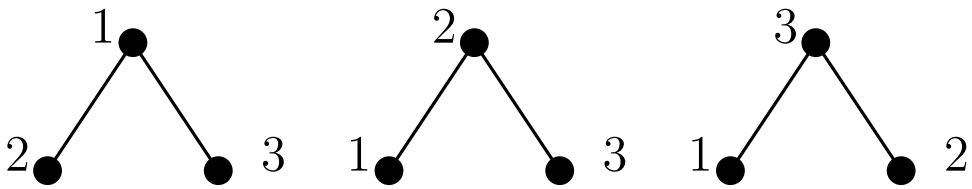

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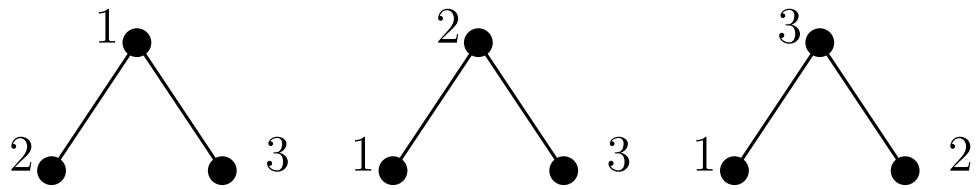

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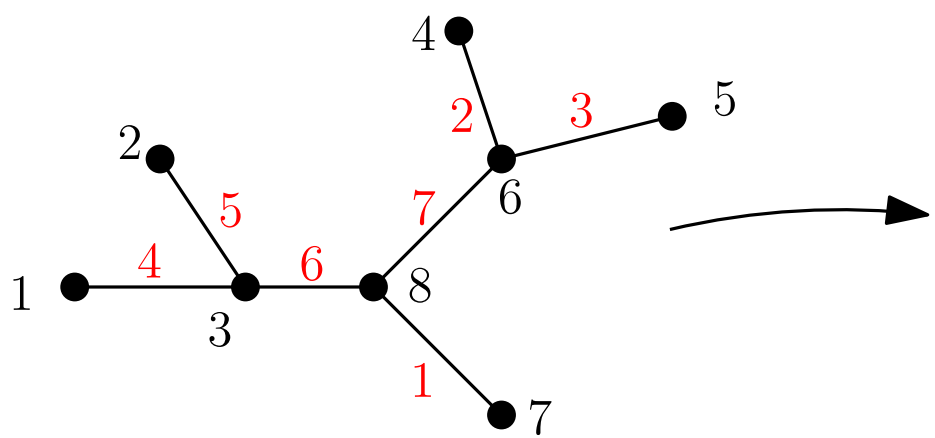
Call a permutation $c \in S_n$ a long cycle if it is conjugate to $(12 \cdots n)$. There are $n^{n-2} \cdot (n-1)!$ minimal length factorizations $\tau_1 \cdots \tau_{n-1} = c$ of long cycles $c \in S_n$ in transpositions τ_i.



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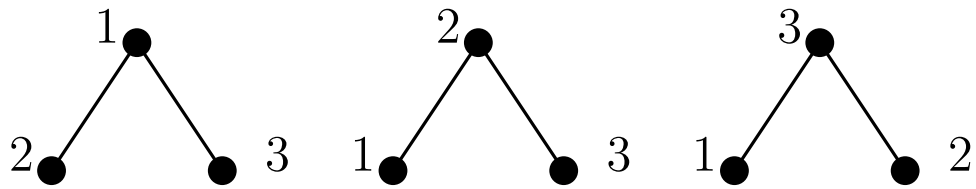

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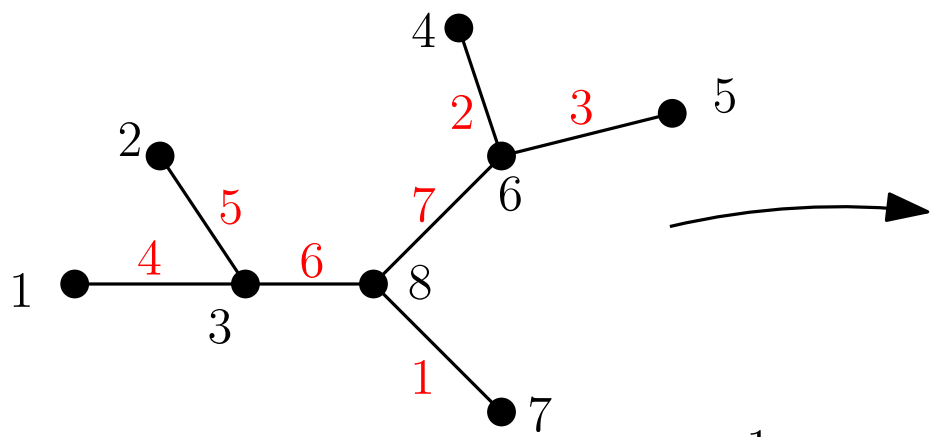

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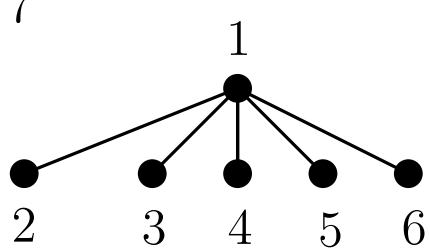
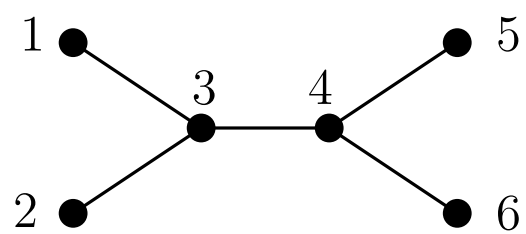


All long cycles appear here!



Some long cycles do not appear here!

What if we add weights?

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$$L_{K_4}(\boldsymbol{\omega}) := \begin{bmatrix} \sum_{j\neq 1}\omega_{1j} & -\omega_{12} & -\omega_{13} & -\omega_{14} \\ -\omega_{12} & \sum_{j\neq 2}\omega_{2j} & -\omega_{23} & -\omega_{24} \\ -\omega_{13} & -\omega_{23} & \sum_{j\neq 3}\omega_{3j} & -\omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & \sum_{j\neq 4}\omega_{4j} \end{bmatrix}$$

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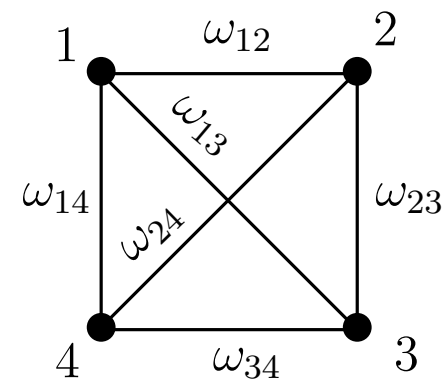

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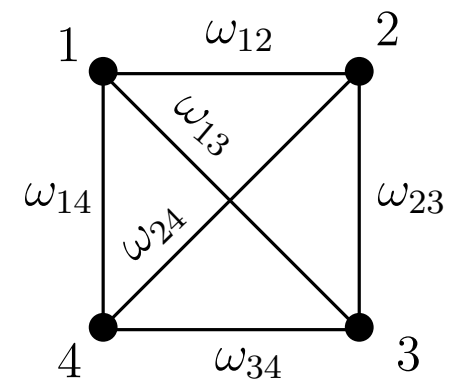

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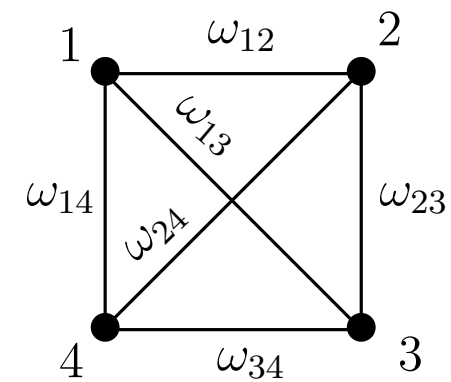

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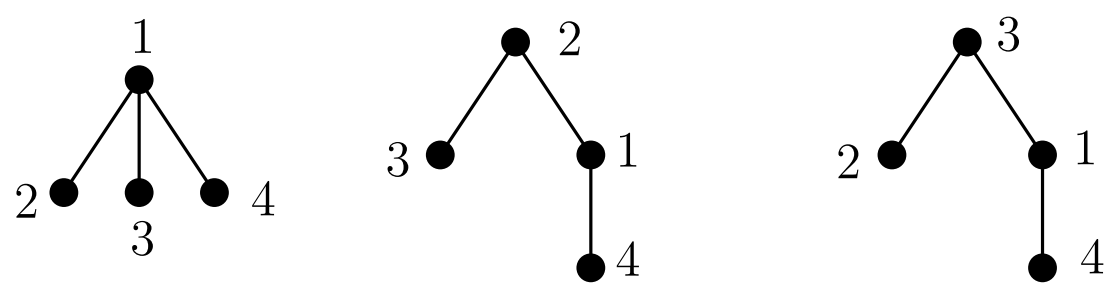
Example for $G = K_4$:

$$\operatorname{qdet}\left(L_{K_4}(\boldsymbol{\omega})\right) := \prod_{\lambda_i \neq 0} \lambda_i(\boldsymbol{\omega}) = 4 \cdot w_{12} \cdot w_{13} \cdot w_{14} + 4 \cdot w_{12} \cdot w_{23} \cdot w_{14} + 4 \cdot w_{13} \cdot w_{23} \cdot w_{14} + \cdots$$



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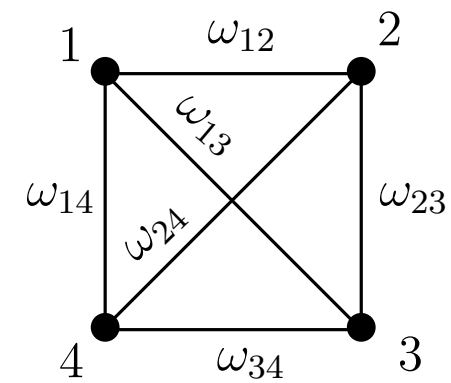

Corollary (the Denes argument)**.**
The weighted count of factorizations of long cycles $c \in S_n$ in tranpositions τ_i is given via

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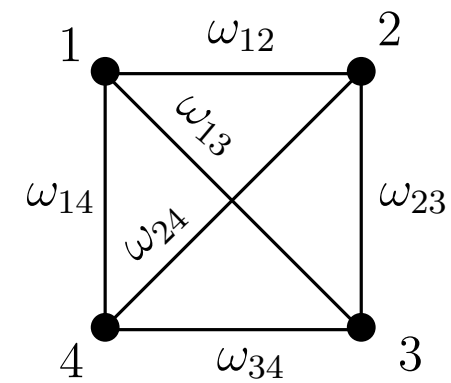

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which is in fact a new proof of the (weighted) Matrix Tree theorem after Denes' argument.

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2. *In a real reflection group W, a product of the simple generators in any order* and *any element conjugate to that.*
3. *Equivalently for real W, c is an element whose order satisfies $|c| \cdot n = 2|\mathcal{R}|$, and which has an invariant plane that is not orthogonal to any root and which it rotates by $2\pi i/|c|$.*

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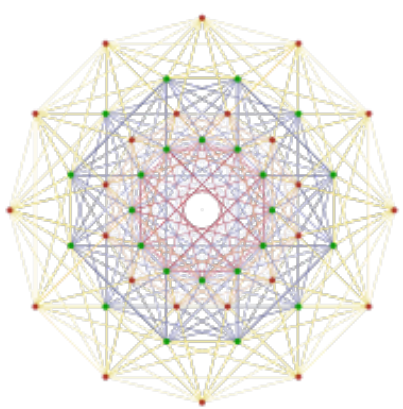
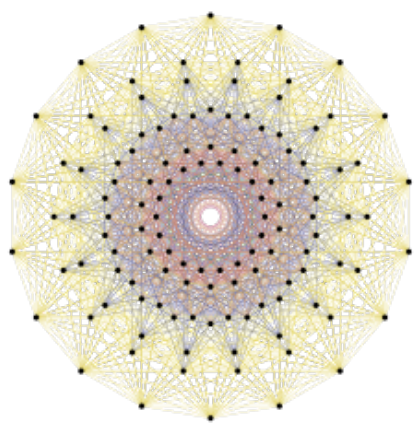


E_6



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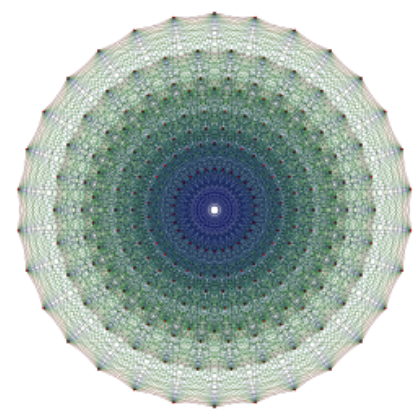

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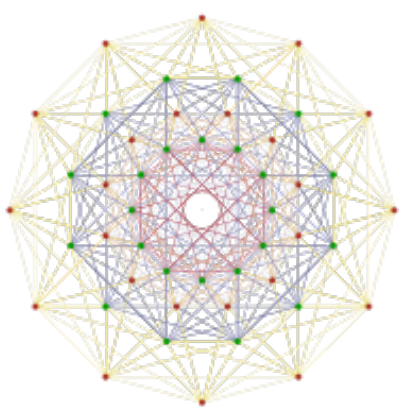

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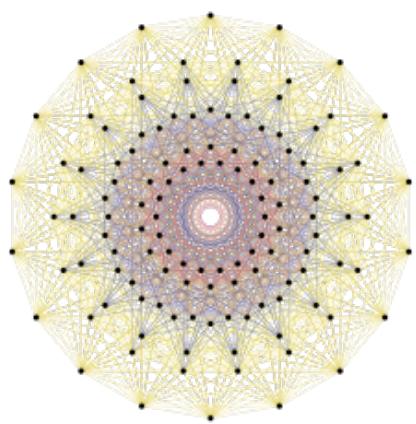

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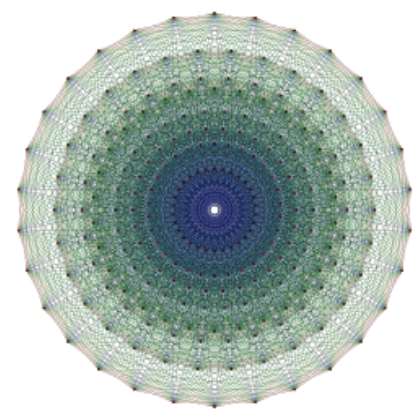

E_8

4. In the general complex case, c is a Springer $e^{2\pi i/h}$-regular element where h is the Gordon-Griffeth Coxeter number $(|\mathcal{R}| + |\mathcal{A}|)/n$.

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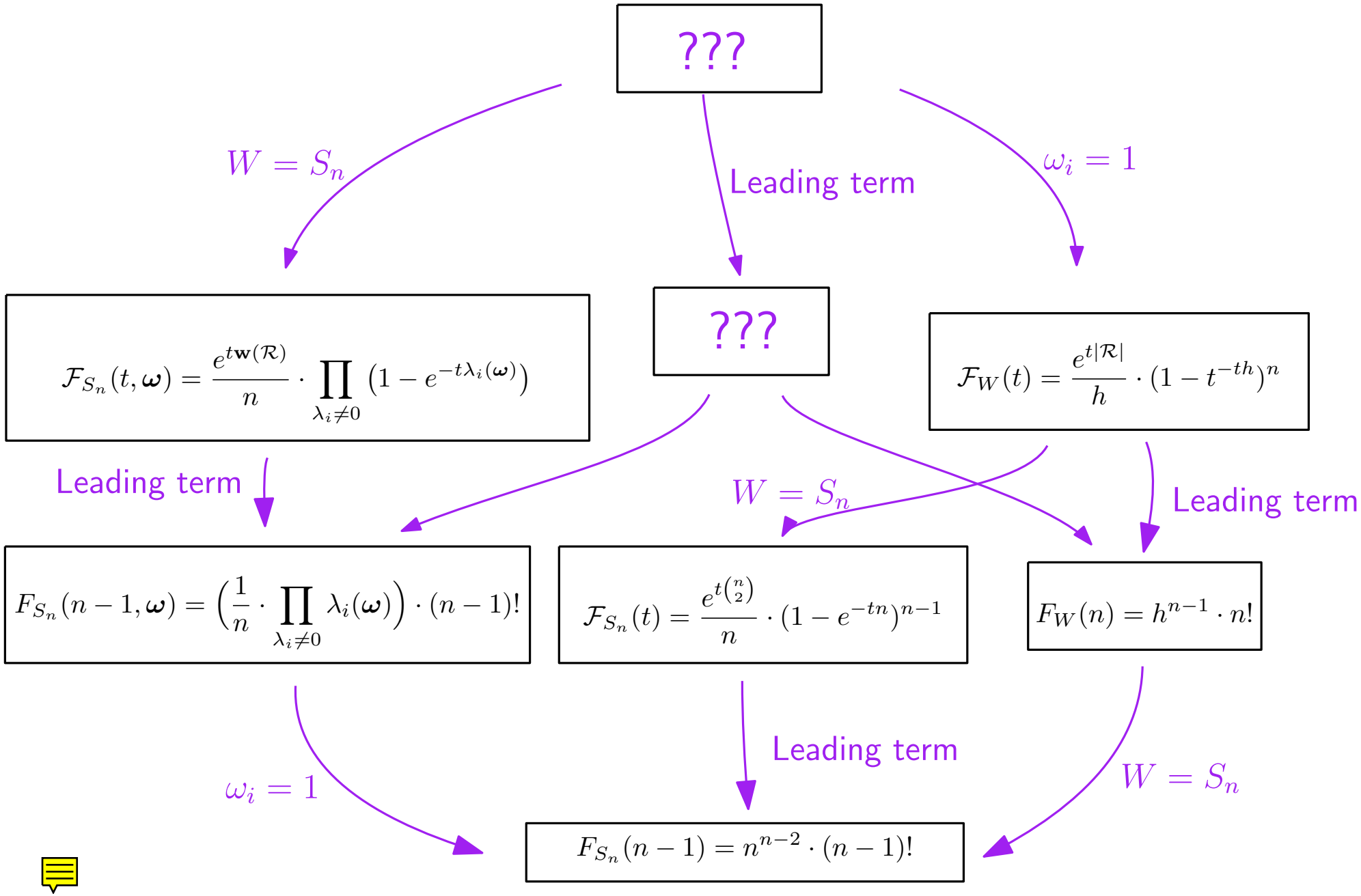

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where $\{\lambda_i^T(\boldsymbol{\omega})\}$ are the eigenvaues of the W-Laplacian:

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(ρ_V is the reflection representation of W)

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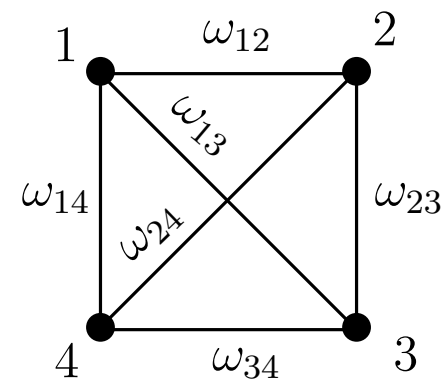
Laplacian Matrix



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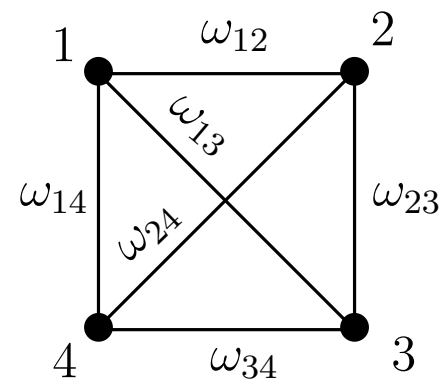

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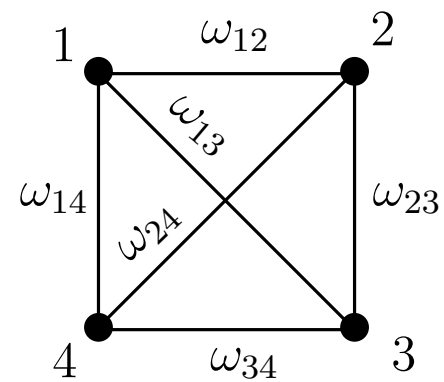

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That the virtual characters agree on the identity $\mathbf{id} \in W$ and the element of the group algebra $\mathbf{R} := \sum_{i=1}^{n} J_i = \sum_{\tau \in \mathcal{R}} \tau$ is in fact equivalent with the Chapuy-Stump formula.

It has relatively difficult uniform proofs.

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Burman's theory of Lie-like elements completely determines the eigenvalues of $\boldsymbol{\mathcal{L}}_W^T(\boldsymbol{\omega})$ on $\bigwedge^k(V_{\mathrm{ref}})$. They are precisely the k-sums of the eigenvalues of the W-Laplacian $L_W^T(\boldsymbol{\omega})$.

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So now, we have

$$\mathcal{F}_W^T(t,\boldsymbol{\omega}) = \frac{e^{t\mathbf{w}(\mathcal{R})}}{h} \sum_{k=0}^{n} (-1)^k \sum_{1\le i_1\le\cdots\le i_k\le n} e^{-t\lambda_{i_1}(\boldsymbol{\omega})-\cdots-t\lambda_{i_k}(\boldsymbol{\omega})} = \frac{e^{t\mathbf{w}(\mathcal{R})}}{h} \cdot \prod_{i=1}^{n} \big(1-e^{-t\lambda_i(\boldsymbol{\omega})}\big).$$

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Their characters and parabolic subgroups are indexed by combinatorial objects and restriction to (parabolic) subgroups can be described via a variant of the Littlewood-Richardson's rules (John Stembridge's notes were very helpful).

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3) Burman's theory of Lie-like elements and our ability to experiment in Sage-Gap-Chevie were key. Also a love for the "Okounkov-Vershik approach" (thanks Vic!).

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The whole characteristic polynomial of the Laplacian of a graph G has a combinatorial interpretation. This is usually referred to as the (weighted) Matrix-forest theorem:

$$\det\left(x + L_G(\boldsymbol{\omega})\right) = \sum_{\mathcal{F}} \mathbf{w}(\mathcal{F}) \cdot x^{c(\mathcal{F})},$$

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Theorem 3 (Chapuy, D. '19)**.** *The characteristic polynomial of the W-Laplacian is given as*

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Corollary (Chapuy, D. '19)**.** *If we set all weights equal to 1 we get a generalization of the Deligne-Arnol'd-Bessis formula $\mathrm{Hur}(W) = \frac{h^n n!}{|W|}$:*

$$(x+h)^n = \sum_{X \in \mathcal{L}_{\mathcal{A}_W}} |W_X| \cdot \mathrm{Hur}(W_X) \cdot \frac{x^{\dim(X)}}{\big(\operatorname{codim}(X)\big)!}.$$

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For each hyperplane $H_i \in \mathcal{A}$ choose an orthogonal vector r_i of unit norm. Then

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The recursion looks very similar to Brieskorn's lemma:

$$\operatorname{Poin}(V\setminus\mathcal{A}, t) = \sum_{X\in\mathcal{L}_{\mathcal{A}}} \operatorname{rank}\left(H^{\mathrm{top}}(V\setminus\mathcal{A}_X)\right) \cdot t^{\dim(X)},$$

which in fact shows furthermore a natural decomposition of the corresponding cohomology spaces. Could the previous proposition be interpreted in a similar way?

A uniform proof of the chain number $\frac{h^n n!}{|W|}$ of $NC(W)$.

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Write $\mathrm{Hur}(W)$ for the number of reduced reflection factorizations of a fixed Coxeter element c:

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Case by case.

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But in fact the recursion for the characteristic polynomial of the W-Laplacian gives us more:

$$(h+x)^n = \sum_{X \in \mathcal{L}_{\mathcal{A}_W}} \Big(\prod_{i=1}^{\mathrm{codim}(X)} h_i(X) \Big) \cdot x^{\dim(X)}.$$

The end!

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Thank you very much!

A combinatorial description of the eigenvalues of the W-Laplacian

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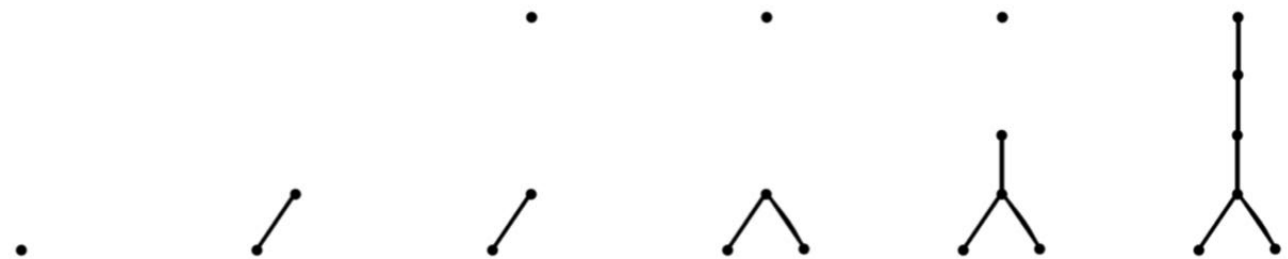

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{cccccc} 2 & 1 & 0 & 1 & 2 & 4 \\ 0 & 3 & 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 0 & 0 & 8 \\ 0 & 0 & 0 & 4 & 2 & 4 \\ 0 & 0 & 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{array} \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{array}$$

$$\{\lambda_i(\boldsymbol{\omega})\} = \left\{ \begin{array}{c} 2\omega_1 + \omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\ 3\omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\ 2\omega_3 + 8\omega_6, \quad 4\omega_4 + 2\omega_5 + 4\omega_6, \\ 6\omega_5 + 4\omega_6, \quad 10\omega_6 \end{array} \right\}$$