Random Number Generation
and Fitting Interval Partitions

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AleaEnAmSud,
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Random Variable Simulation

Objective: given a perfect source of *independent fair bits*

\[ \mathcal{X} = X_1, X_2, \ldots \]

simulate a random variable \( Y \) with a *prescribed distribution*.

– Generating random bits \( \longrightarrow \) costly.
– What if \( X_i \) were not random bits? \( \longrightarrow \) other distributions.

Classical algorithm in probability courses
– given uniform \( U \in [0, 1] \) and a continuous distribution function \( F \) consider the so called *inverse method*

\[ Y := F^{-1}(U) . \]

– in our context we may consider

\[ U := (0.X_1X_2\ldots)_2 . \]
Interval algorithm: intro

Discrete random variable \( Y \in \mathbb{Z}_{>0} \) with distribution vector \( p_1, p_2, \ldots \).

The inverse method gives intervals

\[
I_i(p) := \left[ \sum_{j<i} p_j, \sum_{j \leq i} p_j \right],
\]

and defines

\[
Y = i \iff U \in I_i(p).
\]

Question

How many fair bits \( X_1, X_2, \ldots \) do we need to determine \( Y \)?
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Question

How many fair bits $X_1, X_2, \ldots$ do we need to determine $Y$?

More precisely, given $u \in [0, 1]$ in binary $u = (0.x_1x_2\ldots)_2$ we define

$$k_p(u) := \inf \left\{ k \geq 0 : \exists i \text{ s.t. } (0.x_1\ldots x_k, 0.x_1\ldots x_k+2^{-k}) \subset I_i(p) \right\}$$
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What is

$$\mathbb{E}[k_p(U)]$$?
Theorem

The expected number of bits is bounded from below by the entropy of \( p \)

\[
\mathbb{E}[k_{p}(U)] \geq H(Y),
\]

where \( H(Y) := \sum_{i} p_{i} \log_{2}(1/p_{i}) \).
Lower bound: the entropy

**Theorem**

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where \( H(Y) := \sum_i p_i \log_2(1/p_i) \).

**Proof.**

The code

\[
C := \left\{ x_1 \ldots x_k \in \{0, 1\}^+ : \right. \\
\left. \exists i \text{ s.t. } (0.x_1 \ldots x_k, 0.x_1 \ldots x_k + 2^{-k}) \subset I_i(p) , \\
\forall j (0.x_1 \ldots x_{k-1}, 0.x_1 \ldots x_{k-1} + 2^{-(k-1)}) \subset I_j \right\}
\]

is prefix free and determines \( Y \).
The distribution \( p_N := (1/N, 1/N \ldots, 1/N) \)

Theorem

The redundancy \( \mathbb{E}[k_{p_N}(U)] - H(Y) \) equals

\[
R(x) = 2^x - x + 1 - \frac{2^{\nu(N)} - 1}{N - 1} 2^x - \log_2 \left( 1 + \frac{1}{N-1} \right),
\]

where \( x = \{\log_2(N - 1)\} \), \{\cdot\} denotes the fractional part and \( \nu(N) \) is the greatest \( t \) such that \( 2^t \) divides \( N \).
Example: fair 3-sided dice

First $p_3 = (1/3, 1/3, 1/3)$ divides the interval $[0, 1]$ as follows

![Diagram showing division of interval by $p_3$]

while the subdivision procedure, a binary search for $U$, gives

![Diagram showing binary search for $U$]

where we remark that the number of bits can be deduced from the denominators.
Example: fair 3-sided dice

First $p_3 = (1/3, 1/3, 1/3)$ divides the interval $[0, 1]$ as follows

```
   0   1/3   2/3   1
```
```
  1   2    3
```

while the subdivision procedure, a binary search for $U$, gives

```
0  1/4  5/16  11/32  3/8  21/32  11/16  3/4  1
```
```
  1 1/2  5/8  11/16  3/4  1
```

where we remark that the number of bits can be deduced from the denominators.

In this case we have

$$\mathbb{E}[k_{p_3}(U)] = 3 + \log_2(3) + 1.41503 \ldots .$$
Generic Distribution

For an arbitrary probability vector \( p = (p_1, p_2, \ldots) \)

Theorem

The redundancy \( \mathbb{E}[k_p(U)] - H(Y) \) is at most 2, i.e.

\[
H(Y) \leq \mathbb{E}[k_p(U)] \leq H(Y) + 2.
\]

Furthermore, the +2 is tight by our example \( p_N \).
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Knuth-Yao proved that the optimal algorithm satisfies these bounds.

- Algorithm seen as binary tree.
- Optimal algorithm obtained by decomposing each

\[
p_i = (0.p_1^{(i)}p_2^{(i)}\ldots)_2
\]

in binary and assigning a leaf of probability \( p_j^{(i)} \) for each \( (i, j) \).
Example: continued fractions

Procedures gives way to, given the binary representation of $U$, decide to which interval among the partition

$$[0, p_1), [p_1, p_1 + p_2), [p_1 + p_2, p_1 + p_2 + p_3), \ldots$$

it belongs to. This can be applied to other partitions.
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For example, if we consider the convergents $(q_k)_k$ of $U$, the set $I_k(a, b) := \{ U \in [0, 1] : (q_{k-1}(U), q_k(U)) = (a, b) \}$ is an interval of length $\frac{1}{B(a+b)}$ when $\gcd(a, b) = 1$ and $a \leq b$. 


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is an interval of length $\frac{1}{b(a+b)}$ when $\gcd(a, b) = 1$ and $a \leq b$.

A possible partition (negligible intersection) is given by fixing $k$

$$\mathcal{I}_k := \{ I_k(a, b) : \gcd(a, b) = 1, 1 \leq a \leq b \},$$

which determines the value of $(q_{k-1}(U), q_k(U))$. 
Example: continued fractions

A different partition, relating to our ANALCO paper is:

- Fix $n \in \mathbb{Z}_{>0}$, and consider

$$\mathbb{I}_n := \{(a, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : \gcd(a, b) = 1, \ a \leq n < b\}.$$
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  \]

- Consider \( k := k(U, n) \) such that \( q_{k-1}(U) \leq n < q_k(U) \) and define
  \[
  I_{a,b} := \left\{ U \in [0, 1] : \left( q_{k(U, n)}^{-1}, q_k(U, n) \right) = (a, b) \right\},
  \]
where \((a, b) \in \mathbb{I}_n\), again an interval, forming a partition.
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2. Consider \( k := k(U, n) \) such that \( q_{k-1}(U) \leq n < q_k(U) \) and define

\[
I_{a,b} := \left\{ U \in [0, 1] : (q_k(U,n)-1, q_k(U,n)) = (a, b) \right\},
\]

where \((a, b) \in \mathbb{I}_n\), again an interval, forming a partition.

3. Then the number of bits needed to determine

\[
(q_k(U,n)-1, q_k(U,n))
\]

is roughly

\[
H \left( q_k(U,n)-1, q_k(U,n) \right) = 2 \log_2 n + 1 - \frac{12}{\pi^2} \int_0^1 \int_{x \leq y} \frac{\log_2(y(x+y))}{y(x+y)} \, dx \, dy + o(1)
\]

\[
= 2 \log_2 n - 2.4263 \ldots + o(1)
\]
Interval Algorithm

- **Generalization** of the previous procedure for fair bits

\[ \mathcal{X} = (X_1, X_2, \ldots). \]

- Each \( X_i \) takes values on \([M]\) with vector \((q_1, q_2, \ldots, q_M)\).
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Now the procedure goes as follows:

- Let \( K_k(q) = [A_k, B_k] \) be our **working interval** after processing \( X_1, \ldots, X_k \).
- Partition \( K_k(q) \) into intervals \( K_{k,j}(q) := [A_k + Q_{j-1}(B_k - A_k), A_k + Q_j(B_k - A_k)] \),

according to \( Q_j(q) := \sum_{i \leq j} q_i \).
- Suppose \( X_{k+1}(q) = j \), then set \( K_{k+1}(q) := K_{k,j}(q) \).
Interval $K_k(q)$ corresponds to what before was

$$[0.x_1 \ldots x_k, 0.x_1 \ldots x_k + 2^{-k}).$$

We continue until

$$k_{q,p} := \inf \left\{ k \geq 0 : \exists i \text{ s.t. } K_k(q) \subset I_i(p) \right\},$$

in which case we return $Y = i$ if $K_k(q) \subset I_i(p)$.
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in which case we return $Y = i$ if $K_k(q) \subset I_i(p)$

**Theorem (Lower bound)**

The cost of simulating the random variable $Y$ having prob. vector $p$ by using the Interval Algorithm with an $M$-valued “coin flips” according to the prob. vector $q$ is bounded from below by

$$\frac{H(p)}{H(q)} \leq \mathbb{E}[k_{q,p}].$$
Theorem (Han, Hoshi 95)

For any probability vectors $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_m)$, the expected number of coin tosses in the interval algorithm is upper-bounded

$$\mathbb{E}[k_{\mathbf{q}, \mathbf{p}}] \leq \frac{H(\mathbf{p})}{H(\mathbf{q})} + \frac{\log 2(M - 1)}{H(\mathbf{q})} + \frac{h(q_{\text{max}})}{(1 - q_{\text{max}})H(\mathbf{q})}.$$ 

Proof.

Whiteboard (or blackboard).
Generalization to random processes

We want to simulate a random process

\[ \mathcal{Y} = (Y_1, Y_2, Y_3, \ldots) , \]

rather than a single \( Y \) with a prescribed distribution.

The question now is

what is the asymptotic cost of producing \( \mathcal{Y}_n = (Y_1, Y_2, \ldots, Y_n) \) ?
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Remark

If the “target” source \( Y_1, Y_2, \ldots \) is stationary, \( p^n \) denotes the vector of \( (Y_{j+1}, \ldots, Y_{j+n}) \) for \( j \geq 0 \) and \( k_n := k_{q, p^n} : \)

\[
\lim_{n \to \infty} \frac{\mathbb{E}[k_n]}{n} = \frac{H(\mathcal{Y})}{H(\mathcal{X})}
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\]

Proof.

By independence \( H(\mathcal{X}) = H(q) \), while \( H(\mathcal{Y}_n)/n \to H(\mathcal{Y}). \)
Generalization

- We may imagine now that the variables $X_1, X_2, \ldots$ are not necessarily independent or identically distributed → natural if produced by a dynamical system (e.g. Euclid).
- The question can be framed more purely in terms of sequences of interval partitions.
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- The question can be framed more purely in terms of sequences of interval partitions.

Definition (Interval partition)

An interval partition is a finite or countable partition of $[0, 1]$ into intervals. If $\mathcal{P}$ is an interval partition and $x \in [0, 1]$, we let $\mathcal{P}(x)$ denote the interval $I \in \mathcal{P}$ such that $x \in I$. 
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Now our problem is rephrased in terms of

$$k_{\mathcal{Q}, \mathcal{P}}(n, x) = \inf \left\{ k \geq 0 : Q_k(x) \subset \mathcal{P}_n(x) \right\};$$
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$$ k_{Q, \mathcal{P}}(n, x) = \inf \left\{ k \geq 0 : Q_k(x) \subset \mathcal{P}_n(x) \right\}; $$

compare it with $k_{q, \mathcal{P}_n}$:

▶ $\mathcal{P}_n$ corresponds to the partition according to $(Y_1, \ldots, Y_n)$.
▶ $Q_k$ corresponds to the partition according to $(X_1, \ldots, X_k)$. 
Entropy of an interval partition

Definition (Entropy)

Let $\mathcal{P} := (\mathcal{P}_n)_{n=1}^{\infty}$ be a sequence of interval partitions. We say that $\mathcal{P}$ has entropy $c \geq 0$ with respect to a measure $\lambda$ if

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\lim_{n \to \infty} \frac{1}{n} \log \lambda(\mathcal{P}_n(x)) = c, \quad \lambda - a.e.
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Let $\lambda$ be the Lebesgue measure:

- For $Q_k = \{ \left[ \frac{i}{2^k}, \frac{i+1}{2^k} \right) : i = 0, \ldots, 2^k - 1 \}$ we get

  $$- \frac{1}{k} \log \lambda(Q_k(x)) = \log 2, \quad \forall k \geq 1.$$
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- For $\mathcal{P}_n = \left\{ I_n(a, b) : \gcd(a, b) = 1, 1 \leq a \leq b \right\}$, where
  $$
  I_n(a, b) = \left\{ x \in [0, 1] : (q_{n-1}(x), q_n(x)) = (a, b) \right\},
  $$
  we get (blackboard explanation)

  $$
  \lim_{n \to \infty} - \frac{1}{n} \log \lambda(\mathcal{P}_n(x)) = \frac{\pi^2}{6 \log 2}, \quad \lambda - a.e.
  $$
Asymptotic Cost

Theorem (Dajani, Fieldsteel, 2001)

Let $P := \{P_n\}_{n=1}^{\infty}$ and $Q := \{Q_n\}_{n=1}^{\infty}$ be sequences of interval partitions, and let $\lambda$ be a Borel probability measure on $[0, 1)$.

Assume $P$ and $Q$ have entropies $H(P)$ and $H(Q)$ respectively with respect to $\lambda$, then

$$\lim_{n \to \infty} \frac{1}{n} k_{Q,P}(n, x) = \frac{H(P)}{H(Q)}$$

for $\lambda$-a.e. $x$. 

Example: the number of digits required to determine $(q_n - 1(x), q_n(x))$ from the base 10 expansion of $x$ behaves like $\pi \frac{2}{6 \log 2 \log 10} n$ a.e. $x$, a result previously proved by Lochs.
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Good partition sequences

Theorem (Shannon, McMillan, Breiman)

Let $T$ be an ergodic measure preserving transformation on a probability space $(\Omega, \mathcal{B}, \mu)$ and let $P$ be a finite or countable generating partition for $T$ for which $H_\mu(P) < \infty$. Then for $\mu$-a.e. $x$,

$$\lim_{n \to \infty} -\frac{\log \mu(P_n(x))}{n} = h_\mu(T).$$

Here $H_\mu(P)$ denotes the entropy of the partition $P$, $h_\mu(T)$ the entropy of $T$ and $P_n(x)$ denotes the element of the partition $\bigvee_{i=0}^{n-1} T^{-i} P$ containing $x$. 
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We recall that

$$
h_\mu(T) = \sup \{ h_\mu(T, A) : A \text{ countable partition of } X \},
$$

and

$$
h_\mu(T, A) = \lim_{n \to \infty} \frac{1}{n} H(A(U), A(TU), \ldots, A(T^{n-1}U)),
$$

where $U$ is distributed according to $\mu$. 
Concluding remarks

There are other interesting results/ideas not mentioned in this talk,

- The **optimal algorithm for discrete uniform generation from coin flips** has a fairly simple implementation, see the note by Jérémie Lumbroso [arXiv:1304.1916v1](https://arxiv.org/abs/1304.1916).
Concluding remarks

There are other interesting results/ideas not mentioned in this talk,

- The optimal algorithm for discrete uniform generation from coin flips has a fairly simple implementation, see the note by Jérémie Lumbroso arXiv:1304.1916v1.

- The generalisation of Bosma-Dajani-Kraaikamp of the cost of passing from base 10 to the Continued Fraction Expansion. One considers appropriate maps $T$ and $S$ called “number-theoretic fibered maps” associated with digits and get a limiting result with the quotient of $h(T)$ and $h(S)$. 
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- What about non-discrete random variables? See the original paper by von Neumann, Knuth-Yao and Philippe Duchon.
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