Thomas Garrity 8 juin 2017

Outline
1) CF and transfert operator
2) Triangle map
 - Def
 - Hilbert Space Approach
 - Banach Space Approach
3) Triangle Partition Maps (TRIP)
 Joint work with Ilya Amburg

$\boxed{1}$ CF

$G : [0,1) \to [0,1)$ $\qquad x = [a_0; a_1, a_2, \ldots]$

$x \mapsto \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$

$$P_{n,k}(x) = \frac{\#\{a_m = k : m \leq n\}}{n}, \quad P_k(x) = \lim_{n \to \infty} P_{n,k}(x) = \frac{1}{\ln(2)} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{1+x}\,dx$$

a.e. x

In non obvious ways, the proof comes down to

$$\mathcal{L}_G(f)(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} f\left(\frac{1}{x+n}\right)$$

and $\frac{1}{x+1}$ is an eigenfunction with eigenvalue 1.

$T : X \to X$, Function Spaces on X

$(\text{Functions on } X) \times (\text{Mesurable sets on } X)$

$$(F, A) \to \int_A F\,d\mu$$

and $\langle F, T^{-1}(A) \rangle = \langle L_T F, A \rangle$

Left adjoint $\qquad$ transfert operator

Gauss map

Mayer-Roepstorff $\begin{cases} \text{Hilbert Space} \\ \text{Banach} \end{cases}$

$[0,1]$, $L^2[0,1]$

$$\mathcal{L}_G f = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} f\left(\frac{1}{n+x}\right)$$ is nuclear of trace class zero.

Morally $$\mathcal{L}_G f \sim \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix}$$ and $\sum |\lambda_i|^2 < \infty$

In practice,
$$\mathcal{L}_G f = \sum_{n=1}^{\infty} \langle f, \mu_k \rangle e_k$$
with $\sum |\mu_k| |e_k| < \infty$

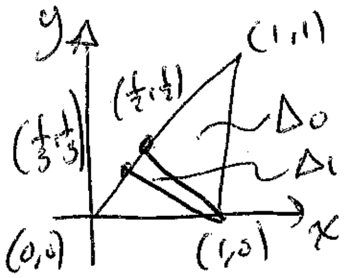
$\boxed{2}$ Triangle map

MCF are trying to capture factoring 2 numbers into a third

Triangle (a,b,c), $a > b > c > 0$

$$\longrightarrow (b, c, a-b-kc)$$

with $a-b-kc > 0$ and $a-b-(k+1)c < 0$



$(a,b,c) \sim \left(\frac{b}{a}, \frac{c}{a}\right) = (x,y)$

$$\Delta_k = \left\{ (x,y) \in \Delta : \begin{array}{l} 1-x-ky \geq 0 \\ 1-x-(k+1)y < 0 \end{array} \right\}$$

$$T : \Delta_k \longrightarrow \Delta$$
$$(x,y) \longmapsto \left(\frac{y}{x}, \frac{1-x-ky}{x}\right)$$

$$T_k^{-1} : \Delta \longrightarrow \Delta$$
$$(x,y) \longmapsto \left(\frac{1}{1+kx+y}, \frac{x}{1+kx+y}\right)$$

$$\mathcal{L}_T f(x,y) = \sum \frac{1}{(1+nx+y)^3} f\left(\frac{1}{1+nx+y}, \frac{x}{1+nx+y}\right)$$

inverse map

looks like $\sum_{n=1}^{\infty} \frac{1}{(n+x)^2} f\left(\frac{1}{n+x}\right)$ → gives problem, because x can get small

$\frac{1}{x(1+y)}$ is eigenvector with eigenvalue one.

But is the dimension 1 ?

Find Banach Space

$$V = \{ f : \Delta \to \mathbb{R} \mid f \text{ cont } |x f(x)| \text{ bounded} \}$$

1) $\mathcal{L}_f : V \to V$
2) 1 is largest eigenvalue
3) Eigenspace of 1 is 1-dim using ergodic

Good Gauss-Kuzman statistics

Hilbert Space Approach

$$\frac{1}{x(1+y)} \notin L^2(dx,dy)$$

So Define measure

$$dm(t) = \frac{t}{e^t - 1} dt \text{ on } \mathbb{R}_{\geq 0}$$

$$\eta_k(s) = \frac{s^k e^{-s}}{(k+1)!}$$

$$e_k(t) = L_k^1(t) = 1^{st} \text{ Laguerre polyonomial}$$

$$E_k(x,y) = \int_0^\infty \frac{1}{x^2} e^{-t\left(\frac{1-x+y}{x}\right)} e_k(t)\, dm(t)$$

$$K_T(e(x,t)) = \int_0^1 \frac{J_1(2\sqrt{st})}{\sqrt{st}}\,\frac{t}{e^t-1}\,e(x,s)\,dm(s)$$

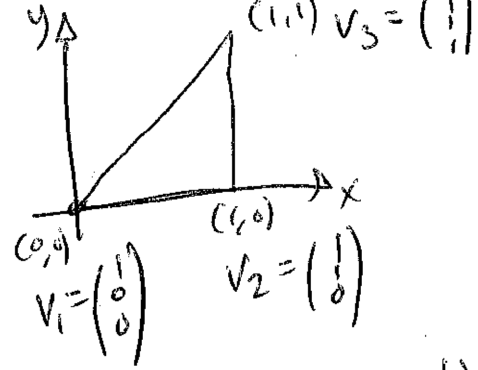
$$\hat{e}(x,y) = \frac{1}{x}\int_{s=1}^{\infty} e^{-sy}\,e(x,s)\,dm(s)$$

$$\langle \alpha(s),\beta(s)\rangle = \int_0^\infty \alpha(s)\beta(s)\,dm(s)$$

$$\mathcal{L}_T f(x,y) = \mathcal{L}_T\,\hat{\varphi}(x,y) = \frac{1}{x^2}\int_0^\infty e^{-t\left(\frac{1-x+y}{x}\right)} K_T(\varphi(x,t))\,dm(t)$$

$$= \sum_{m=0}^{\infty} \langle \varphi(x,s), m_k(s)\rangle E_k(x,y)$$

<u>Triangle Partition Maps (TRIP)</u>



$$F_0 = \begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}, \quad F_1 = \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$$

$$V = (v_1\, v_2\, v_3) \xrightarrow{F_0} (v_2, v_3, v_1+v_3)$$
$$V = (v_1\, v_2\, v_3) \xrightarrow{F_1} (v_1, v_2, v_1+v_3)$$

$$\Delta_k = \begin{pmatrix}k+1 & k+2 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{pmatrix} = V F_1^{k-1} F_0$$

$$T(x,y) = \left(\frac{y}{x}, \frac{1-x-ky}{x}\right) \quad \text{gives} \quad \begin{pmatrix}0&1&0\\0&0&1\\1&-1&-k\end{pmatrix}\begin{pmatrix}1\\x\\y\end{pmatrix} = \begin{pmatrix}x\\y\\1-x-ky\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\0&0&1\\1&-1&-k\end{pmatrix} = V F_0^{-1} F_1^{-(k-1)} V^{-1}, \qquad V F_0^{-1} F_1^{-(k-1)} V^{-1}\begin{pmatrix}1\\x\\y\end{pmatrix}$$

S_3 = perm. matrices

$$F_0(\sigma,\tau_0,\tau_1) = \sigma F_0 \tau_0^{-1}$$
$$F_1(\sigma,\tau_0,\tau_1) = \sigma F_1 \tau_1^{-1}$$

<u>Julien Cassaigne</u>: $(e, (23), (23))$

+ Combinations ...

Transfer operator

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$$\sum_{n=0}^{\infty} \frac{1}{(m(x,y))^3} f(\ldots) \qquad\qquad \sum \frac{1}{(\text{exp. in } K)}$$

$m(x,y)$ linear in K

$V F_1^{\,K} F_0 C^T$

6 diff Jordan canonical forms for F_1

3 of these eigenvalue 1
other eigenvalues, -Fibonacci, 2 diff. cubics.