Introduction to transfer operators

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(I) Dynamical systems of the interval

(II) Euclidean divisions

(III) Instances of dynamical systems of the interval
that arise in the Euclidean Context

(IV) Transfer operators

(I) Dynamical systems of the interval

A dynamical system $(\mathcal{I}, S)$ is defined by four elements:

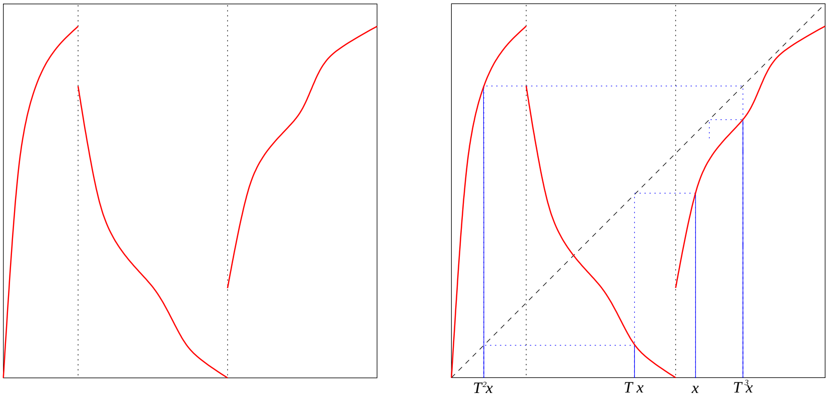
- a finite or infinite denumerable alphabet Σ,
- a topological partition of $\mathcal{I} :=]0,1[$ with open intervals $\mathcal{I}_{m, m \in \Sigma}$,
- an encoding mapping σ equal to m on each $\mathcal{I}_m$,
- a shift mapping T
  s.t. $T|_{\mathcal{I}_m}$ is a bijection of class $\mathcal{C}^2$ from $\mathcal{I}_m$ to $\mathcal{J}_m := S(\mathcal{I}_m)$.

Given an input x of $\mathcal{I}$, this gives rise to the trajectory

$$\mathcal{T}(x) := (x, Tx, T^2x, \dots)$$

and to the word $M(x)$ which encodes the trajectory

$$M(x) := (\sigma x, \sigma T x, \sigma T^2 x, \dots).$$



A dynamical system, with $\Sigma = \{a, b, c\}$ and a word $M(x) = (c, b, a, c \ldots)$.

Correlations between symbols due to

– the geometry of the branches

– the shape of the branches

The geometry of the branches [position of $T(\mathcal{I}_m)$ wrt $\mathcal{I}_\ell$] ;
it describes the set $s(m)$ of possible successors of the symbol m.

Particular cases:

- Complete systems $T(\mathcal{I}_m) = \mathcal{I}$
- Markovian systems $T(\mathcal{I}_m) = \text{union of some } \mathcal{I}_\ell$

give rise to a finite characterization of $s(m)$.

Topological mixing.

$\forall (b,e) \in \Sigma^2$, $\exists n_0 \geq 1$ st $\forall n \geq n_0$, one has: $\mathcal{I}_b \cap T^{-n}(\mathcal{I}_e) \neq \emptyset$

"There is a word of length n which begins with b and ends with e".

Correlations between symbols also due to the shape of the branches

The shape of the branches [derivatives of the branches] also explains how the distribution evolves.

Less correlated systems correspond to systems with affine branches.

Uniform expansiveness.

$\exists \delta > 1$ st , $\forall m \in \Sigma$, $\forall x \in \mathcal{I}$, one has: $|T'(x)| \geq \delta > 1.$

Particular cases...

A memoryless source:=

a complete system with affine branches and uniform initial density

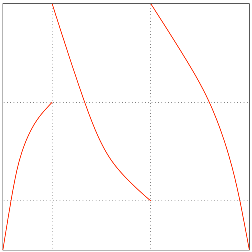
A Markov chain:=

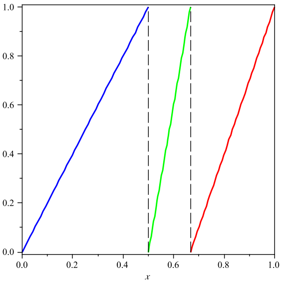
a Markovian system with affine branches,

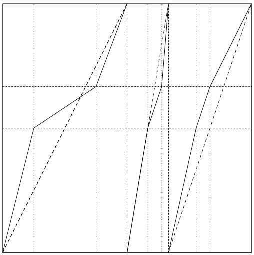
with an initial density which is constant on each $\mathcal{I}_m$.

Examples.

A Markovian system, a memoryless source, a Markov chain.







General case of interest.

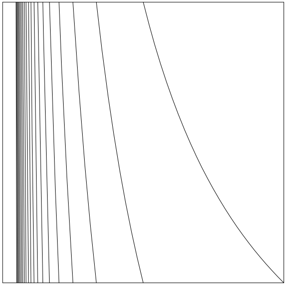
A complete – or a Markovian – system

– with a possible infinite denumerable alphabet

– topologically mixing – and expansive.

Main instance: the Euclidean source defined with the Gauss map

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \qquad T(0) = 0$$



(II) Instances of “natural” instances of dynamical systems

related to “natural” divisions

Various possible types of Euclidean divisions

– MSB divisions [directed by the Most Significant Bits]
shorten integers on the left,
and provide a remainder r smaller than u,
(w.r.t the usual absolute value), i.e. with more zeroes on the left.

– LSB divisions [directed by the Least Significant Bits]
shorten integers on the right,
and provide a remainder r smaller than u
(w.r.t the dyadic absolute value), i.e. with more zeroes on the right.

– Mixed divisions
shorten integers both on the right and on the left,
with new zeroes both on the right and on the left.

Instances of MSB Algorithms.

– Variants according to the position of remainder r,

By Default: $v = mu + r$ with $0 \le r < u$

By Excess: $v = mu - r$ with $0 \le r < u$

Centered: $v = mu + \epsilon r$ with $\epsilon = \pm 1, \;\; 0 \le r \le u/2$

– The Japanese division associated with $\alpha \in [0,1]$

the remainder r belongs to the interval $[(1-\alpha)\cdot u, \alpha \cdot u]$

– Subtractive Algorithm :

A division with quotient m can be replaced by m subtractions

`While` $v \ge u$ `do` $v := v - u$

An instance of a Mixed Algorithm.

The Subtractive Algorithm,

where the zeroes on the right are removed from the remainder defines the Binary Algorithm.

Subtractive Gcd Algorithm.	Binary Gcd Algorithm.
`Input.` u, v; $v \geq u$	`Input.` u, v odd; $v \geq u$
`While` $(u \neq v)$ `do`	`While` $(u \neq v)$ `do`
`While` $v > u$ `do`	`While` $v > u$ `do`
	$k := \nu_2(v-u)$;
$v := v - u$	$v := \dfrac{v-u}{2^k}$;
Exchange u and v.	Exchange u and v.
`Output.` u (or v).	`Output.` u (or v).

The 2-adic valuation ν_2 counts the number of zeroes on the right

An instance of a LSB Algorithm.

On a pair (u, v) with v odd and u even,

with $\nu_2(u) = k$, of the form $u := 2^k\, u'$

the LSB division produces

- a quotient a odd, with $|a| < 2^k$
- and a remainder r with $\nu_2(r) > k,$ of the form $r := 2^k\, r'$,

and writes $\quad v = a \cdot u' + 2^k \cdot r'$.

The pair (r', u') satisfies

$$\nu_2(r') > \nu_2(u') = 0 \text{ and } \gcd(u, v) = \gcd(r', u').$$

It will be the new pair for the next step.

The tortoise and the hare.

A blue tortoise and a red hare

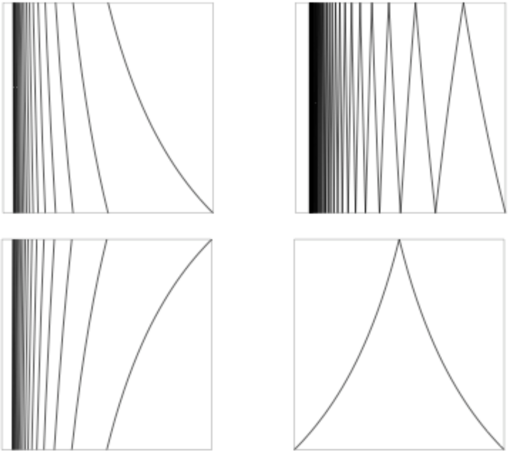
An execution of the LSB Algorithm on $(72001, 2011176)$

i	u_i [base 2]	a_i	k_i
1	1111010110000000101000	-3	3
2	110010011011010100000	1	1
3	1100001100010100000000	1	3
4	100110001111000000000	-1	1
5	1110100101010000000000	-1	1
6	1100000100100000000000	1	1
7	1000100011000000000000	-1	1
8	10000010111000000000000	1	1
9	1100000000000000	1	2
10	10000010000000000000000	-1	1
11	1000100000000000000000	1	1
12	1100000000000000000000	-5	3
13	100000000000000000000000	3	2

(III) Euclidean dynamical systems.

For each MSB Alg., replace the rational u/v by a generic real x:

A continuous dynamical system extends each discrete division



Above, Standard and Centered; On the bottom, By-Excess and Subtractive.

On the bottom, there are indifferent points : $x = 1$ or 0, for which $T(x) = x, |T'(x)| = 1$.

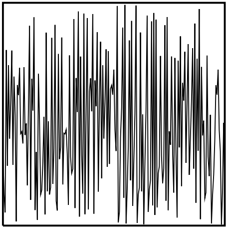
Dynamical Systems relative to MSB Algorithms.

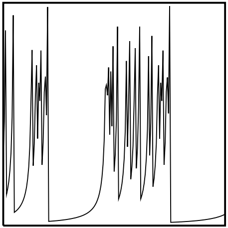
Key Property : Expansiveness of branches

$$|T'(x)| \geq \rho > 1 \text{ for all } x \text{ in } \mathcal{I}$$

When true, this implies a chaotic behaviour for trajectories. The associated algos are Fast and belong to the Good Class

When this condition is violated at only one indifferent point, this leads to intermittency phenomena. The associated algos are Slow.





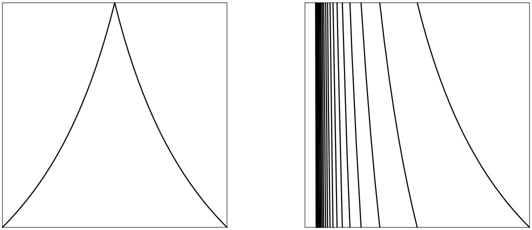
Chaotic Orbit [Fast Class], Intermittent Orbit [SlowClass].

Induction Method

For a DS (I, T) with a "slow" branch relative to a slow interval J, contract each part of the trajectory which belongs to J into one step.

This (often) transforms the slow DS (I, T) into a fast one (I, S):

While $x \in J$ do $x := T(x)$;
$S(x) := T(x)$;



The Induced DS of the Subtractive Alg = the DS of the Standard Alg.

Dynamical systems relative to α Euclidean divisions

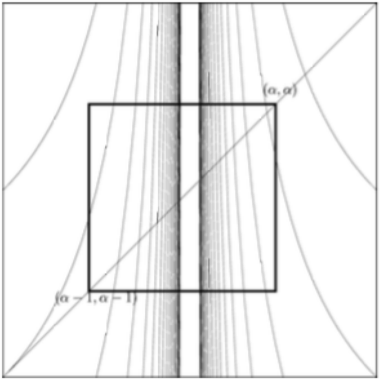


Fig. 1. The family of dynamical systems $\overline{S}_\alpha$.

We can consider the folded or unfolded versions of the DS.

Two other Euclidean dynamical systems, related to mixed or LSB divisions: the Binary Algorithm and the LSB Algorithm.

These algorithms use the 2–adic valuation ν .... only defined on rationals.

The 2–adic valuation ν is extended to a real random variable ν with

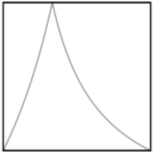
$$\Pr[\nu = k] = 1/2^k \qquad \text{for} \quad k \geq 1.$$

This gives rise to a probabilistic dynamic system.

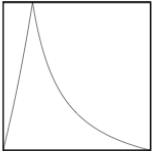
namely, a sequence of dynamical systems $\mathcal{S}_k$

where $\mathcal{S}_k$ is chosen to the probability $1/2^k$.

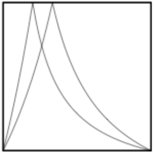
(I) The DS relative to the Binary Algorithm



$k = 1$



$k = 2$



$k = 1$ and $k = 2$

Two other Euclidean dynamical systems, related to mixed or LSB divisions: the Binary Algorithm and the LSB Algorithm.

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namely, a sequence of dynamical systems $\mathcal{S}_k$

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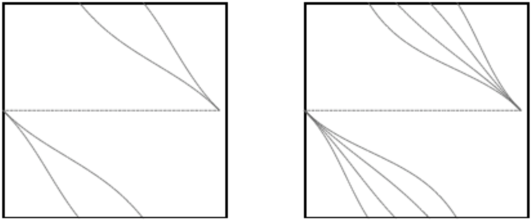
(II) The DS relative to the LSB Algorithm



(IV) Transfer operators.

We will describe the general framework

Geometric properties of the Dynamical System

$\Downarrow$

Spectral properties for the Transfer Operator

in a convenient functional space.

$\Downarrow$

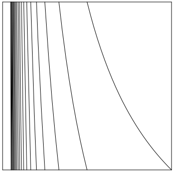
Analytical properties of the (Dirichlet) Generating Functions

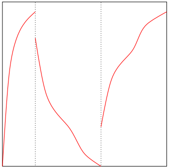
$\Downarrow$

Probabilistic analysis of the execution of the algorithm

or the trajectories of the DS

The transfer operator associated with a complete system.





$\mathcal{H} := \{$inverse branches$\}$

Density Transformer: for an initial density f on $[0,1]$,
$\mathbf{H}[f]$ is the density on $[0,1]$ after one iteration of the shift

$$\mathbf{H}[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|\, f \circ h(x)$$

Transfer operator (Ruelle): extension of $\mathbf{H}$ into $\mathbf{H}_s$.

$$\mathbf{H}_s[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|^s\, f \circ h(x), \qquad \mathbf{H}_1 = \mathbf{H}$$

The k-th iterate : the distribution after k iterations

$$\mathbf{H}_s^k[f](x) = \sum_{h \in \mathcal{H}^k} |h'(x)|^s\, f \circ h(x)$$

The quasi-inverse : all the finite trajectories (stop at 0)

$$(I - \mathbf{H}_s)^{-1}[f](0) = \sum_{h \in \mathcal{H}^\star} |h'(0)|^s\, f \circ h(0)$$

Remark: If the system is not complete, we must add indicator functions.

For a Markovian sytem, we can replace them by an operator matrix.

What is needed on the operator $\mathbf{H}_s$ for the analysis of the DS/algorithm? Sufficient conditions !

For the study of truncated trajectories $(x, Tx, \dots T^k x)$

Quasi-Power Properties of the iterates $\mathbf{H}_s^k$ near $s = 1$

Sufficient Spectral Properties:

Unique Dominant Eigenvalue (UDE) + Spectral Gap (SG)

For the average case analysis of particular trajectories (rational or periodic)

Analytic Properties of $(I - \mathbf{H}_s)^{-1}$ or $\mathrm{Tr}\,(I - \mathbf{H}_s)^{-1}$ for $\Re s \geq 1$

Sufficient Spectral Properties:

UDE +SG+ Aperiodicity

For the distributional analysis of particular trajectories (rational or periodic)

Analytical properties on $(I - \mathbf{H}_s)^{-1}$ also on the left of $\Re s = 1$.

Sufficient Spectral Properties:

UDE + SG+ Strong Aperiodicity (Dolgopyat).

What can be expected for the analysis of generic truncated trajectories?

Decomposable dynamical systems

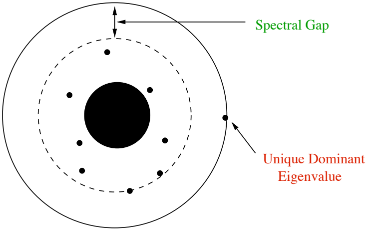
Definition. A DS is decomposable if there is a Banach space $\mathcal{F}$ for which:

(a) the operator $\mathbf{H}_s$ acts on $\mathcal{F}$ for $\Re s > s_0$ (with $s_0 < 1$)

(b) the map $s \mapsto \mathbf{H}_s$ is analytic

(c) the density transformer $\mathbf{H} = \mathbf{H}_1$ admits $\lambda = 1$ as a unique (simple) dominant eigenvalue on the circle $\{\lambda \mid |\lambda| = 1\}$,

(d) there is a spectral gap.



On which functional space $\mathcal{F}$?

The answer depends on the DS, and thus on the division....

A compromise is often needed!

Choice of $\mathcal{F}$: Quasi-Compactness (I)

Some definitions. For an operator $\mathbf{L}$,

– the spectrum $\mathrm{Sp}(\mathbf{L}) := \{\lambda \in \mathbb{C}; \quad \mathbf{L} - \lambda I \quad \text{non invertible}\}$

– the spectral radius $R(\mathbf{L}) := \sup\{|\lambda|, \quad \lambda \in \mathrm{Sp}(\mathbf{L})\}$

– the essential spectral radius $R_e(\mathbf{L}) =$ the smallest $r > 0$ s.t
any $\lambda \in \mathrm{Sp}(\mathbf{L})$ with $|\lambda| > r$ is an isolated eigenvalue of finite multiplicity.

– For compact operators, the essential radius equals 0.

Definition. $\mathbf{L}$ is quasi-compact if the inequality $R_e(\mathbf{L}) < R(\mathbf{L})$ holds.

For a quasi-compact operator:
Outside the closed disk of radius $R_e(\mathbf{L})$, the spectrum of the operator consists of isolated eigenvalues of finite multiplicity.
$\Longrightarrow$ There is an eigenvalue λ with $|\lambda| = R(\mathbf{L})$ and a spectral gap.

Sufficient conditions for quasi-compactness (II)

A theorem, due to Hennion:

Suppose that the Banach space $\mathcal{F}$

- is endowed with two norms, a weak norm $|.|$ and a strong norm $||.||$,
- and the unit ball of $(\mathcal{F}, ||.||)$ is precompact in $(\mathcal{F}, |.|)$.

If $\mathbf{L}$ is a bounded operator on $(\mathcal{F}, ||.||)$

for which there exist two sequences $\{r_n \geq 0\}$ and $\{t_n \geq 0\}$ s.t.

$$||\mathbf{L}^n[f]|| \leq r_n \cdot ||f|| + t_n \cdot |f| \qquad \forall n \geq 1, \forall f \in \mathcal{F},$$

Then: $\qquad R_e(\mathbf{L}) \leq r := \lim_{n\to\infty} \inf (r_n)^{1/n}.$

If $R(\mathbf{L}) > r$, then the operator $\mathbf{L}$ is quasi-compact on $(\mathcal{F}, ||.||)$.

Remark : The magenta inequality is called a Lasota-Yorke inequality.

Some instances of quasi-compactness (III)

Consider systems that are uniformly expansive $|T'(x)| \geq \delta > 1$
with a distortion property

$$\exists K > 0, \forall h \in \mathcal{H}, \forall x \in X, \quad |h''(x)| \leq K\,|h'(x)|.$$

(a) If they are complete, then we can choose: $\mathcal{F} := \mathcal{C}^1(\mathcal{I})$,

- the weak norm is the sup-norm $||f||_0 := \sup |f(t)|$,
- the strong norm is the norm $||f||_1 := \sup |f(t)| + \sup |f'(t)|$.

(b) For any geometry, then we can choose: $\mathcal{F} := BV(\mathcal{I})$

- the weak norm is the sup-norm $||f||_0 := \sup |f(t)|$,
- the strong norm is the BV–norm

In the two cases, there is a Lasota-Yorke inequality :
the density transformer $\mathbf{H}$ satisfies the hypotheses of Hennion's Theorem.

Some instances of quasi-compactness – or compactness (IV)

For the Tortoise and Hare DS (non expansive),

we use two functional spaces

(a) The space of α–Hölder functions $\mathcal{F} := H_\alpha(J) \quad (\alpha \in]0,1])$

- the weak norm is the L^1 norm
- the strong norm is the α-Hölder norm $|\cdot|_\alpha$,

(b) The space $\mathcal{F} := \mathcal{C}^0(J)$ of continuous functions;

- the strong norm is the sup norm $||\cdot||_0$
- the weak norm is the L^1 norm

For the Binary algorithm, there is a functional space $\mathcal{F}$ (a Hardy space) on which the transfer operator is compact.

Spectral properties of $\mathbf{H}$ relative to eigenvalues of modulus 1.

Simpler case for a DS uniformly expansive with distortion

Property. The following holds on $\mathcal{C}^1(\mathcal{I})$:

(a) There exists a density f strictly positive for which $\mathbf{H}[f] = f$

(b) If the eigenvalue $\lambda = 1$ is simple, then the system is ergodic.

(c) The two conditions are equivalent

- $\lambda = 1$ is the only eigenvalue on $\{|\lambda| = 1\}$ (moreover simple),
- the system is mixing.

This property also holds

– in spaces of regular enough functions,

– in the quasi-compacity framework.

What can be expected for the analysis of particular trajectories?

Supplementary Property: Aperiodicity

On the punctured vertical line $\{s \mid \Re s = 1, s \neq 1\}$, $1 \notin \mathrm{Sp}\,\mathbf{H}_s$.

Why the name? Because of the converse property

The two conditions are equivalent (in the quasi-compacity framework...)

- There exists $t_0 \neq 0$ for which $1 \in \mathrm{Sp}\,\mathbf{H}_{1+it_0}$
- The spectrum $s \mapsto \mathrm{Sp}\,\mathbf{H}_s$ is periodic of period it_0.

There exist systems with affine branches which are periodic.
A periodic system has affine branches (up to conjugaison)

Then the Euclidean systems (with LFT branches) are all aperiodic.

The triple [UDE+ SG + Aperiodicity] entails good properties for $(I - \mathbf{H}_s)^{-1}$, sufficient for applying Tauberian Theorems

$s = 1$ is the only pole on the line $\Re s = 1$

Expansion near the pole $s = 1$

$$(I - \mathbf{H}_s)^{-1} \sim \frac{a}{s-1}$$

s=1

Half–plane of convergence $\Re s > 1$

No hypothesis needed on the half–plane $\Re s < 1$.

The functional spaces where the triple [UDE+ SG + Aperiodicity] holds.

Algs	Geometry of branches	Convenient Functional space
Good Class (Standard, Centered)	Expansive	$\mathcal{C}^1(\mathcal{I})$
Binary	Not expansive	The Hardy space $\mathcal{H}(\mathcal{D})$
LSB	Expansive on average	Various spaces: $\mathcal{C}^0(J), \mathcal{C}^1(J)$ Hölder $\mathbb{H}_\alpha(J)$
Slow Class (Subtractive, By-Excess)	An indifferent point	Induction $+ \mathcal{C}^1(\mathcal{I})$

In each case, the aperiodicity holds since the branches have not "all the same form".

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