# A Framework for Multidimensional Continued Fractions

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December 11, 2016

**Abstract**

This is not a paper. It is an attempt to fix, for multidimensional continued fractions, notation that many of us can then use. The first section is for $\mathbb{R}^n$. Section two gives the eight triangle partition map analogs of the classical Gauss map, the $\mathbb{R}^2$ case. I have not seen them before but would not be at all surprised if they are already known. If anyone has seen one of the seven "new" maps before, please let me know. The third section is fixing notation for the $\mathbb{R}^3$ case. The original triangle map is worked out, as well as showing that the Cassaigne algorithm is a TRIP map. The final section is about what a common framework might look like. These notes would not be appropriate for someone who did not already know something about multi-dimensional continued fractions. Comments and corrections welcome.

## 1 A general framework

### 1.1 Initial notation

We start with a quite general framework. In later sections we will see how the standard continued fraction algorithm and various multi-dimensional continued fraction algorithms can be put into this language. None of this should be viewed as conceptually deep. We are just being careful with notation.

Start with $n$ linearly independent vectors, written as column vectors,

$$v_1, v_2, \ldots, v_n$$

in $\mathbb{R}^n$ We write these as a single $n \times n$ matrix

$$V = (v_1, \ldots v_n).$$

These $n$ vectors define a cone in $\mathbb{R}^N$:

$$\triangle = \{\mathbf{x} \in \mathbb{R}^n : \exists \text{ non-negative real numbers } a, b, c \text{ with } \mathbf{x} = av_1 + v_2 + v_3\}.$$

Suppose we have two $n \times n$ matrices $F_0$ and $F_1$ such that for all $\mathbf{x} \in \triangle$,

$$\mathbf{x}F_0, \mathbf{x}F_1 \in \triangle$$

and such that

$$\triangle F_0 \cap \triangle F_1$$

is a cone in a hyperplane and such that

$$\triangle = \triangle F_0 \cup \triangle F_1.$$

Set

$$\begin{aligned} \triangle_0 &= \triangle F_0 \\ \triangle_1 &= \triangle F_1. \end{aligned}$$

Then we can define two linear maps:

$$T_0 : \triangle_0 \to \triangle$$
$$T_1 : \triangle_1 \to \triangle,$$

both one-to-one and onto. For

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \triangle_{\mathbf{0}} - (\triangle_{\mathbf{0}} \cap \triangle_{\mathbf{1}}),$$

define

$$T_0(\mathbf{x}) = VF_0^{-1}V^{-1}\mathbf{x} = VF_0^{-1}V^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and for

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \triangle_{\mathbf{1}} - (\triangle_{\mathbf{0}} \cap \triangle_{\mathbf{1}}),$$

define

$$T_1(\mathbf{x}) = VF_1^{-1}V^{-1}\mathbf{x} = VF_1^{-1}V^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In the natural way we define

$$T\mathbf{x}) = \begin{cases} T_0(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \triangle_{\mathbf{0}} - (\triangle_{\mathbf{0}} \cap \triangle_{\mathbf{1}}) \\ T_1(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \triangle_{\mathbf{1}} - (\triangle_{\mathbf{0}} \cap \triangle_{\mathbf{1}}) \end{cases}$$

**Definition 1.** *A non-zero vector* $\mathbf{x} \in \triangle$ *has* $(F_0, F_1)$ *sequence* $(i_0, i_1, i_2, \ldots)$*, consisting of zeros and ones, if*

$$\mathbf{x} \in \triangle_{i_0}, T(\mathbf{x}) \in \triangle_{i_1}, T(T(\mathbf{x})) \in \triangle_{i_2}, \ldots.$$

*If it ever happens that* $T^{(n)}(\mathbf{x})$ *is in both* $\triangle_0$ *and* $\triangle_1$*, the sequence stops and we declare, by fiat, that* $i_n = 0$.

There is always a problem for how to handle points in the intersection $\triangle_0 \cap \triangle_1$. Since this is a set of measure zero, these intersection points don't really matter for most of our concerns. We will be somewhat sloppy in how we treat these types of points.

There is also a multiplicative version. Here the difference between the 0 map and the 1 map becomes important.

For any non-negative integer $k$, define the cones

$$\triangle_k^G = \triangle F_1^k F_0.$$

The $G$ is for Gauss, for this is a generalization of the classical Gauss map, as we will see. Then for any $\mathbf{x} \in \triangle_k^G$, define

$$T : \triangle_k^G \to \triangle$$

(which if we want to emphasis the domain $\triangle_k^G$, we will sometimes denote with a subscript as $T_k$) by setting

$$T(\mathbf{x}) = V(VF_1^k F_0)^{-1}\mathbf{x}.$$

**Definition 2.** *A non-zero vector* $\mathbf{x} \in \triangle$ *has G-sequence (or Gauss sequence)* $(k_0, k_1, k_2, \ldots)$*, consisting of non-negative integers, if*

$$\mathbf{x} \in \triangle_{k_0}, T^G(\mathbf{x}) \in \triangle_{i_1}, T^G(T^G(\mathbf{x})) \in \triangle_{i_2}, \ldots.$$

*If it ever happens that* $T^{(n)}(\mathbf{x})$ *is in both* $\triangle_0$ *and* $\triangle_1$*, the sequence stops.*

## 1.2 Adding permutations

Associated to the above, we have an additional $n! \cdot n! \cdot n!$ algorithms, constructed as follows. Let $S_n$ denote the permutation group on $n$ letters and write each element of $S_n$ as an $n \times n$ matrix acting on the right. For example, we write $S_3$ as the following matrices:

$$\begin{aligned} e &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ (12) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ (13) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ (23) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ (123) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ (132) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Start again with $n$ linearly independent vectors, written as column vectors,

$$v_1, v_2, \ldots, v_n$$

in $\mathbb{R}^n$, which we will again write as an $n \times n$ matrix

$$V = (v_1, \ldots v_n).$$

And again fix two $n \times n$ matrices $F_0$ and $F_1$ such that for all $\mathbf{x} \in \triangle$,

$$\mathbf{x}F_0, vF_1 \in \triangle$$

and such that

$$\triangle F_0 \cap \triangle F_1$$

is a cone in a hyperplane and such that

$$\triangle = \triangle F_0 \cup \triangle F_1.$$

Now choose a $(\sigma, \tau_0, \tau_1) \in S_n \times S_n \times S_n$. Define

$$\begin{aligned} F_0(\sigma, \tau_0, \tau_1) &= \sigma F_0 \tau_0 \\ F_1(\sigma, \tau_0, \tau_1) &= \sigma F_1 \tau_1 \end{aligned}$$

and set

$$\begin{aligned} \triangle_0(\sigma, \tau_0, \tau_1) &= \triangle F_0(\sigma, \tau_0, \tau_1) \\ \triangle_1(\sigma, \tau_0, \tau_1) &= \triangle F_1(\sigma, \tau_0, \tau_1) \end{aligned}$$

For each choice of $(\sigma, \tau_0, \tau_1) \in S_n \times S_n \times S_n$, we have two linear maps:

$$T_0(\sigma,\tau_0,\tau_1) : \triangle_0(\sigma,\tau_0,\tau_1) \to \triangle$$
$$T_1(\sigma,\tau_0,\tau_1) : \triangle_0(\sigma,\tau_0,\tau_1) \to \triangle,$$

both one-to-one and onto. For

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \triangle_{\mathbf{0}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}}) - (\triangle_{\mathbf{0}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}}) \cap \triangle_{\mathbf{1}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}})),$$

define

$$T_0(\sigma,\tau_0,\tau_1)(\mathbf{x}) = VF_0^{-1}(\sigma,\tau_0,\tau_1)V^{-1}\mathbf{x} = VF_0^{-1}(\sigma,\tau_0,\tau_1)V^{-1}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and for

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \triangle_{\mathbf{1}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}}) - (\triangle_{\mathbf{0}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}}) \cap \triangle_{\mathbf{1}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}})),$$

define

$$T_1(\sigma,\tau_0,\tau_1)(\mathbf{x}) = VF_1^{-1}(\sigma,\tau_0,\tau_1)^{-1}V^{-1}\mathbf{x} = VF_1(\sigma,\tau_0,\tau_1)V^{-1}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In the natural way we define

$$T(\sigma,\tau_0,\tau_1)\mathbf{x}) = \begin{cases} T_0(\sigma,\tau_0,\tau_1)(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \triangle_{\mathbf{0}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}}) - (\triangle_{\mathbf{0}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}}) \cap \triangle_{\mathbf{1}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}})) \\ T_1(\sigma,\tau_0,\tau_1)(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \triangle_{\mathbf{1}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}}) - (\triangle_{\mathbf{0}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}}) \cap \triangle_{\mathbf{1}}(\sigma,\tau_{\mathbf{0}},\tau_{\mathbf{1}})) \end{cases}$$

**Definition 3.** *A non-zero vector* $\mathbf{x} \in \triangle$ *has Farey-*$(\sigma,\tau_0,\tau_1)$ *sequence* $(i_0, i_1, i_2, \ldots)$*, consisting of zeros and ones, if*

$$\mathbf{x} \in \triangle_{i_0}(\sigma,\tau_0,\tau_1), T(\mathbf{x}) \in \triangle_{i_1}(\sigma,\tau_0,\tau_1), T(T(\mathbf{x})) \in \triangle_{i_2}(\sigma,\tau_0,\tau_1), \ldots.$$

*If it ever happens that* $T^{(n)}(\mathbf{x})$ *is in both* $\triangle_0$ *and* $\triangle_1$*, the sequence stops*

There is still a problem of handling points in the intersection $\triangle_0(\sigma,\tau_0,\tau_1) \cap \triangle_1(\sigma,\tau_0,\tau_1)$. Similar to before, this is still a set of measure zero, so again these intersection points don't really matter for us. We will continue to be s sloppy in how we treat these types of points.

Here is the multiplicative version.

For any non-negative integer $k$, define the cones

$$\triangle_k^G(\sigma,\tau_0,\tau_1) = \triangle F_1^k(\sigma,\tau_0,\tau_1)F_0(\sigma,\tau_0,\tau_1).$$

Then for any $\mathbf{x} \in \triangle_k(\sigma,\tau_0,\tau_1)^G$, define

$$T^G(\sigma,\tau_0,\tau_1) : \triangle_k^G(\sigma,\tau_0,\tau_1) \to \triangle$$

(which if we want to emphasis the domain $\triangle_k^G$, we will sometimes denote with a subscript as $T_k(\sigma,\tau_0,\tau_1)$) by setting

$$T^G(\sigma,\tau_0,\tau_1)(\mathbf{x}) = V(VF_1^k(\sigma,\tau_0,\tau_1)F_0(\sigma,\tau_0,\tau_1))^{-1}\mathbf{x}.$$

**Definition 4.** *A non-zero vector* $\mathbf{x} \in \triangle$ *has G-*$(\sigma,\tau_0,\tau_1)$*-sequence (or Gauss sequence)* $(k_0, k_1, k_2, \ldots)$*, consisting of non-negative integers, if*

$$\mathbf{x} \in \triangle_{k_0(\sigma,\tau_0,\tau_1)}, T^G(\sigma,\tau_0,\tau_1)(\mathbf{x}) \in \triangle_{i_1}(\sigma,\tau_0,\tau_1), T^G(\sigma,\tau_0,\tau_1)(T^G(\sigma,\tau_0,\tau_1)(\mathbf{x})) \in \triangle_{i_2}(\sigma,\tau_0,\tau_1), \ldots.$$

*If it ever happens that* $T^G(\sigma,\tau_0,\tau_1)^{(n)}(\mathbf{x})$ *is in both* $\triangle_0(\sigma,\tau_0,\tau_1)$ *and* $\triangle_1(\sigma,\tau_0,\tau_1)$*, the sequence stops.*

# 2 Gauss map and TRIP-Gauss Maps

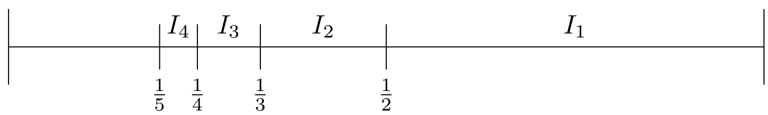
## 2.1 Classical Gauss Map and Farey Map

We start with a classical overview of the Farey map and the Gauss map, and then put it into the language of the above.

**Definition 5.** *For any* $0 < x \leq 1$*, the classical Gauss map* $G : (0, 1] \to [0, 1)$ *is*

$$G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

(Here $\lfloor x \rfloor$ is the floor function, or the greatest integer less than or equal to $x$.)
We can interpret this as a map on a partitioning of the unit interval. Consider




We define $I_k = (\frac{1}{k+1}, \frac{1}{k}]$. If $x \in I_k$, then the Gauss map is simply,

$$G(x) = \frac{1}{x} - k = \frac{1 - kx}{x}.$$

**Definition 6.** *A number* $x$ *has a continued fraction sequence* $a_0, a_1, a_2, \ldots$ *if*

$$x \in I_{a_0}, G(x) \in I_{a_1}, G(G(x)) \in I_{a_2}, \ldots .$$

It can be shown that

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

**Definition 7.** *For* $0 < x \leq 1$*, the Farey map* $F : (0, 1] \to [0, 1)$ *is*

$$F(x) = \begin{cases} \frac{1-x}{x} & \text{if } \frac{1}{2} < x \leq 1 \\ \frac{x}{1-x} & \text{if } 0 < x < \frac{1}{2} \end{cases}$$

Set

$$I_0 = \left(\frac{1}{2}, 1\right), \; I_1 = \left(0, \frac{1}{2}\right).$$

**Definition 8.** *A number* $x$ *has Farey-sequence* $i_0, i_1, i_2, \ldots$ *, with each* $i_k$ *a zero or a one, if*

$$x \in I_{i_0}, F(x) \in I_{i_1}, F(F(x)) \in I_{i_2}, \ldots .$$

## 2.2 Translating into $2 \times 2$ matrices:

We know put this into the language of Section 1.1. Set

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$V = (v_1, v_2) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

Then set

$$F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, F_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

We will identify $2 \times 1$ column vectors, up to multiplying by a scalar, with elements in $\mathbb{R}$ by setting

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \frac{x}{y},$$

provided of course that $y \neq 0$. Then

$$v_1 \sim 0, \; v_2 \sim 1.$$

We will identify a matrix

$$\begin{pmatrix} x & z \\ y & w \end{pmatrix}$$

with the interval

$$\left(\frac{x}{y}, \frac{z}{w}\right), \text{ or } \left(\frac{z}{w}, \frac{x}{y}\right)$$

if $x/y < z/w$ or if $z/w < x/y$.

Note that

$$F_1^k = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned}
\triangle_0 &= VF_0 \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
&\sim \left(\frac{1}{2}, 1\right) \\
&= I_0 \\
\triangle_1 &= VF_1 \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \\
&\sim \left(0, \frac{1}{2}\right) \\
&= I_1.
\end{aligned}$$

We now define two maps

$$T_0 : \triangle_0 \to \triangle$$

$$T_1 : \triangle_1 \to \triangle,$$

each one-to-one and onto. For

$$\mathbf{x} = \begin{pmatrix} x \\ 1 \end{pmatrix} \in \triangle_{\mathbf{0}} - (\triangle_{\mathbf{0}} \cap \triangle_{\mathbf{1}}),$$

$$\begin{aligned}
T_0(\mathbf{x}) &= VF_0^{-1}V^{-1}\mathbf{x} \\
&= VF_0^{-1}V^{-1}\begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1-x \\ x \end{pmatrix} \\
&\sim \frac{1-x}{x}
\end{aligned}$$

and for

$$\mathbf{x} = \begin{pmatrix} x \\ 1 \end{pmatrix} \in \triangle_{\mathbf{1}} - (\triangle_{\mathbf{0}} \cap \triangle_{\mathbf{1}}),$$

$$\begin{aligned}
T_1(\mathbf{x}) &= VF_1^{-1}V^{-1}\mathbf{x} \\
&= VF_1^{-1}V^{-1}\begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x \\ 1-x \end{pmatrix} \\
&\sim \frac{x}{1-x}.
\end{aligned}$$

This is simply the Farey map.

The multiplicative version will give us the classical Gauss map, as follows.

For any non-negative integer $k$, define the cones

$$\begin{aligned}
\triangle_k^G &= \triangle F_1^k F_0 \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ k+1 & k+2 \end{pmatrix} \\
&\sim \left(\frac{1}{k+2}, \frac{1}{k+1}\right) \\
&= I_{k+1}.
\end{aligned}$$

The $G$ is for Gauss. Then for any

$$\mathbf{x} = \begin{pmatrix} x \\ 1 \end{pmatrix} \in \triangle_k^G,$$

( which means that $1/(k+2) < x < 1/(k+1)$, we have

$$\begin{aligned}
T^G(\mathbf{x}) &= V(VF_1^kF_0)^{-1}\mathbf{x} \\
&= VF_0^{-1}F_1^{-1}V^{-1}\mathbf{x} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} \\
&= \begin{pmatrix} -1-k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1-(k+1)x \\ x \end{pmatrix} \\
&\sim \frac{1-(k+1)x}{x},
\end{aligned}$$

giving us the Gauss map.

## 2.3 Adding permutations the additive case

Since we are working in $\mathbb{R}^2$ and with $2\times 2$ matrices, we will be concerned with the permutation group

$$S_2 = \{e, (12)\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\|.$$

We will set $\sigma = (12)$. From Section 1.2, we can create eight maps (seven new ones), from the two matrices

$$F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, F_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We have the following cones and subintervals

$$
\begin{aligned}
\triangle_0(e,e,e) &= V_0(e,e,e) \\
&= VF_0(e,e,e) \\
&= (v_1,v_2)\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= (v_2, v_1+v_2) \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
&\sim \left(\frac{1}{2},1\right) \\
\triangle_1(e,e,e) &= V_1(e,e,e) \\
&= VF_1(e,e,e) \\
&= (v_1,v_2)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= (v_1, v_1+v_2) \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \\
&\sim \left(0,\frac{1}{2}\right) \\
\triangle_0(e,e,\sigma) &= V_0(e,e,\sigma) \\
&= VF_0(e,e,\sigma) \\
&= (v_1,v_2)\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= (v_2, v_1+v_2) \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
&\sim \left(\frac{1}{2},1\right) \\
\triangle_1(e,e,\sigma) &= V_1(e,e,\sigma) \\
&= VF_1(e,e,\sigma) \\
&= (v_1,v_2)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= (v_1+v_2, v_1) \\
&= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\
&\sim \left(0,\frac{1}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\triangle_0(e,\sigma,e) &= V_0(e,\sigma,e)\\
&= VF_0(e,\sigma,e)\\
&= (v_1,v_2)\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}\\
&= (v_1+v_2,v_2)\\
&= \begin{pmatrix} 1 & 1\\ 2 & 1\end{pmatrix}\\
&\sim \left(\frac{1}{2},1\right)\\
\triangle_1(e,\sigma,e) &= V_1(e,\sigma,e)\\
&= VF_1(e,\sigma,e)\\
&= (v_1,v_2)\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}\\
&= (v_1,v_1+v_2)\\
&= \begin{pmatrix} 0 & 1\\ 1 & 2\end{pmatrix}\\
&\sim \left(0,\frac{1}{2}\right)\\
\triangle_0(e,\sigma,\sigma) &= V_0(e,\sigma,\sigma)\\
&= VF_0(e,\sigma,\sigma)\\
&= (v_1,v_2)\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}\\
&= (v_1+v_2,v_2)\\
&= \begin{pmatrix} 1 & 1\\ 2 & 1\end{pmatrix}\\
&\sim \left(\frac{1}{2},1\right)\\
\triangle_1(e,\sigma,\sigma) &= V_1(e,\sigma,\sigma)\\
&= VF_1(e,\sigma,\sigma)\\
&= (v_1,v_2)\begin{pmatrix} 1 & 1\\ 1 & 0\end{pmatrix}\\
&= (v_1+v_2,v_1)\\
&= \begin{pmatrix} 1 & 0\\ 2 & 1\end{pmatrix}\\
&\sim \left(0,\frac{1}{2}\right)
\end{aligned}
$$

By calculation, we have

$$F_0(e,e,e) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = F_1(\sigma,e,e)$$
$$F_1(e,e,e) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = F_0(\sigma,e,e)$$

$$F_0(e,e,\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = F_1(\sigma,\sigma,e)$$
$$F_1(e,e,\sigma) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = F_0(\sigma,\sigma,e)$$

$$F_0(e,\sigma,e) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = F_1(\sigma,e,\sigma)$$
$$F_1(e,\sigma,e) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = F_0(\sigma,e,\sigma)$$

$$F_0(e,\sigma,\sigma) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = F_1(\sigma,\sigma,\sigma)$$
$$F_1(e,\sigma,\sigma) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = F_0(\sigma,\sigma,\sigma)$$

Then

$$
\begin{array}{rcl}
\triangle_0(\sigma,e,e) & = & V_0(\sigma,e,e) \\
& = & VF_0(\sigma,e,e) \\
& = & (v_1,v_2)\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right) \\
& = & (v_1,v_1+v_2) \\
& = & \left(\begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array}\right) \\
& \sim & \left(0,\dfrac{1}{2}\right) \\
\triangle_1(\sigma,e,e) & = & V_1(\sigma,e,e) \\
& = & VF_1(\sigma,e,e) \\
& = & (v_1,v_2)\left(\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right) \\
& = & (v_2,v_1+v_2) \\
& = & \left(\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right) \\
& \sim & \left(\dfrac{1}{2},1\right) \\
\triangle_0(\sigma,e,\sigma) & = & V_0(\sigma,e,\sigma) \\
& = & VF_0(\sigma,e,\sigma) \\
& = & (v_1,v_2)\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right) \\
& = & (v_1,v_1+v_2) \\
& = & \left(\begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array}\right) \\
& \sim & \left(0,\dfrac{1}{2}\right) \\
\triangle_1(\sigma,e,\sigma) & = & V_1(\sigma,e,\sigma) \\
& = & VF_1(\sigma,e,\sigma) \\
& = & (v_1,v_2)\left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right) \\
& = & (v_1+v_2,v_2) \\
& = & \left(\begin{array}{cc} 1 & 1 \\ 2 & 1 \end{array}\right) \\
& \sim & \left(\dfrac{1}{2},1\right)
\end{array}
$$

$$
\begin{aligned}
\triangle_0(\sigma,\sigma,e) &= V_0(\sigma,\sigma,e) \\
&= VF_0(\sigma,\sigma,e) \\
&= (v_1,v_2)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= (v_1+v_2,v_1) \\
&= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\
&\sim \left(0,\frac{1}{2}\right) \\
\triangle_1(\sigma,\sigma,e) &= V_1(\sigma,\sigma,e) \\
&= VF_1(\sigma,\sigma,e) \\
&= (v_1,v_2)\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= (v_2,v_1+v_2) \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
&\sim \left(\frac{1}{2},1\right) \\
\triangle_0(\sigma,\sigma,\sigma) &= V_0(\sigma,\sigma,\sigma) \\
&= VF_0(\sigma,\sigma,\sigma) \\
&= (v_1,v_2)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= (v_1+v_2,v_1) \\
&= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\
&\sim \left(0,\frac{1}{2}\right) \\
\triangle_1(\sigma,\sigma,\sigma) &= V_1(\sigma,\sigma,\sigma) \\
&= VF_1(\sigma,\sigma,\sigma) \\
&= (v_1,v_2)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
&= (v_1+v_2,v_2) \\
&= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \\
&\sim \left(\frac{1}{2},1\right)
\end{aligned}
$$

We now want to define the appropriate Farey maps, which we will denote by $T(\mu,\tau_0,\tau_1)$. Each of these maps is made up of two other maps :$T_0(\mu,\tau_0,\tau_1)$ and $T_1(\mu,\tau_0,\tau_1)$, with each

$$T_i(\mu,\tau_0,\tau_1) : I_i(\mu,\tau_0,\tau_1) \to I.$$

All of these maps can also be written as $2\times 2$ matrices giving the following mappings:

$$T_i(\mu,\tau_0,\tau_1) : \triangle_i(\mu,\tau_0,\tau_1) \to \triangle.$$

In all, we have, as matrices,

$$T_i(\mu,\tau_0,\tau_1) = V(VF_i(\mu,\tau_0,\tau_1))^{-1}.$$

The maps as matrices are

$$
\begin{aligned}
T_0(e,e,e) &= V(VF_0(e,e,e))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\
T_1(e,e,e) &= V(VF_1(e,e,e))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
T_0(e,e,\sigma) &= V(VF_0(e,e,\sigma))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\
T_1(e,e,\sigma) &= V(VF_1(e,e,\sigma))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix} \\
T_0(e,\sigma,e) &= V(VF_0(e,\sigma,e))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \\
T_1(e,\sigma,e) &= V(VF_1(e,\sigma,e))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
T_0(e,\sigma,\sigma) &= V(VF_0(e,\sigma,\sigma))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \\
T_1(e,\sigma,\sigma) &= V(VF_1(e,\sigma,\sigma))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}
\end{aligned}
$$

$$\begin{aligned}
T_0(\sigma,e,e) &= V(VF_0(\sigma,e,e))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
T_1(\sigma,e,e) &= V(VF_1(\sigma,e,e))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\
T_0(\sigma,e,\sigma) &= V(VF_0(\sigma,e,\sigma))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\
T_1(\sigma,e,\sigma) &= V(VF_1(\sigma,e,\sigma))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \\
T_0(\sigma,\sigma,e) &= V(VF_0(\sigma,\sigma,e))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix} \\
T_1(\sigma,\sigma,e) &= V(VF_1(\sigma,\sigma,e))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\
T_0(\sigma,\sigma,\sigma) &= V(VF_0(\sigma,\sigma,\sigma))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix} \\
T_1(\sigma,\sigma,\sigma) &= V(VF_1(\sigma,\sigma,\sigma))^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}(\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix})^{-1} \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

Then the maps are

$$
\begin{aligned}
T(e,e,e)(x) &= \begin{cases} T_0(e,e,e)(x) & \text{if } \frac{1}{2} < x < 1 \\ T_1(e,e,e)(x) & \text{if } 0 < x < \frac{1}{2} \end{cases} \\
&= \begin{cases} \frac{1-x}{x} & \text{if } \frac{1}{2} < x < 1 \\ \frac{x}{1-x} & \text{if } 0 < x < \frac{1}{2} \end{cases} \\
T(e,e,\sigma)(x) &= \begin{cases} T_0(e,e,\sigma)(x) & \text{if } \frac{1}{2} < x < 1 \\ T_1(e,e,\sigma)(x) & \text{if } 0 < x < \frac{1}{2} \end{cases} \\
&= \begin{cases} \frac{1-x}{x} & \text{if } \frac{1}{2} < x < 1 \\ \frac{1-2x}{1-x} & \text{if } 0 < x < \frac{1}{2} \end{cases} \\
T(e,\sigma,e)(x) &= \begin{cases} T_0(e,\sigma,e)(x) & \text{if } \frac{1}{2} < x < 1 \\ T_1(e,\sigma,e)(x) & \text{if } 0 < x < \frac{1}{2} \end{cases} \\
&= \begin{cases} \frac{2x-1}{x} & \text{if } \frac{1}{2} < x < 1 \\ \frac{x}{1-x} & \text{if } 0 < x < \frac{1}{2} \end{cases} \\
T(e,\sigma,\sigma)(x) &= \begin{cases} T_0(e,\sigma,\sigma)(x) & \text{if } \frac{1}{2} < x < 1 \\ T_1(e,\sigma,\sigma)(x) & \text{if } 0 < x < \frac{1}{2} \end{cases} \\
&= \begin{cases} \frac{2x-1}{x} & \text{if } \frac{1}{2} < x < 1 \\ \frac{1-2x}{1-x} & \text{if } 0 < x < \frac{1}{2} \end{cases} \\
T(\sigma,e,e)(x) &= \begin{cases} T_0(\sigma,e,e)(x) & \text{if } 0 < x < \frac{1}{2} \\ T_1(\sigma,e,e)(x) & \text{if } \frac{1}{2} < x < 1 \end{cases} \\
&= \begin{cases} \frac{x}{1-x} & \text{if } 0 < x < \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} < x < 1 \end{cases} \\
T(\sigma,e,\sigma)(x) &= \begin{cases} T_0(\sigma,e,\sigma)(x) & \text{if } 0 < x < \frac{1}{2} \\ T_1(\sigma,e,\sigma)(x) & \text{if } \frac{1}{2} < x < 1 \end{cases} \\
&= \begin{cases} \frac{x}{1-x} & \text{if } 0 < x < \frac{1}{2} \\ \frac{2x-1}{x} & \text{if } \frac{1}{2} < x < 1 \end{cases} \\
T(\sigma,\sigma,e)(x) &= \begin{cases} T_0(\sigma,\sigma,e)(x) & \text{if } 0 < x < \frac{1}{2} \\ T_1(\sigma,\sigma,e)(x) & \text{if } \frac{1}{2} < x < 1 \end{cases} \\
&= \begin{cases} \frac{1-2x}{1-x} & \text{if } 0 < x < \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} < x < 1 \end{cases} \\
T(\sigma,\sigma,\sigma)(x) &= \begin{cases} T_0(\sigma,\sigma,\sigma)(x) & \text{if } 0 < x < \frac{1}{2} \\ T_1(\sigma,\sigma,\sigma)(x) & \text{if } \frac{1}{2} < x < 1 \end{cases} \\
&= \begin{cases} \frac{1-2x}{1-x} & \text{if } 0 < x < \frac{1}{2} \\ \frac{2x-1}{x} & \text{if } \frac{1}{2} < x < 1 \end{cases}
\end{aligned}
$$

The symmetry and pairing between those maps that start with an $e$ and those that start with a $\sigma$ simply reflect our somewhat arbitrary choice of which map we denote by a subscript of 0 and which map we denote by a subscript of 1.

## 2.4 The multiplicative verson

This symmetry, though, is not nearly so clean when we turn to the multiplicative versions.

For $(\tau, \mu_1, \mu_2) \in S_2 \times S_2 \times S_2$, we set

$$
\begin{aligned}
F_0(\tau,\mu_1,\mu_2) &= \tau \cdot F_0 \cdot \mu_1 \\
F_1(\tau,\mu_1,\mu_2) &= \tau \cdot F_1 \cdot \mu_2 \\
\triangle_k^G(\tau,\mu_1,\mu_2) &= VF_1(\tau,\mu)^{k-1}F_0(\tau,\mu) \\
T_k^G(\tau,\mu_1,\mu_2)(bfx) &\sim VF_0(\tau,\mu_1,\mu_2)^{-1}F_1(\tau,\mu_1,\mu_2)^{1-k}V^{-1}bfx \\
&\text{for} \quad \mathbf{x} \in \triangle_k(\tau,\mu_1,\mu_2)^G
\end{aligned}
$$

### 2.5 The explicit matrices $F_0(\mu, \tau_0, \tau_1)$ and $F_1(\mu, \tau_0, \tau_1)$

As matrices we have

$$\begin{aligned}
F_0(e,e,e) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
F_1(e,e,e) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
F_0(e,e,\sigma) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
F_1(e,e,\sigma) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
F_0(e,\sigma,e) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
F_1(e,\sigma,e) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
F_0(e,\sigma,\sigma) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
F_1(e,\sigma,\sigma) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
F_0(\sigma,e,e) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
F_1(\sigma,e,e) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
F_0(\sigma,e,\sigma) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
F_1(\sigma,e,\sigma) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
F_0(\sigma,\sigma,e) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
F_1(\sigma,\sigma,e) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
F_0(\sigma,\sigma,\sigma) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
F_1(\sigma,\sigma,\sigma) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\end{aligned}$$

We have each of these matrices act on the column vectors $(v_1, v_2)$ by multiplication on the right. For example, we have $F_0(e,e,e)$ acting on $(v_1, v_2)$ as

$$(v_1, v_2)F_0(e,e,e) = (v_1, v_2)\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (v_2, v_1 + v_2),$$

which we will write as

$$F(e,e,e) : (v_1, v_2) \to (v_2, v_1 + v_2).$$

Acting on the vertices $v_1, v_2$, we have

$$F_0(e,e,e) : (v_1, v_2) \to (v_2, v_1 + v_2)$$

$$F_1(e,e,e) : (v_1, v_2) \to (v_1, v_1 + v_2)$$

$$F_0(e,e,\sigma) : (v_1, v_2) \to (v_2, v_1 + v_2)$$
$$F_1(e,e,\sigma) : (v_1, v_2) \to (v_1 + v_2, v_1)$$
$$F_0(e,\sigma,e) : (v_1, v_2) \to (v_1 + v_2, v_2)$$
$$F_1(e,\sigma,e) : (v_1, v_2) \to (v_1, v_1 + v_2)$$
$$F_0(e,\sigma,\sigma) : (v_1, v_2) \to (v_1 + v_2, v_2)$$
$$F_1(e,\sigma,\sigma) : (v_1, v_2) \to (v_1 + v_2, v_1)$$
$$F_0(\sigma,e,e) : (v_1, v_2) \to (v_1, v_1 + v_2)$$
$$F_1(\sigma,e,e) : (v_1, v_2) \to (v_2, v_1 + v_2)$$
$$F_0(\sigma,e,\sigma) : (v_1, v_2) \to (v_1, v_1 + v_2)$$
$$F_1(\sigma,e,\sigma) : (v_1, v_2) \to (v_1 + v_2, v_2)$$
$$F_0(\sigma,\sigma,e) : (v_1, v_2) \to (v_1 + v_2, v_1)$$
$$F_1(\sigma,\sigma,e) : (v_1, v_2) \to (v_2, v_1 + v_2)$$
$$F_0(\sigma,\sigma,\sigma) : (v_1, v_2) \to (v_1 + v_2, v_1)$$
$$F_1(\sigma,\sigma,\sigma) : (v_1, v_2) \to (v_1 + v_2, v_2)$$

Now let us see about iterations of the $F_1(\mu, \tau_0, \tau_1)$s as applied to the two vectors $V = (v_1, v_2)$.

Consider the Fibonacci sequence

$$f_{k+1} = f_k + f_{k-1},$$

with $f_{-1} = 0, f_0 = 1$. Then we have, starting with $f_{-1} = 0$, the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots.$$

$$
\begin{aligned}
VF_1^k(e,e,e) &= (v_1,v_2)\begin{pmatrix} 0 & 1 \\ k & 1 \end{pmatrix} \\
&= (v_1, kv_1+v_2) \\
&= \begin{pmatrix} 0 & 1 \\ 1 & k+1 \end{pmatrix} \\
VF_1^k(e,e,\sigma) &= (v_1,v_2)\begin{pmatrix} f_k & f_{k-1} \\ f_{k-1} & f_{k-2} \end{pmatrix} \\
&= (f_kv_1+f_{k-1}v_2, f_{k-1}v_1+f_{k-2}v_2) \\
&= \begin{pmatrix} f_{k-1} & f_{k-2} \\ f_{k+1} & f_k \end{pmatrix} \\
VF_1^k(e,\sigma,e) &= (v_1,v_2)\begin{pmatrix} 0 & k \\ 1 & 1 \end{pmatrix} \\
&= (v_1, kv_1+v_2) \\
&= \begin{pmatrix} 0 & 1 \\ 1 & k+1 \end{pmatrix} \\
VF_1^k(e,\sigma,\sigma) &= (v_1,v_2)\begin{pmatrix} f_k & f_{k-1} \\ f_{k-1} & f_{k-2} \end{pmatrix} \\
&= (f_kv_1+f_{k-1}v_2, f_{k-1}v_1+f_{k-2}v_2) \\
&= \begin{pmatrix} f_{k-1} & f_{k-2} \\ f_{k+1} & f_k \end{pmatrix} \\
VF_1^k(\sigma,e,e) &= (v_1,v_2)\begin{pmatrix} f_{k-2} & f_{k-1} \\ f_{k-1} & f_k \end{pmatrix} \\
&= (f_{k-2}v_1+f_{k-1}v_2, f_{k-1}v_1+f_kv_2) \\
&= \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix} \\
VF_1^k(\sigma,e,\sigma) &= (v_1,v_2)\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \\
&= (v_1+kv_2, v_2) \\
&= \begin{pmatrix} k & 1 \\ k+1 & 1 \end{pmatrix} \\
VF_1^k(\sigma,\sigma,e) &= (v_1,v_2)\begin{pmatrix} f_{k-2} & f_{k-1} \\ f_{k-1} & f_k \end{pmatrix} \\
&= (f_{k-2}v_1+f_{k-1}v_2, f_{k-1}v_1+f_kv_2) \\
&= \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix} \\
VF_1^k(\sigma,\sigma,\sigma) &= (v_1,v_2)\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \\
&= (v_1+kv_2, v_2) \\
&= \begin{pmatrix} k & 1 \\ k+1 & 1 \end{pmatrix}
\end{aligned}
$$

Of course, the somewhat drastic different forms of the various $F_1(\mu,\tau_0,\tau_1)$ depends on the different types of eigenvalues for the matrices.

## 2.6 The natural subintervals $\triangle_k^G(\mu,\tau_0,\tau_1) = VF_1^kF_0$

Now to list the various subintervals $\triangle_k^G(\mu,\tau_0,\tau_1)$.

$$
\begin{aligned}
\triangle_k^G(e,e,e) &\sim VF_1^k(e,e,e)F_0(e,e,e) \\
&= \begin{pmatrix} 0 & 1 \\ 1 & k+1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ k+1 & k+2 \end{pmatrix} \\
\triangle_k^G(e,e,\sigma) &\sim VF_1^k(e,e,\sigma)F_0(e,e,\sigma) \\
&= \begin{pmatrix} f_{k-1} & f_{k-2} \\ f_{k+1} & f_k \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} f_{k-2} & f_k \\ f_k & f_{k+2} \end{pmatrix} \\
\triangle_k^G(e,\sigma,e) &\sim VF_1^k(e,\sigma,e)F_0(e,\sigma,e) \\
&= \begin{pmatrix} 0 & 1 \\ 1 & k+1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2+k & 1+k \end{pmatrix} \\
\triangle_k^G(e,\sigma,\sigma) &\sim VF_1^k(e,\sigma,\sigma)F_0(e,\sigma,\sigma) \\
&= \begin{pmatrix} f_{k-1} & f_{k-2} \\ f_{k+1} & f_k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} f_k & f_{k-2} \\ f_{k+2} & f_k \end{pmatrix} \\
\triangle_k^G(\sigma,e,e) &\sim VF_1^k(\sigma,e,e)F_0(\sigma,e,e) \\
&= \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} f_{k-1} & f_{k+1} \\ f_k & f_{k+2} \end{pmatrix} \\
\triangle_k^G(\sigma,e,\sigma) &\sim VF_1^k(\sigma,e,\sigma)F_0(\sigma,e,\sigma) \\
&= \begin{pmatrix} k & 1 \\ k+1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix} \\
\triangle_k^G(\sigma,\sigma,e) &\sim VF_1^k(\sigma,\sigma,e)F_0(\sigma,\sigma,e) \\
&= \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f_{k+1} & f_{k-1} \\ f_{k+2} & f_k \end{pmatrix} \\
\triangle_k^G(\sigma,\sigma,\sigma) &\sim VF_1^k(\sigma,\sigma,\sigma)F_0(\sigma,\sigma,\sigma) \\
&= \begin{pmatrix} k & 1 \\ k+1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} k+1 & k \\ k+2 & k+1 \end{pmatrix}
\end{aligned}
$$

### 2.7 The actual maps in the $\mathbb{R}^2$ case

To find the appropriate analog of the Gauss map, we need to find the various $V(VF_1^k(\mu,\tau_0,\tau_1)F_0(\mu,\tau_0,\tau_1))^{-1}$. We have

$$\begin{aligned}
V(VF_1^k(e,e,e)F_0(e,e,e))^{-1} &= \begin{pmatrix} -k-1 & 1 \\ 1 & 0 \end{pmatrix} \\
V(VF_1^k(e,e,\sigma)F_0(e,e,\sigma))^{-1} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^k f_{k+2} & -(-1)^k f_k \\ -(-1)^k f_k & (-1)^k f_{k-2} \end{pmatrix} \\
&= \begin{pmatrix} -(-1)^k f_k & (-1)^k f_{k-2} \\ -(-1)^k f_k + (-1)^k f_{k+2} & (-1)^k f_{k-2} - (-1)^k f_k \end{pmatrix} \\
V(VF_1^k(e,\sigma,e)F_0(e,\sigma,e))^{-1} &= \begin{pmatrix} 2+k & -1 \\ 1 & 0 \end{pmatrix} \\
V(VF_1^k(e,\sigma,\sigma)F_0(e,\sigma,\sigma))^{-1} &= \begin{pmatrix} (-1)^k f_{k+2} & -(-1)^k f_k \\ (-1)^k(f_{k+2}-f_k) & (-1)^k(f_{k-2}-f_k) \end{pmatrix} \\
V(VF_1^k(\sigma,e,e)F_0(\sigma,e,e))^{-1} &= \begin{pmatrix} (-1)^k f_k & -(-1)^k f_{k-1} \\ (-1)^k(-f_{k+2}+f_k) & (-1)^k(-f_{k-1}+f_{k+1}) \end{pmatrix} \\
V(VF_1^k(\sigma,e,\sigma)F_0(\sigma,e,\sigma))^{-1} &= \begin{pmatrix} k+1 & -k \\ -1 & 1 \end{pmatrix} \\
V(VF_1^k(\sigma,\sigma,e)F_0(\sigma,\sigma,e))^{-1} &= \begin{pmatrix} (-1)^k f_{k+2} & -(-1)^k f_{k+1} \\ (-1)^k(f_{k+2}-f_k) & (-1)^k(f_{k-1}-f_{k+1}) \end{pmatrix} \\
V(VF_1^k(\sigma,\sigma,\sigma)F_0(\sigma,\sigma,\sigma))^{-1} &= \begin{pmatrix} -(k+2) & k+1 \\ -1 & 1 \end{pmatrix}
\end{aligned}$$

Then the actual maps are

$$
\begin{aligned}
T_k^G(e,e,e)(x) &= \frac{1-(k+1)x}{x}\\
T_k^G(e,e,\sigma)(x) &= \frac{-f_kx+f_{k-2}}{(-f_k+f_{k+2})x+f_{k-2}-f_k}\\
&= \frac{-f_kx+f_{k-2}}{f_{k+1}x-f_{k-1}}\\
T_k^G(e,\sigma,e)(x) &= \frac{-1+(k+2)x}{x}\\
T_k^G(e,\sigma,\sigma)(x) &= \frac{-f_k+f_{k+2}x}{(f_{k+2}-f_k)x+(f_{k-2}-f_k)}\\
&= \frac{-f_k+f_{k+2}x}{f_{k+1}x+-f_{k+1}}\\
T_k^G(\sigma,e,e)(x) &= \frac{-f_{k-1}+f_kx}{(f_k-f_{k+2})x+(f_{k+1}-f_{k-1})}\\
&= \frac{-f_{k-1}+f_kx}{-f_{k+1}x+f_k}\\
T_k^G(\sigma,e,\sigma)(x) &= \frac{(k+1)x-k}{1-x}\\
T_k^G(\sigma,\sigma,e)(x) &= \frac{-f_{k+1}+f_{k+2}x}{(f_{k-1}-f_{k+1})+(f_{k+2}-f_k)x}\\
&= \frac{-f_{k+1}+f_{k+2}x}{-f_k+f_{k+1}x}\\
T_k^G(\sigma,\sigma,\sigma)(x) &= \frac{k+1-(k+2)x}{1-x}
\end{aligned}
$$

# 3 $\mathbb{R}^3$ version: Triangle Partiton Maps

TRIP maps originally appeared in [4]. More about them are in [5, 8, 2].

## 3.1 Additive version of TRIP maps

Using the notation from 1.1, we start with three linearly independent vectors $v_1, v_2, \ldots, v_n$, written as column vectors, in $\mathbb{R}^3$, and we write these as a single $3 \times 3$ matrix

$$V = (v_1, v_2, v_3).$$

We assume that the vectors are positively oriented, meaning that

$$\det(V) > 0.$$

These 3 vectors define a cone in $\mathbb{R}^3$:

$$\triangle = \{\mathbf{x} \in \mathbb{R}^n : \exists \text{ non-negative real numbers } a, b, c \text{ with } \mathbf{x} = av_1 + v_2 + v_3\}.$$

We will often depict not the cone but a slice of the cone, which will be a triangle.

Though the choice vectors $v_1, v_2, \ldots, v_n$ is not that important, as long as all are in $\mathbb{Z}^3$, there are two popular choices, either

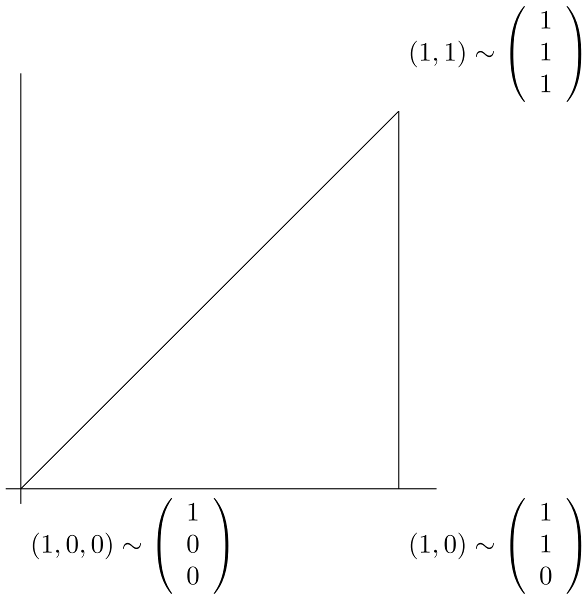
$$V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

or the identity matrix

$$V = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

If we choose the first, then it is typical to represent the cone as the triangle

$$\triangle = \{(x,y) : 1 \geq x \geq y \geq 0\}$$




Set

$$F_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \; F_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

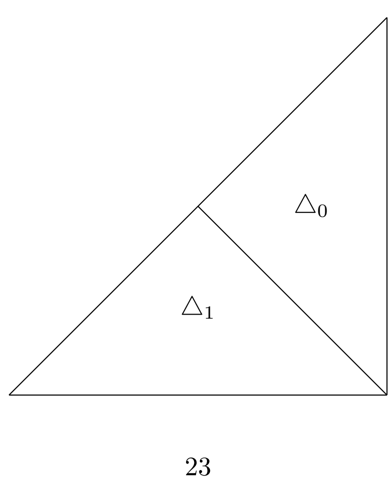
Still using the first choice of $(v_1, v_2, v_3)$, then the subcones $\triangle_0$, spanned by

$$VF_0 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

and $\triangle_1$, spanned by

$$VF_1 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

are



We write the group $S_3$ as the following matrices:

$$\begin{aligned} e &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ (12) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ (13) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ (23) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ (123) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ (132) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

We will now basically repeat what we did in section 1.2, setting $n = 3$.
Choose a $(\sigma, \tau_0, \tau_1) \in S_3 \times S_3 \times S_3$. Define

$$\begin{aligned} F_0(\sigma, \tau_0, \tau_1) &= \sigma F_0 \tau_0 \\ F_1(\sigma, \tau_0, \tau_1) &= \sigma F_1 \tau_1 \end{aligned}$$

and set

$$\begin{aligned} \triangle_0(\sigma, \tau_0, \tau_1) &= \triangle F_0(\sigma, \tau_0, \tau_1) \\ \triangle_1(\sigma, \tau_0, \tau_1) &= \triangle F_1(\sigma, \tau_0, \tau_1) \end{aligned}$$

For each choice of $(\sigma, \tau_0, \tau_1) \in S_3 \times S_3 \times S_3$, we have the two linear maps:

$$T_0(\sigma, \tau_0, \tau_1) : \triangle_0(\sigma, \tau_0, \tau_1) \to \triangle$$

$$T_1(\sigma, \tau_0, \tau_1) : \triangle_0(\sigma, \tau_0, \tau_1) \to \triangle,$$

both one-to-one and onto. For

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \triangle_{\mathbf{0}}(\sigma, \tau_{\mathbf{0}}, \tau_{\mathbf{1}}) - (\triangle_{\mathbf{0}}(\sigma, \tau_{\mathbf{0}}, \tau_{\mathbf{1}}) \cap \triangle_{\mathbf{1}}(\sigma, \tau_{\mathbf{0}}, \tau_{\mathbf{1}})),$$

define

$$T_0(\sigma, \tau_0, \tau_1)(\mathbf{x}) = V F_0(\sigma, \tau_0, \tau_1)^{-1} V^{-1} \mathbf{x} = V F_0(\sigma, \tau_0, \tau_1)^{-1} V^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and for

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \triangle_{\mathbf{1}}(\sigma, \tau_{\mathbf{0}}, \tau_{\mathbf{1}}) - (\triangle_{\mathbf{0}}(\sigma, \tau_{\mathbf{0}}, \tau_{\mathbf{1}}) \cap \triangle_{\mathbf{1}}(\sigma, \tau_{\mathbf{0}}, \tau_{\mathbf{1}})),$$

define

$$T_1(\sigma,\tau_0,\tau_1)(\mathbf{x}) = VF_1(\sigma,\tau_0,\tau_1)^{-1}V^{-1}\mathbf{x} = VF_1(\sigma,\tau_0,\tau_1)^{-1}V^{-1}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

**Definition 9.** *The additve, or the Farey, TRIP maps are the maps*

$$T(\sigma,\tau_0,\tau_1)\mathbf{x}) = \begin{cases} T_0(\sigma,\tau_0,\tau_1)(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \triangle_{\mathbf{0}}(\sigma,\boldsymbol{\tau_0},\boldsymbol{\tau_1}) - (\triangle_{\mathbf{0}}(\sigma,\boldsymbol{\tau_0},\boldsymbol{\tau_1}) \cap \triangle_{\mathbf{1}}(\sigma,\boldsymbol{\tau_0},\boldsymbol{\tau_1})) \\ T_1(\sigma,\tau_0,\tau_1)(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \triangle_{\mathbf{1}}(\sigma,\boldsymbol{\tau_0},\boldsymbol{\tau_1}) - (\triangle_{\mathbf{0}}(\sigma,\boldsymbol{\tau_0},\boldsymbol{\tau_1}) \cap \triangle_{\mathbf{1}}(\sigma,\boldsymbol{\tau_0},\boldsymbol{\tau_1})) \end{cases}$$

**Definition 10.** *A non-zero vector $\mathbf{x} \in \triangle$ has Farey-$(\sigma,\tau_0,\tau_1)$ sequence $(i_0,i_1,i_2,\ldots)$, consisting of zeros and ones, if*

$$\mathbf{x} \in \triangle_{i_0}(\sigma,\tau_0,\tau_1), T(\mathbf{x}) \in \triangle_{i_1}(\sigma,\tau_0,\tau_1), T(T(\mathbf{x})) \in \triangle_{i_2}(\sigma,\tau_0,\tau_1), \ldots.$$

*If it ever happens that $T^{(n)}(\mathbf{x})$ is in both $\triangle_0$ and $\triangle_1$, the sequence stops*

As always, there is the problem of handling points in the intersection $\triangle_0(\sigma,\tau_0,\tau_1) \cap \triangle_1(\sigma,\tau_0,\tau_1)$. Similar to before, this is still a set of measure zero, so again these intersection points don't really matter for us. We will continue to be s sloppy in how we treat these types of points.

## 3.2 Multiplicative version of TRIP maps

For any non-negative integer $k$, define the cones

$$\triangle_k^G(\sigma,\tau_0,\tau_1) = \triangle F_1(\sigma,\tau_0,\tau_1)^k F_0(\sigma,\tau_0,\tau_1).$$

**Definition 11.** *For any $\mathbf{x} \in \triangle_k^G(\sigma,\tau_0,\tau_1)$, define the multiplicative, or Gauss, TRIP map to be*

$$T^G(\sigma,\tau_0,\tau_1) : \triangle_k^G(\sigma,\tau_0,\tau_1) \to \triangle$$

*(which if we want to emphasis the domain $\triangle_k^G(\sigma,\tau_0,\tau_1)$, we will sometimes denote with a subscript as $T_k^G(\sigma,\tau_0,\tau_1)$) by setting*

$$T^G(\sigma,\tau_0,\tau_1)(\mathbf{x}) = V(VF_1(\sigma,\tau_0,\tau_1)^k F_0(\sigma,\tau_0,\tau_1))^{-1}\mathbf{x}.$$

**Definition 12.** *A non-zero vector $\mathbf{x} \in \triangle$ has G-$(\sigma,\tau_0,\tau_1)$-sequence (or Gauss sequence) $(k_0,k_1,k_2,\ldots)$, consisting of non-negative integers, if*

$$\mathbf{x} \in \triangle_{k_0(\sigma,\tau_0,\tau_1)}, T^G(\sigma,\tau_0,\tau_1)(\mathbf{x}) \in \triangle_{i_1}(\sigma,\tau_0,\tau_1), T^G(\sigma,\tau_0,\tau_1)(T^G(\sigma,\tau_0,\tau_1)(\mathbf{x})) \in \triangle_{i_2}(\sigma,\tau_0,\tau_1), \ldots.$$

*If it ever happens that $T^G(\sigma,\tau_0,\tau_1)^{(n)}(\mathbf{x})$ is in both $\triangle_0(\sigma,\tau_0,\tau_1)$ and $\triangle_1(\sigma,\tau_0,\tau_1)$, the sequence stops.*

## 3.3 Triangle Maps: the $(e,e,e)$ case:

For more on the triangle map, see [6, 3, 10, 13]

We now look at the $(e,e,e)$ case, which goes by the name of the triangle map algorithm.

We have

$$
\begin{aligned}
F(e,e,e) &= e\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}e \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\
&= F_0 \\
F_1(e,e,e) &= e\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}e \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= F_1.
\end{aligned}
$$

We have

$$
\begin{aligned}
\triangle_0(e,e,e) &= VF_0 \\
&= (v_1,v_2,v_3)\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\
&= (v_2,v_3,v_1+v_3) \\
\triangle_1(e,e,e) &= VF_1 \\
&= (v_1,v_2,v_3)\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= (v_1,v_2,v_1+v_3).
\end{aligned}
$$

We want a description of those vectors $v \in \triangle_0(e,e,e)$ and those vectors $v \in \triangle_1(e,e,e)$.

**Proposition 1.** *Suppose*

$$v = xv_1 + yv_2 + zv_3$$

*with* $x, y, z > 0$. *Then we have* $v \in \triangle_0(e,e,e)$ *if and only if* $x < z$, *which means that* $v \in \triangle_1(e,e,e)$ *if and only if* $z < x$.

It is here where we are using our assumption that $\det(v_1,v_2,v_3) > 0$.

*Proof.* We will be using the following linear algebra lemma.

**Lemma 1.** *Let* $w_1, w_2, w_3$ *be three column vectors in* $\mathbb{R}^3$ *such that* $\det(w_1,w_2,w_3) > 0$. *Then a vector* $w = xw_1 + yw_2 + zw_3$ *is in the cone spanned by* $w_1, w_2, w_3$, *meaning that* $x, y, z > 0$ *if and only if*

$$
\begin{aligned}
\det(w,w_1,w_2) &> 0 \\
\det(w,w_2,w_3) &> 0 \\
\det(w,w_3,w_1) &> 0.
\end{aligned}
$$

Thus $v \in \triangle_0(e,e,e)$ if

$$
\begin{aligned}
\det(v,v_2,v_3) &> 0 \\
\det(v,v_3,v_1+v_3) &> 0 \\
\det(v,v_1+v_3,v_2) &> 0
\end{aligned}
$$

and $v \in \triangle_1(e,e,e)$ if

$$\begin{aligned} \det(v, v_1, v_2) &> 0 \\ \det(v, v_2, v_1+v_3) &> 0 \\ \det(v, v_1+v_3, v_2) &> 0. \end{aligned}$$

As we are assuming that $v \in \triangle$, we can show that most of the above equations are always true. The inequality that distinguishes the two cones is

$$\det(v, v_1+v_3, v_2).$$

We have

$$\begin{aligned} \det(v, v_1+v_3, v_2) &= \det(v, v_1, v_2) + \det(v, v_3, v_2) \\ &= \det(xv_1+yv_2+zv_3, v_1, v_2) + \det(xv_1+yv_2+zv_3, v_3, v_2) \\ &= z\det(v_3, v_1, v_2) + x\det(v_1, v_3, v_2) \\ &= z - x. \end{aligned}$$

Thus $v \in \triangle_0$ if $z - x > 0$ and $v \in \triangle_1$ if $z - x < 0$. □

We then know that our Farey map is

$$T(e,e,e)(v) = \begin{cases} V(VF_0(e,e,e))^{-1}v & \text{if} \quad z-x>0 \\ V(VF_1(e,e,e))^{-1}v & \text{if} \quad x-z>0 \end{cases}$$

Note that the coefficients $x$ and $z$ are not how we write the column vector $v$ in $\mathbb{R}^3$ but are what is needed to have $v = xv_1 + yv_2 + zv_3$.

We now want to write out what happens for the two standard cases for $V$.

**The** $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ **Case**

Let

$$v = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}.$$

We want to find the constants $\alpha, \beta, \gamma$ so that

$$v = \alpha v_1 + \beta v_2 + \gamma v_3.$$

(Of course, what we are now calling $\alpha, \beta, \gamma$ are what earlier we denoted by $x, y, z$.) We can see that

$$v = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = (1-x)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (x-y)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and hence

$$\begin{aligned} \alpha &= 1-x \\ \beta &= x-y \\ \gamma &= y. \end{aligned}$$

This means that

$$\begin{aligned} V^{-1} &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

We know that the vector $v$ is in $\triangle_0(e,e,e)$ if $\gamma - \alpha > 0$, which means that

$$0 < \gamma - \alpha = y - (1-x) = y + x - 1$$

which means that

$$\triangle_0(e,e,e) = \left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} : y + x > 1 \right\}$$

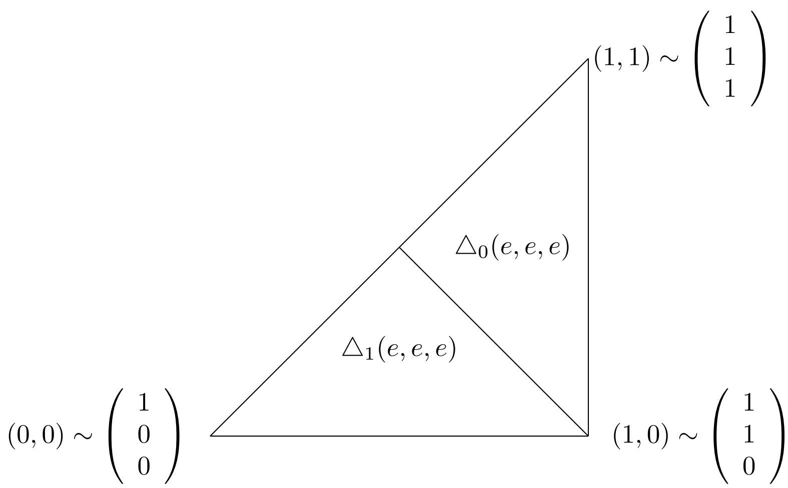
and that $v$ is in $\triangle_1(e,e,e)$ if $\gamma - \alpha < 0$, meaning

$$0 > \gamma - \alpha = y - (1-x) = y + x - 1$$

which means that

$$\triangle_1(e,e,e) = \left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} : y + x < 1 \right\}$$

This gives us




We have

$$
\begin{aligned}
T(e,e,e)(v) &= \begin{cases} T_0(e,e,e)(v) & \text{if} \quad \gamma - \alpha > 0 \\ T_1(e,e,e)(v) & \text{if} \quad \alpha - \gamma > 0 \end{cases} \\
&= \begin{cases} V(VF_0(e,e,e))^{-1}v & \text{if} \quad x+y>1 \\ V(VF_1(e,e,e))^{-1}v & \text{if} \quad x+y<1 \end{cases} \\
&= \begin{cases} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} & \text{if} \quad x+y>1 \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} & \text{if} \quad x+y<1 \end{cases} \\
&= \begin{cases} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} & \text{if} \quad x+y>1 \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} & \text{if} \quad x+y<1 \end{cases} \\
&= \begin{cases} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} & \text{if} \quad x+y>1 \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} & \text{if} \quad x+y<1 \end{cases} \\
&= \begin{cases} \begin{pmatrix} x \\ y \\ 1-x \end{pmatrix} & \text{if} \quad x+y>1 \\ \begin{pmatrix} 1-y \\ x \\ y \end{pmatrix} & \text{if} \quad x+y<1 \end{cases}
\end{aligned}
$$

In the standard normalization for this basis, we divide by the first term to get

$$
T(e,e,e)(x,y) = \begin{cases} \left(\frac{y}{x}, \frac{1-x}{x}\right) & \text{if} \quad x+y>1 \\ \left(\frac{x}{1-y}, \frac{y}{1-y}\right) & \text{if} \quad x+y<1 \end{cases}
$$

**The** $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ **Case**

Let

$$
v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
$$

In this case it is easy to find constants $\alpha, \beta, \gamma$ so that

$$
v = \alpha v_1 + \beta v_2 + \gamma v_3,
$$

namely $\alpha = x, \beta = y, \gamma = z$.

We know that the vector $v$ is in $\triangle_0(e,e,e)$ if $\gamma - \alpha > 0$, which means that

$$
0 < z - x
$$

which means that

$$\triangle_0(e,e,e) = \left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} : z > x \right\}$$

and that $v$ is in $\triangle_1(e,e,e)$ if $\gamma - \alpha < 0$, meaning

$$0 > z - x$$

which means that

$$\triangle_1(e,e,e) = \left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} : z < x \right\}$$

We have

$$\begin{aligned}
T(e,e,e)(v) &= \begin{cases} T_0(e,e,e)(v) & \text{if} \quad \gamma - \alpha > 0 \\ T_1(e,e,e)(v) & \text{if} \quad \alpha - \gamma > 0 \end{cases} \\
&= \begin{cases} V(VF_0(e,e,e))^{-1}v & \text{if} \quad z > x \\ V(VF_1(e,e,e))^{-1}v & \text{if} \quad z < x \end{cases} \\
&= \begin{cases} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{if} \qquad z > x \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{if} \quad x + y < 1 \end{cases} \\
&= \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{if} \quad z > x \\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{if} \quad z < x \end{cases} \\
&= \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{if} \quad z > x \\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{if} \quad z < x \end{cases} \\
&= \begin{cases} \begin{pmatrix} y \\ z - x \\ x \end{pmatrix} & \text{if} \quad z > x \\ \begin{pmatrix} x - z \\ y \\ z \end{pmatrix} & \text{if} \quad z < x \end{cases}
\end{aligned}$$

In the standard normalization for this basis, we divide the first two terms by the $l_1$ norm of the entries, to get

$$T(e,e,e)(x,y) = \begin{cases} \left(\frac{y}{y+z}, \frac{z-x}{y+z}\right) & \text{if} \quad z > x \\ \left(\frac{x-z}{x+y}, \frac{y}{x+y}\right) & \text{if} \quad z < x \end{cases}$$

## 3.4 The multiplicative versions for $(e,e,e)$

For both choices of basis, we will need that

$$\begin{aligned} F_1(e,e,e)^n &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n \\ &= \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Recall that

$$\begin{aligned} \triangle_k^G(e,e,e) &= VF_1^k(e,e,e)F_0 \\ &= (v_1,v_2,v_3)\begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= (v_1,v_2,kv_1+v_3)\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= (v_2, kv_1+v_3, (k+1)v_1+v_3.) \end{aligned}$$

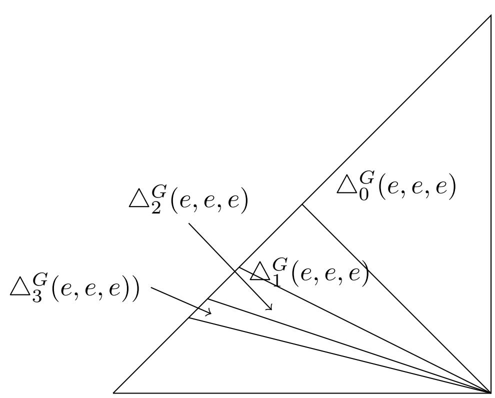
Then for $v \in \triangle_k^G(e,e,e)$, we have

$$T_k^G(v) = V(VF_1^k(e,e,e)F_0(e,e,e))^{-1}v$$

When $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, we have

$$\triangle_k^G(e,e,e) = \begin{pmatrix} 1 & k+1 & k+2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

which means we have



This gives us

$$\begin{aligned}
T_k^G(v) &= V(VF_1^k(e,e,e)F_0(e,e,e))^{-1}v \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k+1 & k+2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 1+k \\ 1 & -1 & -k \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -k \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \\
&= \begin{pmatrix} x \\ y \\ 1-x-ky \end{pmatrix}
\end{aligned}$$

With the other basis, namely when $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we have

$$\triangle_k^G(e,e,e) = \begin{pmatrix} 0 & k & k+1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Then we have, since here $V = I$ is the identity matrix,

$$\begin{aligned}
T_k^G(v) &= V(VF_1^k(e,e,e)F_0(e,e,e))^{-1}v \\
&= \begin{pmatrix} 0 & k & k+1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1+k \\ 1 & 0 & -k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} y \\ (k+1)z-x \\ x-kz \end{pmatrix}
\end{aligned}$$

## 3.5 The Cassaigne Algorithm: $(e,(23),(23))$

The Cassaigne algorithm [9] is

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &\rightarrow \begin{cases} \begin{pmatrix} x-z \\ z \\ y \end{pmatrix} & \text{if} \quad x > z \\ \begin{pmatrix} y \\ x \\ z-x \end{pmatrix} & \text{if} \quad x < z \end{cases} \\
&= \begin{cases} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{if} \quad x > z \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{if} \quad x < z \end{cases}
\end{aligned}$$

We will show that this is the TRIP map $(e,(23),(23))$ with the basis $V = I$ We will show that

$$
\begin{aligned}
F_0(e,(23),(23)) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \\
F_1(e,(23),(23)) &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1}
\end{aligned}
$$

We have

$$
\begin{aligned}
F_0(e,(23),(23)) &= e\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1} \\
F_1(e,(23),(23)) &= e\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1}
\end{aligned}
$$

### 3.6 Other multi-dimensional continued fraction algorithms of the form $(\sigma, \tau_0, \tau_1)$

In [4] some other algorithms are discussed that can be put in the form of a $(\sigma, \tau_0, \tau_1)$ algorithm.

### 3.7 Combination TRIP Maps

There are $|S_3 \times S_3 \times S_3| = 216$ different $(\sigma, \tau_0, \tau_1)$ algorithms. But we can use these to generate an infinite family, which we call combination TRIP maps. Here would be an example. Apply the $((13),(123),(23))$ algorithm five times, then the $(e,(12),(132))$ algorithm twice, then repeat. This method seems to generate many of the well known multidimensional continued fraction algorithms. In [4] it is shown that these include the Brun algorithm, the Fully Subtractive algorithm and the algorihtm (which is linked to Jacobi-Perron). I believe that Selmer will also fall into this language.

## 4 Common Frameworks

People study multi-dimensional continued fractions for many reasons. And there are many muti-dimensional continued fractions out there, as can be seen in [15, 8]. Each of these have their own advantages and disadvantages. What is needed is to understand the overall structure of different types of combination TRIP maps.

So far, there are only a few partial results. For example, one of the primary reasons for studying multi-dimensional continued fractions is to find ways to understand cubic irrationals, namely to answer the Hermite problem. For example,we like to able to prove that a pair of real numbers $(\alpha, \beta)$ has a periodic $(\sigma, \tau_0, \tau_1)$

expansion if and only if $\alpha$ and $\beta$ are at worse cubic irrationals in the same cubic number field. This is still quite open, and in fact, I bellieve, to likely not be true. But we would least like to be able to show that if $(\alpha, \beta)$ has a periodic $(\sigma, \tau_0, \tau_1)$ expansion if and only if $\alpha$ and $\beta$ are at worse cubic irrationals in the same cubic number field. It is known when this is true and when it is not, for TRIP maps, as discussed [4]. To some extent it is even known why, though this understanding could be deepened.

In the multiplicative side, in finding the vertices of the various $\triangle_k^G(\sigma, \tau_0, \tau_1)$, the maps $T_k^G(\sigma, \tau_0, \tau_1)$ and the associated transfer operators, there are times when there is linear growth in $k$ and times when there is exponential growth in $k$. This can be explicitly seen in the eight $\mathbb{R}^2$ algorithms. In all of these the key part is

$$F_1^k(\sigma, \tau_0, \tau_1).$$

Thus the eigenvalues of these matrices are important. In work with Ilya Amburg [2], we show that there are six types of behavior of the eigenvalues. Among the 216 TRIP maps, we look at Jordan canonical form of all of the $F_1(\sigma, \tau_0, \tau_1)$ and show that are only six possibilities:

$$J_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(1-\sqrt{5}) & 0 \\ 0 & 0 & \frac{1}{2}(1+\sqrt{5}) \end{pmatrix},$$

$$J_5 = D(roots(-1 - t^2 + t^3)),$$

and

$$J_6 = D(roots(-1 - t + t^3)).$$

where $D(roots(-1 - t^2 + t^3))$ corresponds to a three-by-three square matrix with diagonal entries defined by the roots of $-1 - t^2 + t^3 = 0$; similarly for $D(roots(-1 - t + t^3))$.

It can be shown that

$$(J_1)^k = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(J_2)^k = \begin{pmatrix} (-1)^k & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$(J_3)^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$(J_4)^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\frac{1}{2}(1-\sqrt{5}))^k & 0 \\ 0 & 0 & (\frac{1}{2}(1+\sqrt{5}))^k \end{pmatrix},$$

$$(J_5)^k = D(roots(-1 - t^2 + t^3)^k),$$

and

$$(J_6)^k = D(roots(-1 - t + t^3)^k).$$

This tells us why we at times have linear growth in $k$ and at times exponential growth.

Of course, most of the natural questions are still open. In fact, we probably do not even know what are the right questions that we should be asking. Much structure is awaiting discovery.

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