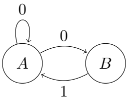
Recurrence function of Sturmian sequences: A probabilistic study over *quadratic irrationals*

Pablo Rotondo[1], Brigitte Vallée[2]



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Study in combinatorics of words.

Objective: description of the finite factors of an infinite word $\boldsymbol{u}$

– How many factors of length n? $\longrightarrow$ Complexity

– What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

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what are the extreme bounds for the recurrence function?

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Here, in an appropriate model,

we perform a probabilistic study:

For a "random" sturmian word stemming from a reduced quadratic irrational

– what is the mean value of the recurrence?

– what is the limit distribution of the recurrence?

Plan of the talk

Complexity

Definition

Complexity function of an infinite word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$

$$p_{\boldsymbol{u}} \colon \mathbb{N} \to \mathbb{N}\,, \qquad p_{\boldsymbol{u}}(n) = \#\{\text{factors of length } n \text{ in } \boldsymbol{u}\}\,.$$

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$\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is not eventually periodic

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$\iff p_{\boldsymbol{u}}(n+1) > p_{\boldsymbol{u}}(n)$ for all $n \in \mathbb{N}$

$\implies p_{\boldsymbol{u}}(n) \geq n+1$

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Definition (Uniform recurrence)

A word $\boldsymbol{u} \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent $\Leftrightarrow$ each finite factor appears infinitely often and with bounded gaps.

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Consider $\boldsymbol{u}$ uniformly recurrent. Its recurrence function is:

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- Related to the complexity function

$$R_{\boldsymbol{u}}(n) \geq n + p_{\boldsymbol{u}}(n) - 1.$$

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$$R_{\boldsymbol{u}}(n) \geq \underbrace{n}_{\text{length of first factor}} + \underbrace{p_{\boldsymbol{u}}(n) - 1}_{\text{count } +1 \text{ for every other factor}} .$$

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$\boldsymbol{u} \in \{0,1\}^{\mathbb{N}}$ is Sturmian $\iff p_{\boldsymbol{u}}(n) = n+1$ for each $n \geq 0$.

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Explicit construction

Given $\alpha, \beta \in [0,1)$ we define

$$\underline{\mathfrak{S}}_{\alpha,\beta}(n) = \lfloor (n+1)\,\alpha + \beta \rfloor - \lfloor n\,\alpha + \beta \rfloor \ ,$$
$$\overline{\mathfrak{S}}_{\alpha,\beta}(n) = \lceil (n+1)\,\alpha + \beta \rceil - \lceil n\,\alpha + \beta \rceil \ ,$$

for $n \geq 0$.

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for $n \geq 0$.

- $\boldsymbol{u}$ is Sturmian $\Longleftrightarrow$ there are $\alpha, \beta \in [0,1)$, α irrational, such that

$u_i = \underline{\mathfrak{S}}_{\alpha,\beta}(i)\,,$ for all $i \geq 0\,,$ or $u_i = \overline{\mathfrak{S}}_{\alpha,\beta}(i)\,,$ for all $i \geq 0\,.$

Digital lines

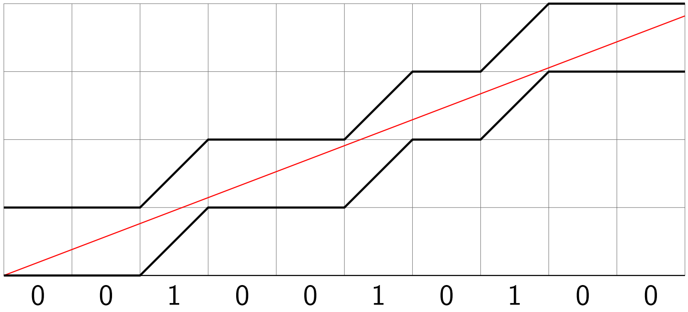

Figure : In digital geometry $\underline{\mathfrak{S}}$ and $\overline{\mathfrak{S}}$ code discrete lines. In the picture we see $\underline{\mathfrak{S}}(\alpha, 0)$ written below, where α is the slope.

Recurrence of Sturmian words

Properties

Let $\boldsymbol{u}$ be Sturmian of the form $\underline{\mathfrak{S}}(\alpha, \beta)$ or $\overline{\mathfrak{S}}(\alpha, \beta)$. Then

- $\boldsymbol{u}$ is uniformly recurrent

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- Further $(R_{\alpha}(n))_{n\in\mathbb{N}}$ only depends on the **continuants** of α.

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- Further $(R_\alpha(n))_{n\in\mathbb{N}}$ only depends on the **continuants** of α.

Reminder: Consider the continued fraction expansion (CFE) of α

$$\alpha = \cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_k + \cfrac{1}{\ddots}}}},$$

The continuant $q_n(\alpha)$ is the denominator of the truncated CFE

$$\cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_n}}}.$$

Recurrence of Sturmian words: Morse, Hedlund

Theorem (Morse, Hedlund, 1940)

The recurrence function is piecewise affine and satisfies

$$R_\alpha(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha)\,, \qquad \text{for } n \in [q_{k-1}(\alpha), q_k(\alpha)[.$$

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Remark

- (α, n) determines a unique k with $n \in [q_{k-1}(\alpha), q_k(\alpha)[$.

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Let us see what they look like...

Recurrence function for two Sturmian words

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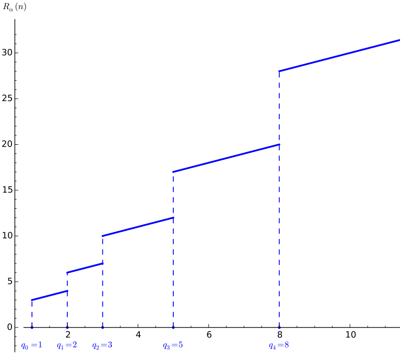
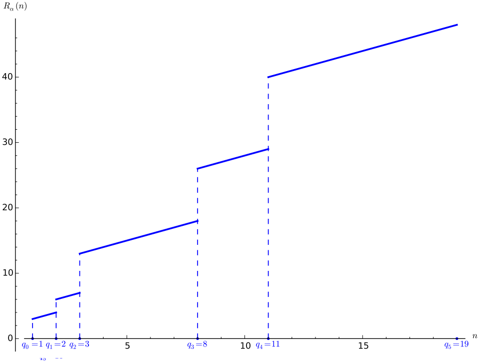


Recurrence function for $\alpha = \varphi^2$,
with $\varphi = (\sqrt{5}-1)/2$.



Recurrence function for $\alpha = 1/e$.

Recurrence function of Sturmian words: classical results.

Theorem (Morse, Hedlund, 1940)

For almost every irrational α, one has

$$\limsup_{n\to\infty} \frac{R_\alpha(n)}{n\log n} = \infty, \qquad \lim_{n\to\infty} \frac{R_\alpha(n)}{n\,(\log n)^{1+\varepsilon}} = 0 \quad \textit{for any } \varepsilon > 0.$$

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But from below

$$\liminf_{n\to\infty} \frac{R_\alpha(n)}{n} \le 3\,,$$

consider $n \approx \frac{1}{2}\left(q_{k-1}(\alpha) + q_k(\alpha)\right)$.

Our first model: uniform α

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In our probabilistic setting we

- fix an integer n (we want $n \to \infty$ ...)
- pick an irrational α uniformly from $[0, 1]$.

$\Longrightarrow$ we perform the probabilistic study of the normalised recurrence quotient

$$S(\alpha, n) = \frac{R_\alpha(n) + 1}{n},$$

as $n \to \infty$.

We consider the recurrence quotient

$$S_n(\alpha) := S(\alpha, n) = \frac{R_\alpha(n) + 1}{n}.$$

We perform a probabilistic study

- for expected values: $\mathbb{E}[S_n]$
- for distributions : $\mathbb{P}\left(S_n \in J\right)$

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Worst case of $S(\alpha, n)$ is roughly $\log n$ (Morse-Hedlund).

$\Longrightarrow$ We wish to obtain this $\log n$ behaviour in our study of $S(\alpha, n)$.

Study of the recurrence quotient S

Theorem

The random variable $S_n(\alpha) := S(\alpha, n)$ admits a limiting distribution when $n \to \infty$, which is given by

$$\lim_{n\to\infty} \mathbb{P}(\alpha : S_n(\alpha) \le \lambda) = \int_{[2,\lambda]} g(y)dy\,,$$

for $t \ge 2$ (and 0 otherwise), where the density g equals

$$g(\lambda) = \begin{cases} \frac{12}{\pi^2}\,\frac{1}{\lambda-1}\,\log(\lambda-1) & \textit{if } \lambda \in [2,3] \\ \frac{12}{\pi^2}\,\frac{1}{\lambda-1}\,\log\left(1+\frac{1}{\lambda-2}\right) & \textit{if } \lambda \in [3,\infty) \end{cases}.$$

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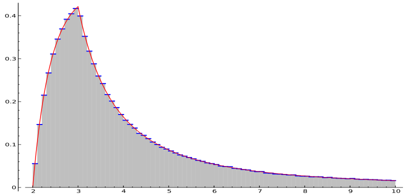
Figure : The limit density $g(x)$ in red and a scaled experimental histogram for $S(\alpha, n)$ in blue, produced with $N = 10^6$.



Principles of the proof

For $n \in [q_{k-1}(\alpha), q_k(\alpha))$, let $x(\alpha, n) = \frac{q_{k-1}(\alpha)}{n}, y(\alpha, n) = \frac{q_k(\alpha)}{n}$. Then

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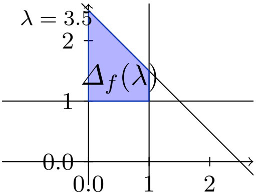
Distribution $\mathbb{P}(S_n \leq \lambda)$ is expressed as the *coprime* Riemann sum of step $\frac{1}{n}$ of

$$\omega(x, y) = \frac{2}{y(x+y)}, \text{ over } \Delta_f(\lambda) := \{(x, y) : 0 < x \leq 1 < y,\ f(x, y) \leq \lambda\}.$$



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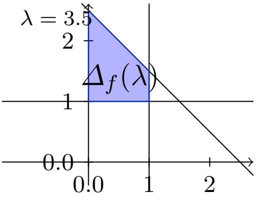

These converge to the integral

$$\lim_{n\to\infty} \mathbb{P}(S_n \leq \lambda) = \frac{6}{\pi^2} \iint_{\Delta_f(\lambda)} \omega(x, y) dx dy$$

Reduced quadratic irrationals

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- Given $w = (w_1, \ldots, w_p) \in \mathbb{Z}^p_{>0}$ consider the inverse branch

$$h_w(x) = \cfrac{1}{w_1 + \cfrac{1}{\ddots + \cfrac{1}{w_p + x}}},$$

then w^∞ is the only real in $[0,1]$ satisfying $h_w(w^\infty) = w^\infty$.

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  then w^∞ is the only real in $[0,1]$ satisfying $h_w(w^\infty) = w^\infty$.
- Consider $w = (w_1, \ldots, w_p) \in \mathbb{N}^p$ primitive, then this is the smallest period of the continued fraction expansion of w^∞.

Reduced quadratic irrationals

Consider $w = (w_1, \ldots, w_p) \in \mathbb{N}^p$, not necessarily primitive

- An appropriate notion of size for w is given by $1/\alpha(w)$, where

$$\alpha(w) := \left|h'_w(w^\infty)\right|^{1/2} = \left(q_p(w) + w^\infty\, q_{p-1}(w)\right)^{-1} .$$

- By the *chain rule*

$$\alpha(w^m) = (\alpha(w))^m$$

  for $m \in \mathbb{Z}_{\geq 0}$, where $w^m = w \cdot w \cdot \ldots w$ concatenated m times.
- If $T(x) = \left\{\frac{1}{x}\right\}$ is the shift of the Euclidean System

$$\alpha(w) = \prod_{k=0}^{p-1} T^k(w^\infty)\,.$$

The generating function

Fix $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$.

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$$S(w^\infty, n) \leq \lambda$$

over the r.q.i w^∞ with w primitive, and $-\log \alpha(w) \leq x$.

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Dirichlet series

To compute with such quantities $\rightarrow$ DGF

$$P_n(s) = \sum_{\substack{w \in \mathbb{Z}^*_{>0}, \\ w \text{ primitive}}} (\alpha(w))^s \left[\!\left[S(w^\infty, n) \leq \lambda \right]\!\right],$$

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As explained by Eda, it is enough to study the non-primitive case

$$G_n(s) = \sum_{w \in \mathbb{Z}^*_{>0}} (\alpha(w))^s \left[\!\left[S(w^\infty, n) \leq \lambda \right]\!\right],$$

as $P_n(s) + \sum_{k \geq 2} P_n(ks) = G_n(s)$.

Tauberian Theorem

Let $(a_i)_{i \in I}$ be a family of non-negative numbers indexed on a numerable set I, and let $h \colon I \to \mathbb{R}_{>0}$ be a function such that

$$D(s) = \sum_{i \in I} a_i h(i)^{-s}$$

converges absolutely for $\Re(s) > 1$.

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Suppose $D(s)$ can be extended analytically to $\Re(s) = 1$ except for $s = 1$, where it satisfies

$$D(s) = \frac{\rho}{(s-1)^k} + \frac{H(s)}{(s-1)^{k-1}}, \qquad \Re(s) > 1,$$

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with H analytic at $s = 1$.
Then

$$\frac{1}{N} \sum_{i\in I : h(i)\le N} a_i \sim \rho \frac{\log^{k-1} N}{(k-1)!}.$$

The first cycle: positioning n

Objective. Given n, we want to study the function

$$\left[\!\left[1 + \frac{q_{k-1}(x) + q_k(x)}{n} \leq \lambda \right]\!\right]$$

where $k := k(x, n)$ is such that $q_{k-1}(x) \leq n < q_k(x)$.

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Write $x = w^\infty$ for some $w \in \mathbb{Z}_{>0}^+$, and $v = w_1 \dots w_k$, **the** prefix of w^∞ needed to compute $(q_{k-1}(w^\infty), q_k(w^\infty))$

$\circledast$ observe we may decompose $v = w^\ell u$, with $u \neq \epsilon$, $u \preceq w$ a prefix of w, and $\ell \in \mathbb{Z}_{\geq 0}$.

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⊛ observe we may decompose $v = w^\ell u$, with $u \neq \epsilon$, $u \preceq w$ a prefix of w, and $\ell \in \mathbb{Z}_{\geq 0}$.

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$\Longrightarrow$ Here we will explain the case $\ell = 0$.

equivalently $k(w^\infty, n) \leq |w|$ holds.

The generating function 2.0

Adding the condition that we be on the first cycle

$$F_n(s) = \sum_{w \in \mathbb{Z}_{>0}^*} (\alpha(w))^s \left[\!\left[S(w^\infty, n) \leq \lambda,\ k(w^\infty, n) \leq |w| \right]\!\right],$$

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here we prove that the sum, even though the integrals depends on p_k, q_{k-1} and q_k simultaneously, can be computed as $n \to \infty$.

Reminder. We want the dominant pole and residue for

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Now we have to *sum* over all w such that $v \preceq w$.

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Pole and residue

- Adding over all $w : v \preceq w$ we obtain

$$\text{Tr}\left((I - H_s)^{-1} \circ H_{[v],s}\right) = F_{n,v}(s) + \text{less significat series}\,,$$

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- Thus we have, as $s \to 2$, dominant eigenvector $\sim \psi$ and

$$\begin{aligned}\text{Tr}\left((I - H_s)^{-1} \circ H_{[v],s}\right) &\sim \frac{1}{s-2} \frac{12 \log 2}{\pi^2} \int_0^1 H_{[v],2}[\psi](t) dt \\ &= \frac{1}{s-2} \frac{12 \log 2}{\pi^2} \int_{h_v([0,1])} \psi(u) du\,.\end{aligned}$$

The real case strikes back

Finally we concentrate on

$$\sum_{w' \in \mathbb{Z}^{+}_{>0}} \left[\!\!\left[S(v^{\infty}, n) \le \lambda,\, k(v^{\infty}, n) = |v| \right]\!\!\right] \int_{h_v([0,1])} \psi(x)dx\,.$$

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- Rewrite the condition in the brackets, giving

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- This is exactly the probability $\mathbb{P}\left(t : S_n(t) \le \lambda\right)$ when t is distributed according to the law

$$\mathbb{P}(t \le T) = \int_{[0,T]} \psi(u)du\,.$$

Declaration of independence

Theorem (Independence from the initial law)

Consider a probability measure μ that is absolutely continuous with respect to the Lebesgue measure m, in symbols $\mu \ll m$.
Then for each fixed $\lambda \in \mathbb{R}$, the limit

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Remark

In particular the resulting limiting distribution is that of the real model (ANALCO).

Convergence speed may vary though.

Independent? How come?

⊛ Suppose $g(x) = \frac{d\mu}{dm}(x) \in C^1([0,1])$, and write

$$\mathbb{P}_{\mu}\left(S_n(\alpha) \le \lambda\right) =$$

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- Not the case, but q_{k-1}/q_k and p_k/q_k are

"asymptotically independent".

Independence of the inverses

We recall the classic

$$p_{k-1}q_k - p_k q_{k-1} = (-1)^k \Rightarrow p_k = \left((-1)^{k+1} q_{k-1}^{-1}\right) \bmod q_k \, .$$

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So q_{k-1} and p_k behave almost like modular inverses, up to a sign change depending on the parity of the depth k.

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Theorem (see e.g. Shparlinski)

Let $q \in \mathbb{Z}_{>0}$ and let $[a_1, b_1], [a_2, b_2] \subset [0, 1]$, then for any $\epsilon > 0$

$$\frac{1}{\varphi(q)} \sum_{\substack{1\le a\le q\,,\\ \gcd(a,q)=1}} \mathbf{1}_{\left(\frac{a}{q},\frac{a^{-1} \bmod q}{q}\right)\in[a_1,b_1]\times[a_2,b_2]}$$

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- Similar studies in other dimensions (?). Brun (?)