# Coboundaries and balance in words

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Let  $\mathcal{A}$  be a finite alphabet and consider  $x \in \mathcal{A}^{\mathbb{Z}}$ ,

$$x = \cdots x_{-2} x_{-1} x_0 x_1 x_2 x_3 \cdots$$

Let v be a factor of x. We are interested in the behavior of the function

$$B^{v}(n) = \max_{|w|=|w'|=n} \{||w|_{v} - |w'|_{v}|\}$$

If this quantity is bounded, x is balanced on v.

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#### Question: given x and v, is x balanced on v?

# Outline

- Balance
- Coboundaries
- Substitution systems
  - Rational Frequencies
  - Examples

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- (X, T) is a minimal subshift on the finite alphabet A.
  - $\rightsquigarrow$  Every  $x \in X$  is uniformly recurrent: every factor of x occurs infinitely often with boundend gaps.
  - $\rightsquigarrow \mathcal{L}(x) = \mathcal{L}(X) \quad \forall x \in X$

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A word x ∈ A<sup>ℤ</sup> is balanced on the factor v ∈ L(x) if there exists a constant C<sub>v</sub> such that for every pair of factors u, w in L(x) with |u| = |w|,

$$|u|_{v}-|w|_{v}|\leq C_{v}.$$

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• Since (X, T) is minimal, balance is a property of the language.

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- $\rightsquigarrow$  (X, T) uniquely ergodic: X admits a unique T-invariant measure.
- A word x ∈ A<sup>ℤ</sup> has uniform frequencies if for every w ∈ L(x), the ratio

$$\frac{|x_k\cdots x_{k+n}|_w}{n+1}$$

has a limit when *n* tends to  $\infty$ , uniformly in *k*.

#### Fact

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 In this case, for all x ∈ X the frequency of the factor w is equal to µ([w]), where

$$[w] = \{x \in X : x_0 \cdots x_{|w|-1} = w\}$$

#### Proposition

The language  $\mathcal{L}(X)$  is balanced in the factor v if and only if v has a frequency  $\mu_v$  and there exists a constant  $B_v$  such that for any factor  $w \in \mathcal{L}(X)$ , we have

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• Equivalently, v has a frequency  $\mu_v$  and there exists  $B_v$  such that for all  $x \in X$  and for all n > 1,

$$||\mathbf{x}_{[0,n)}|_{\mathbf{v}} - \mu_{\mathbf{v}}\mathbf{n}| \le B_{\mathbf{v}}$$

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$$||w|_v - \mu_v|w|| \le B_v$$

 Equivalently, ν has a frequency μ<sub>ν</sub> and there exists B<sub>ν</sub> such that for all x ∈ X and for all n ≥ 1,

$$|\underbrace{|x_{[0,n)}|_{v} - \mu_{v}n}_{Discrepancy}| \le B_{v}$$

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• Let (X, T) a topological dynamical system.

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#### • Dimension group

 $\mathcal{K}^{0}(X,T) = (\mathcal{C}(X,\mathbb{Z})/\partial \mathcal{C}(X,\mathbb{Z}), (\mathcal{C}(X,\mathbb{Z})/\partial \mathcal{C}(X,\mathbb{Z}))^{+}, \mathbf{1})$ 

is a complete invariant of (strong) orbit equivalence.

#### Theorem (Gotschalk-Hedlund '55)

Let (X, T) be a minimal topological dynamical system. The map  $f \in C(X, \mathbb{R})$  is a coboundary if and only if there exists  $x_0 \in X$  such that the sequence  $(f^{(n)}(x_0))_{n\geq 1}$  is bounded, where

$$f^{(n)}(x) = f(x) + f \circ T(x) + \cdots + f \circ T^{n-1}(x).$$

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Remark. If  $f = g \circ T - g$ ,  $(f^{(n)}(x))_{n \in \mathbb{N}}$  is bounded for every  $x \in X$ .

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 $\rightsquigarrow$  f coboundary  $\Leftrightarrow$   $(f^{(n)})_{n \in \mathbb{N}}$  bounded.

# Coboundaries and balance

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# Coboundaries and balance

#### Remark

Let (X, T) a subshift on A. Suppose (X, T) is minimal and uniquely ergodic with measure  $\mu$ . Given a factor  $v \in \mathcal{L}(X)$ , define

$$f_{\mathsf{v}} = \chi_{[\mathsf{v}]} - \mu_{\mathsf{v}} \in C(\mathsf{X}, \mathbb{R})$$

Then, (X, T) is balanced in the factor v if and only if  $f_v$  is a coboundary.

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Then, (X, T) is balanced in the factor v if and only if  $f_v$  is a coboundary.

#### Proof.

Note that for all  $x \in X$ , for all  $n \ge 1$ ,

$$\begin{aligned} f_{v}^{(n)}(x) &= \chi_{[v]}(x) - \mu_{v} + \dots + \chi_{[v]}(T^{n-1}x) - \mu_{v} \\ &= |x_{[0,n+|v|)}|_{v} - n\mu_{v} \end{aligned}$$

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# Substitution systems

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## Substitution systems

Consider a primitive substitution  $\sigma : \mathcal{A} \to \mathcal{A}^*$ .

The symbolic system generated by  $\sigma$  is  $(X_{\sigma}, T)$ , where

$$X_{\sigma} = \{ x \in \mathcal{A}^{\mathbb{Z}} : \forall w \prec x, \exists a \in \mathcal{A}, \exists n \in \mathbb{N} : w \prec \sigma^{n}(a) \}.$$

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Theorem (Queffélec '87)

 $(X_{\sigma}, T)$  is minimal and uniquely ergodic.

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### Partitions in towers

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A partition in towers of the symbolic system (X, T) is

$$\mathcal{P} = \{T^j B_i : 1 \le i \le m, 0 \le j < h_i\}$$

where the  $B_i$ 's are clopen and nonempty,  $m, h_i \in \mathbb{N}$ .

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#### Proposition

Let  $(X_{\sigma}, T)$  be the symbolic system generated by  $\sigma$ . Let  $\mathcal{L}_2(X_{\sigma}) = \mathcal{L}(X_{\sigma}) \cap \mathcal{A}^2$ . For all  $n \in \mathbb{N}$ , define

$$\mathcal{P}_n = \{T^j \sigma^n([ab]) : ab \in \mathcal{L}_2(X_\sigma), 0 \le j < |\sigma^n(a)|\}$$

The sequence  $(\mathcal{P}_n)_{n \in \mathbb{N}}$  is a sequence of partitions in towers of (X, T), such that for all  $n \in \mathbb{N}$ ,  $\mathcal{P}_{n+1}$  is finer than  $\mathcal{P}_n$ .

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is also a nested sequence of partitions in towers.

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• Question: Why not to use

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 We need the atoms of the partition to determine as many starting letters as we want! → cocycle of f<sub>v</sub>.

## Coboundaries and partitions

Define  $R_n(X) = \{ \phi : \mathcal{L}_n(X) \to \mathbb{R} \}$  and  $\beta : \underbrace{R_1(X)}_{\mathbb{R}^{\mathcal{A}}} \to \underbrace{R_2(X)}_{\mathbb{R}^{\mathcal{L}_2(X)}}; \varphi \longmapsto (\beta \varphi)(ab) = \varphi(b) - \varphi(a) \quad \forall ab \in \mathcal{L}_2(X)$ 

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#### Proposition (\*)

Let  $f \in C(X_{\sigma}, \mathbb{Z})$  such that there exists  $k \in \mathbb{N}$  for which f is constant in the atoms of  $\mathcal{P}_k$ . For all  $n \ge k$ , define  $\phi_n \in \mathbb{R}^{\mathcal{L}_2(X_{\sigma})}$  by

$$\phi_n(ab) = \sum_{j=0}^{|\sigma^n(a)|-1} f \mid_{T^j \sigma^n([ab])} \quad \forall ab \in \mathcal{L}_2(X_\sigma).$$

If f is a coboundary, then  $\phi_n \in \beta(R_1(X_{\sigma}))$  for all n large enough.

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## • The subspace $\beta(R_1(X_{\sigma}))$ is easy to handle.

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- The subspace  $\beta(R_1(X_{\sigma}))$  is easy to handle.
- Examples:
  - $\blacktriangleright$  Thue-Morse substitution on  $\{0,1\}$   $\sigma: 0\mapsto 01, 1\mapsto 10$

$$\beta(R_1(X_{\sigma})) = \left\langle \begin{pmatrix} 0\\1\\-1\\0 \end{pmatrix} \right\rangle \leqslant \mathbb{R}^4$$

 $C_{2}(X) = \{00 \ 01 \ 10 \ 11\}$ 



$$\mathcal{L}_2(X_\sigma) = \{00, 01, 10\}$$
  
 $eta(R_1(X_\sigma)) = \left\langle \left(egin{array}{c} 0 \ 1 \ -1 \end{array}
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• We need that  $f \in C(X_{\sigma}, \mathbb{Z})$ .

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Proposition

Let  $v \in \mathcal{L}(X_{\sigma})$  and suppose that  $\mu_{v} \in \mathbb{Q}$ . Then, there exists  $k \geq 1$  such that  $f_{v}$  is constant in the atoms of  $\mathcal{P}_{k}$  and if  $(X_{\sigma}, T)$  is balanced on v,  $\phi_{n}$  (defined for  $f_{v}$ ) belongs to  $\beta(R_{1}(X_{\sigma}))$  for all n large enough.

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Proof.

• There exists  $m \in \mathbb{N}$  such that  $mf_{\nu} \in C(X, \mathbb{Z})$ .

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Proof.

- There exists  $m \in \mathbb{N}$  such that  $mf_v \in C(X, \mathbb{Z})$ .
- For all n, the elements in an atom of P<sub>n</sub> share at least their
   L<sub>n</sub> + 1 letters, where L<sub>n</sub> = min{|σ<sup>n</sup>(a)| : a ∈ A} ⇒ f<sub>v</sub> constant.

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- There exists  $m \in \mathbb{N}$  such that  $mf_v \in C(X, \mathbb{Z})$ .
- For all n, the elements in an atom of P<sub>n</sub> share at least their L<sub>n</sub> + 1 letters, where L<sub>n</sub> = min{|σ<sup>n</sup>(a)| : a ∈ A} ⇒ f<sub>v</sub> constant.
- $f_v$  constant in the atoms of  $\mathcal{P}_k \Leftrightarrow mf_v$  constant in the atoms of  $\mathcal{P}_k$ .
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- $f_v$  coboundary  $\Leftrightarrow mf_v$  coboundary.
- By Proposition (\*), φ<sub>n</sub> belongs to β(R<sub>1</sub>(X<sub>σ</sub>)) for all n large enough.

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Coboundaries and balance in words

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#### Corollary

Let  $\sigma$  be the Thue-Morse substitution and let (X, T) the symbolic system generated by  $\sigma$ . For every  $\ell \geq 2$ , (X, T) is not balanced in the factors of length  $\ell$ .

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$$\phi_{n}(ab) = \alpha_{ab} \left(1 - \frac{1}{q_{v}}\right) + \left(|\sigma^{n}(a)| - \alpha_{ab}\right) \cdot -\frac{1}{q_{v}}$$

$$\alpha_{ab} = \#\{0 \le j < |\sigma^{n}(a)| : T^{j}\sigma^{n}([ab]) \subseteq [v]\}.$$

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## Return words

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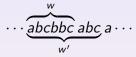
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### Return words

• Let  $\sigma$  be a primitive substitution on  $\mathcal{A}$ ,  $a \in \mathcal{A}$ . A return word to a in  $\mathcal{L}(X_{\sigma})$  is a finite word  $w \in \mathcal{L}(X_{\sigma})$  such that  $w_0 = a$  and  $wa \in \mathcal{L}(X_{\sigma})$ .

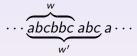
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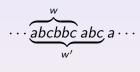
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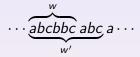
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Coboundaries and balance in words

Let  $v \in \mathcal{L}(X_{\sigma})$  such that  $\mu_{v} \in \mathbb{Q}$ . Write  $\mu_{v} = p_{v}/q_{v}$  with  $(p_{v}, q_{v}) = 1$ . There exists  $N \ge 1$  such that for all  $a \in \mathcal{A}$  and for all return word w to a,  $q_{v}$  divides  $|\sigma^{n}(w)|$  for all  $n \ge N$ . In particular, if  $aa \in \mathcal{L}_{2}(X)$ , then  $q_{v}$  divides  $|\sigma^{n}(a)|$  for all  $n \ge N$ .

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where  $\alpha_{ab} = \#\{0 \leq j < |\sigma^n(a)| : T^j \sigma^n([ab]) \subseteq [v]\}.$ 

#### • We obtain

$$0 = \alpha_{w_{|w|-1}a}(q_{v} - p_{v}) - (|\sigma^{n}(w_{|w|-1})| - \alpha_{w_{|w|-1}a}) \cdot p_{v} + \sum_{i=1}^{|w|-1} \alpha_{w_{i-1}w_{i}}(q_{v} - p_{v}) - (|\sigma^{n}(w_{i-1})| - \alpha_{w_{i-1}w_{i}}) \cdot p_{v}$$

### • which implies

$$q_{v}\left(\alpha_{w_{|w|-1}a} + \sum_{i=1}^{|w|-1} \alpha_{w_{i-1}w_{i}}\right) = p_{v}\left(|\sigma^{n}(w_{|w|-1})| + \sum_{i=1}^{|w|-1} |\sigma^{n}(w_{i-1})|\right)$$
$$= p_{v}|\sigma^{n}(w)|$$

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• Example:  $\sigma: \mathbf{0} \to \mathbf{001}, \mathbf{1} \to \mathbf{101}.$ 

$$M_{\sigma} = \left( egin{array}{cc} 2 & 1 \ 1 & 2 \end{array} 
ight) \quad E_{\sigma} = \{1,3\} \quad f_0 = f_1 = 1/2 \quad q_0 = q_1 = 2,$$

 $00 \in \mathcal{L}_2(X_{\sigma}) \Rightarrow 2 \text{ divides } 3^n \text{ for } n \text{ large enough } \rightarrow \leftarrow$ .

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$$\sigma : \{0, 1, 2, 3\} \to \{0, 1, 2, 3\}^* \text{ given by}$$
$$0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 23, 3 \mapsto 20$$
$$M_{\sigma} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad E_{\sigma} = \{0, -1, 1, 2\}$$
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- There are no aa's.
- But 01 is a return word to 0. If (X, T) is balanced on i,

$$\Rightarrow q_i$$
 divides  $|\sigma^n(01)| = 2^{n+1}$   $n \gg$ 

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