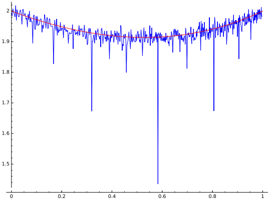
Random Number Generation and Fitting Interval Partitions

Pablo Rotondo[1]



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Random Variable Simulation

Objective: given a perfect source of *independent fair bits*

$$\mathcal{X} = X_1, X_2, \ldots$$

simulate a random variable Y with a *prescribed distribution*.

– Generating random bits $\longrightarrow$ costly.
– What if X_i were not random bits? $\longrightarrow$ other distributions.

Classical algorithm in probability courses
– given uniform $U \in [0, 1]$ and a continuous distribution function F consider the so called inverse method

$$Y := F^{-1}(U)\,.$$

– in our context we may consider

$$U := (0.X_1 X_2 \ldots)_2 \ .$$

Interval algorithm: intro

Discrete random variable $Y \in \mathbb{Z}_{>0}$ with distribution vector $p_1, p_2, \ldots$

The inverse method gives intervals

$$I_i(\mathbf{p}) := \left[\sum_{j<i} p_j, \sum_{j\le i} p_j \right),$$

and defines

$$Y = i \iff U \in I_i(\mathbf{p}).$$

Question

How many fair bits $X_1, X_2, \ldots$ do we need to determine Y?

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How many fair bits $X_1, X_2, \ldots$ do we need to determine Y?

More precisely, given $u \in [0,1]$ in binary $u = (0.x_1 x_2 \ldots)_2$ we define

$$k_{\mathbf{p}}(u) := \inf \left\{ k \geq 0 : \exists i \text{ s.t. } \left(0.x_1 \ldots x_k, 0.x_1 \ldots x_k + 2^{-k}\right) \subset I_i(\mathbf{p}) \right\}$$

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What is

$$\mathbb{E}[k_{\mathbf{p}}(U)] \quad ?$$

Lower bound: the entropy

Theorem

The expected number of bits is bounded from below by the entropy of $\mathbf{p}$

$$\mathbb{E}[k_{\mathbf{p}}(U)] \geq H(Y)\,,$$

where $H(Y) := \sum_i p_i \log_2(1/p_i)$.

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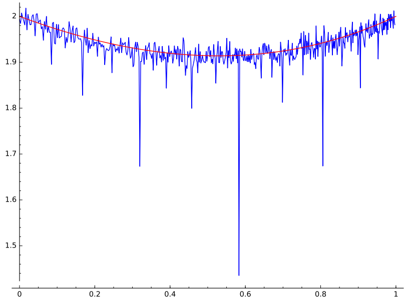
Proof.

The code

$$\begin{aligned}\mathcal{C} := \Big\{ & x_1 \dots x_k \in \{0,1\}^+ : \\ & \exists i \text{ s.t. } \big(0.x_1 \dots x_k, 0.x_1 \dots x_k + 2^{-k}\big) \subset I_i(\mathbf{p})\,, \\ & \nexists j \big(0.x_1 \dots x_{k-1}, 0.x_1 \dots x_{k-1} + 2^{-(k-1)}\big) \subset I_j \Big\}\end{aligned}$$

is prefix free and determines Y. □

The distribution $\mathbf{p}_N := (1/N, 1/N \ldots, 1/N)$



Theorem

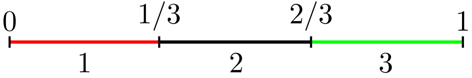
The redundancy $\mathbb{E}[k_{\mathbf{p}_N}(U)] - H(Y)$ equals

$$R(x) = 2^x - x + 1 - \frac{2^{\nu(N)} - 1}{N-1}\, 2^x - \log_2\left(1 + \tfrac{1}{N-1}\right),$$

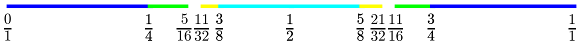
where $x = \{\log_2(N-1)\}$, $\{\cdot\}$ denotes the fractional part and $\nu(N)$ is the greatest t such that 2^t divides N.

Example: fair 3-sided dice

First $\mathbf{p}_3 = (1/3, 1/3, 1/3)$ divides the interval $[0, 1]$ as follows



while the subdivision procedure, a binary search for U, gives



where we remark that the number of bits can be deduced from the denominators.

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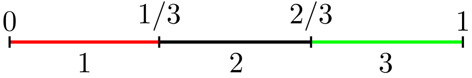

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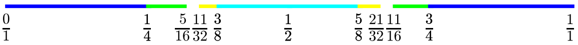

where we remark that the number of bits can be deduced from the denominators.
In this case we have

$$\mathbb{E}[k_{\mathbf{p}_3}(U)] = 3 \doteq \log_2(3) + 1.41503\ldots .$$

Generic Distribution

For an arbitrary probability vector $\mathbf{p} = (p_1, p_2, \ldots)$

Theorem

The redundancy $\mathbb{E}[k_{\mathbf{p}}(U)] - H(Y)$ is at most 2, i.e.

$$H(Y) \le \mathbb{E}[k_{\mathbf{p}}(U)] \le H(Y) + 2\,.$$

Furthermore, the $+2$ is tight by our example $\mathbf{p}_N$.

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Knuth-Yao proved that the optimal algorithm satisfies these bounds.

- Algorithm seen as binary tree.
- Optimal algorithm obtained by decomposing each

$$p_i = (0.p_1^{(i)} p_2^{(i)} \ldots)_2$$

in binary and assigning a leaf of probability $p_j^{(i)}$ for each (i, j).

Example: continued fractions

Procedures gives way to, given the binary representation of U, decide to which interval among the partition

$$[0, p_1), [p_1, p_1 + p_2), [p_1 + p_2, p_1 + p_2 + p_3), \ldots$$

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For example, if we consider the convergents $(q_k)_k$ of U, the set

$$I_k(a, b) := \{U \in [0, 1] : (q_{k-1}(U), q_k(U)) = (a, b)\}$$

is an interval of length $\frac{1}{b(a+b)}$ when $\gcd(a, b) = 1$ and $a \leq b$.

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A possible partition (negligible intersection) is given by fixing k

$$\mathcal{I}_k := \{I_k(a, b) : \gcd(a, b) = 1,\ 1 \leq a \leq b\},$$

which determines the value of $(q_{k-1}(U), q_k(U))$.

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A different partition, relating to our ANALCO paper is:

- Fix $n \in \mathbb{Z}_{>0}$, and consider

$$\mathbb{I}_n := \left\{(a,b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : \gcd(a,b) = 1,\, a \leq n < b\right\}.$$

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- Consider $k := k(U,n)$ such that $q_{k-1}(U) \leq n < q_k(U)$ and define

$$I_{a,b} := \left\{U \in [0,1] : \left(q_{k(U,n)-1}, q_{k(U,n)}\right) = (a,b)\right\},$$

where $(a,b) \in \mathbb{I}_n$, again an interval, forming a partition.

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where $(a,b) \in \mathbb{I}_n$, again an interval, forming a partition.

- Then the number of bits needed to determine

$$\left(q_{k(U,n)-1}, q_{k(U,n)}\right),$$

is roughly

$$\begin{aligned} H\left(q_{k(U,n)-1}, q_{k(U,n)}\right) &= 2\log_2 n + 1 \\ &\quad - \frac{12}{\pi^2} \iint_{0<x<1\le y} \frac{\log_2(y(x+y))}{y(x+y)} dx dy + o(1) \\ &= 2\log_2 n - 2.4263\ldots + o(1) \end{aligned}$$

Interval Algorithm

- Generalization of the previous procedure for fair bits

$$\mathcal{X} = (X_1, X_2, \ldots) .$$

- Each X_i takes values on $[M]$ with vector $(q_1, q_2, \ldots, q_M)$.

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- Each X_i takes values on $[M]$ with vector $(q_1, q_2, \ldots, q_M)$.

Now the procedure goes as follows:

⊛ Let $K_k(\mathbf{q}) = [A_k, B_k)$ be our **working interval** after processing $X_1, \ldots, X_k$.

⊛ Partition $K_k(\mathbf{q})$ into intervals

$$K_{k,j}(\mathbf{q}) := [A_k + Q_{j-1}(B_k - A_k), A_k + Q_j(B_k - A_k)),$$

according to $Q_j(\mathbf{q}) := \sum_{i \le j} q_i$.

⊛ Suppose $X_{k+1}(\mathbf{q}) = j$, then set $K_{k+1}(\mathbf{q}) := K_{k,j}(\mathbf{q})$.

Interval $K_k(\mathbf{q})$ corresponds to what before was

$$[0.x_1 \dots x_k, 0.x_1 \dots x_k + 2^{-k})\,.$$

We continue until

$$k_{\mathbf{q},\mathbf{p}} := \inf \left\{ k \geq 0 : \exists i \text{ s.t. } K_k(\mathbf{q}) \subset I_i(\mathbf{p}) \right\},$$

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Theorem (Lower bound)

The cost of simulating the random variable Y having prob. vector $\mathbf{p}$ by using the Interval Algorithm with an M-valued "coin flips" according to the prob. vector $\mathbf{q}$ is bounded from below by

$$\frac{H(\mathbf{p})}{H(\mathbf{q})} \leq \mathbb{E}[k_{\mathbf{q},\mathbf{p}}] \, .$$

Interval Algorithm: efficiency

Theorem (Han,Hoshi 95)

For any probability vectors $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, \ldots, q_m)$*, the expected number of coin tosses in the interval algorithm is upper-bounded*

$$\mathbb{E}[k_{\mathbf{q},\mathbf{p}}] \leq \frac{H(\mathbf{p})}{H(\mathbf{q})} + \frac{\log 2(M-1)}{H(\mathbf{q})} + \frac{h(q_{\max})}{(1-q_{\max})H(\mathbf{q})} \, .$$

Proof.

Whiteboard (or blackboard). □

Generalization to random processes

We want to simulate a random process

$$\mathcal{Y} = (Y_1, Y_2, Y_3, \ldots),$$

rather than a single Y with a prescribed distribution.

The question now is

what is the asymptotic cost of producing $\mathcal{Y}_n = (Y_1, Y_2, \ldots, Y_n)$?

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Remark

If the "target" source $Y_1, Y_2, \ldots$ is stationary, $\mathbf{p}^n$ denotes the vector of $(Y_{j+1}, \ldots, Y_{j+n})$ for $j \geq 0$ and $k_n := k_{\mathbf{q},\mathbf{p}^n}$:

$$\lim_{n\to\infty} \frac{\mathbb{E}[k_n]}{n} = \frac{H(\mathcal{Y})}{H(\mathcal{X})}$$

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Proof.

By independence $H(\mathcal{X}) = H(\mathbf{q})$, while $H(\mathcal{Y}_n)/n \to H(\mathcal{Y})$. $\square$

Generalization

- We may imagine now that the variables $X_1, X_2, \ldots$ are not necessarily independent or identically distributed
  $\rightarrow$ natural if produced by a dynamical system (e.g. Euclid).
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Definition (Interval partition)

An interval partition is a finite or countable partition of $[0, 1]$ into intervals. If $\mathcal{P}$ is an interval partition and $x \in [0, 1]$, we let $\mathcal{P}(x)$ denote the interval $I \in \mathcal{P}$ such that $x \in I$.

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compare it with $k_{\mathbf{q},\mathbf{p}^n}$:

- $\mathcal{P}_n$ corresponds to the partition according to $(Y_1, \dots, Y_n)$.
- $\mathcal{Q}_k$ corresponds to the partition according to $(X_1, \dots, X_k)$.

Entropy of an interval partition

Definition (Entropy)

Let $\mathcal{P} := (\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of interval partitions. We say that $\mathcal{P}$ has *entropy* $c \geq 0$ with respect to a measure λ if

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Let λ be the Lebesgue measure:

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$$I_n(a,b) = \{x \in [0,1] : (q_{n-1}(x), q_n(x)) = (a,b)\}\,,$$

we get (blackboard explanation)

$$\lim_{n\to\infty} -\frac{1}{n} \log \lambda(\mathcal{P}_n(x)) = \frac{\pi^2}{6\log 2}\,, \quad \lambda - a.e.$$

Asymptotic Cost

Theorem (Dajani, Fieldsteel, 2001)

Let $\mathcal{P} := \{\mathcal{P}_n\}_{n=1}^{\infty}$ *and* $\mathcal{Q} := \{\mathcal{Q}_n\}_{n=1}^{\infty}$ *be sequences of interval partitions, and let* λ *be a Borel probability measure on* $[0,1)$*.*

Assume $\mathcal{P}$ *and* $\mathcal{Q}$ *have entropies* $H(\mathcal{P})$ *and* $H(\mathcal{Q})$ *respectively with respect to* λ*, then*

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for λ*-a.e.* x*.*

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for λ*-a.e.* x.

Example: the number of digits required to determine $(q_{n-1}(x), q_n(x))$ from the base 10 expansion of x behaves like $\frac{\pi^2}{6\log 2 \log 10} n$ a.e. x, a result previously proved by Lochs.

Good partition sequences

Theorem (Shannon,McMillan,Breiman)

Let T be an ergodic measure preserving transformation on a probability space $(\Omega, \mathcal{B}, \mu)$ and let P be a finite or countable generating partition for T for which $H_\mu(P) < \infty$. Then for μ-a.e. x,

$$\lim_{n\to\infty} -\frac{\log \mu\left(P_n(x)\right)}{n} = h_\mu(T)\,.$$

Here $H_\mu(P)$ denotes the entropy of the partition P, $h_\mu(T)$ the entropy of T and $P_n(x)$ denotes the element of the partition $\bigvee_{i=0}^{n-1} T^{-i}P$ containing x.

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We recall that

$$h_\mu(T) = \sup\{h_\mu(T, \mathcal{A}) : \mathcal{A} \text{ countable partition of } X\}\,,$$

and

$$h_\mu(T, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H\left(\mathcal{A}(U), \mathcal{A}(TU), \ldots, \mathcal{A}\left(T^{n-1}U\right)\right)\,,$$

where U is distributed according to μ.

Concluding remarks

There are other interesting results/ideas not mentioned in this talk,

- The optimal algorithm for discrete uniform generation from coin flips has a fairly simple implementation, see the note by Jérémie Lumbroso arXiv:1304.1916v1.

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- What about non-discrete random variables? See the original paper by von Neumann, Knuth-Yao and Philippe Duchon.

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