Discrete planes, topology and multidimensional continued fractions

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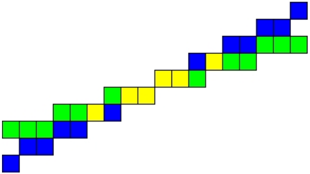
with Valérie Berthé, Éric Domenjoud, Timo Jolivet, Nadia Lafrenière, Xavier Provençal, Jean-Luc Toutant

Outline

Motivations of digital geometry



Discrete datas

⇓

Properties of euclidean geometry are not satisfied

Freeman, H. (1961)
Techniques for the digital computer analysis of chain-encoded arbitrary plane curves,
In *Proceedings of the National Electronics Conference*, Vol. 17, 1961, pp. 421-432.

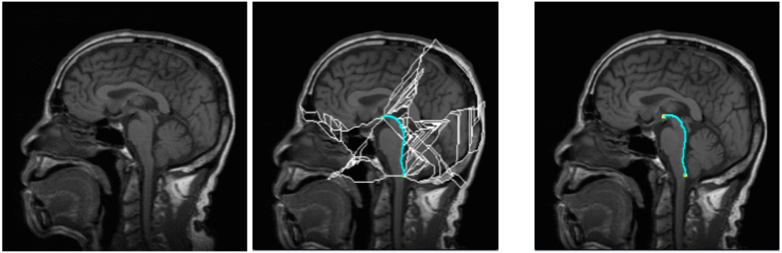
Rosenfeld, A. (1974).
Digital straight line segments.
In *IEEE Transactions on Computers*, pages 1264–1369.

Reveillès, J.-P. (1991).
Géométrie discrète, calcul en nombres entiers et algorithmique.
Thèse d'état, Université Louis Pasteur, Strasbourg.

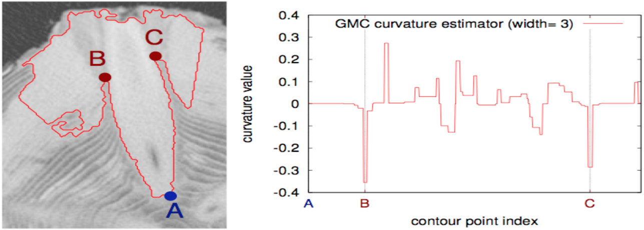
The computer representation of lines and curves has been an active subject of research for nearly half a century. "Related work even earlier on the theory of words (specifically, on mechanical or Sturmian words) remained unnoticed in the pattern recognition community."

Klette, R. and Rosenfeld, A (2004).
Digital Straightness : a review.
Discrete Applied Mathematics, 197:230–139.

Motivations of digital geometry



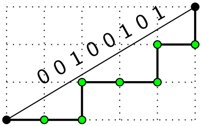
(a) patient (1) (b) all candidates (c)curvature value C=0.0540354



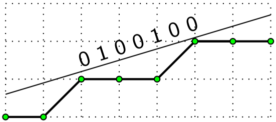
(a) (b)

(Source : Bertrand Kerautret and Adrien Krahenbühl, Loria, Nancy)

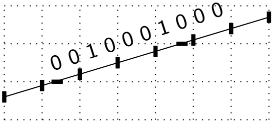
Mathematicians deal with. . .



Christoffel words



Mechanical words



Cutting sequences

abaababaabaababaababa . . .

Sturmian words. . .

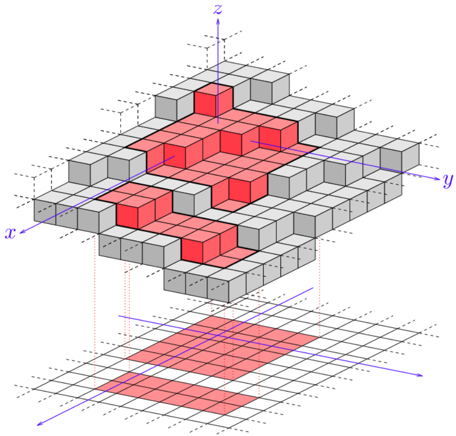
M. Morse and G. A. Hedlund (1940).
Symbolic Dynamics II. Sturmian trajectories.
Amer J. Math., 1-42(62)

E. Coven and G. A. Hedlund (1973).
Sequences with minimal block growth.
Math. Systems Theory, 8:138–153.

Tools

Combinatorics on words, arithmetic, number theory, dynamical systems. . .

What about higher dimensions ?



$\mathbb{Z}^2$-action over $[0,1[$: $\{\alpha i + \beta j + \mu\} \in [0, 0.5[\cup [0.5, 1[$

- Complexity : $p(m,n) \leq mn$
- The language is minimal
- Frequencies : length of an interval
- The set of configurations is closed under central symmetry...

Outline

Motivations

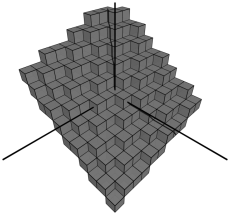
Connecting thickness

Generation of discrete planes

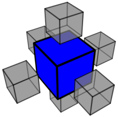
What about higher dimensions ?

Discrete hyperplanes (J.-P. Reveillès, 1995)

$$\mathrm{H}(\boldsymbol{\alpha}, \mu, \omega) = \{\mathbf{x} \in \mathbb{Z}^n \mid 0 \leq \langle \boldsymbol{\alpha}, \mathbf{x} \rangle + \mu < \omega\}$$



A discrete plane in dimension 3



2-adjacency in dimension 3

Adjacency and Connectedness

1. Let $\kappa \in \{0, \ldots, n-1\}$. Two distinct elements $\mathbf{x}$ and $\mathbf{x'}$ in $\mathbb{Z}^n$ are κ-adjacent if

$$\|\mathbf{x} - \mathbf{x'}\|_2^2 = (x_1 - x_1')^2 + \cdots + (x_n - x_n')^2 \leq n - \kappa$$

2. A subset S of $\mathbb{Z}^n$ is connected if, for each pair $(\mathbf{x}, \mathbf{x'})$, there exists a finite sequence $(\mathbf{x} = \mathbf{u}_1, \ldots, \mathbf{u}_k = \mathbf{x'})$ in S where $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$ are adjacent

The n-dimensional case : the separatingness is easy

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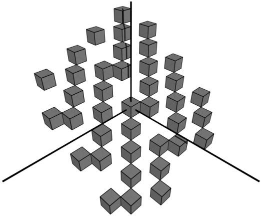
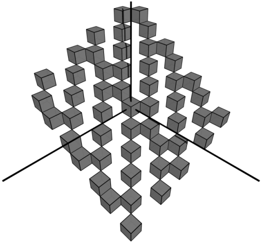


A 0-disconnected discrete plane



A 0-connected discrete plane

In the present work, we investigate the $(n-1)$-connectedness : "facet"-connectedness

Separatingness

$\mathrm{H}(\boldsymbol{\alpha}, \mu, \omega)$ is :

- 2-separating iff $\omega \geq \|\boldsymbol{\alpha}\|_\infty = \max\{|\alpha_1|, |\alpha_2|, |\alpha_3|\}$
- 1-separating iff $\omega \geq \max\{|\alpha_1| + |\alpha_2|, |\alpha_1| + |\alpha_3|, |\alpha_2| + |\alpha_3|\}$
- 0-separating iff $\omega \geq \|\boldsymbol{\alpha}\|_1 = |\alpha_1| + |\alpha_2| + |\alpha_3|$

The 2-dimensional case: the connectedness

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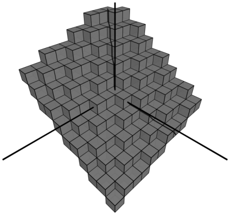

An exemple of 0-connectedness : 0-connected

In the present work, we are interested in the 2-connectedness : "facet"-connectedness

Question

How much is $\Omega(\boldsymbol{\alpha}, \mu) = \inf \{\omega \in \mathbb{R} \mid \mathrm{H}(\mathbf{v}, \mu, \omega) \text{ is 2-connected}\}$?

First properties

1. $\Omega(\mathbf{v}, \mu) \geq \|\mathbf{v}\|_\infty$
2. $\Omega((0, 0, x), \mu) = \mu \bmod |x|$

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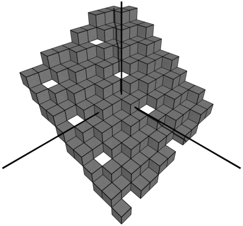

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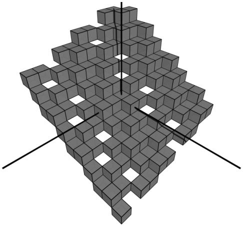

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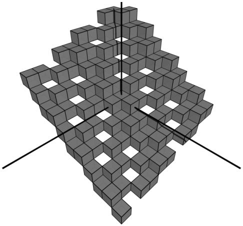

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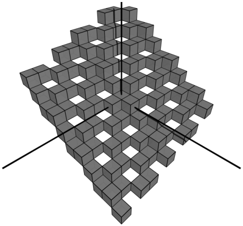

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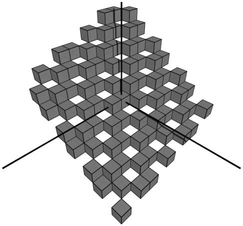

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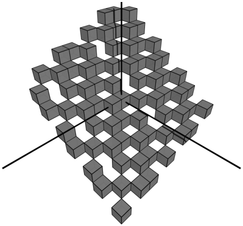

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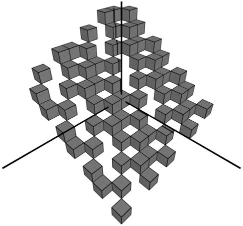

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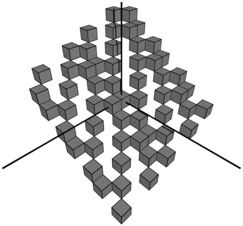

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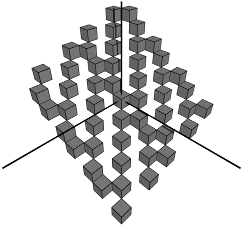

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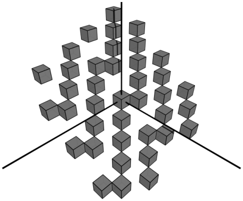

An exemple of 0-connectedness : 0-unconnected

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The fully subtractive reduction

Theorem (J.-Toutant 2009)

Let $\mathbf{v} \in \mathbb{R}_+^3$ *,* $\mu \in \mathbb{R}_+$ *and* $M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Longrightarrow \Omega(\mathbf{v}, \mu) = \Omega(M \cdot \mathbf{v}, \mu) - v_1$.

$$
\begin{array}{ccccc}
\mathbf{v} & \longmapsto & \mathbf{v}' = M \cdot \mathbf{v} & \longmapsto & \mathbf{v} \\
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} & \overset{M}{\longmapsto} & \begin{pmatrix} v_1 \\ v_1 + v_2 \\ v_1 + v_3 \end{pmatrix} = \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} & \overset{M^{-1}}{\longmapsto} & \begin{pmatrix} v_1' \\ v_2' - v_1' \\ v_3' - v_1' \end{pmatrix}
\end{array}
$$

Proof.

Since $\langle \mathbf{x}, \mathbf{v} \rangle = \langle \, {}^t M^{-1} \cdot \mathbf{x}, M \cdot \mathbf{v} \rangle$, then it remains to show that, for each $\omega \in \mathbb{R}_+$:

$$\mathbf{H}(\mathbf{v}, \mu, \omega) \text{ is } 2-\text{connected} \iff \mathbf{H}(M \cdot \mathbf{v}, \mu, \omega + v_1) \text{ is } 2-\text{connected}$$

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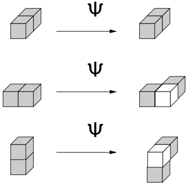

$$\begin{array}{ccc} \mathbf{H}(\mathbf{v}, \mu, \omega) & \longrightarrow & \mathbf{H}(M \cdot \mathbf{v}, \mu, \omega + v_1) \\ \mathbf{x} & \longmapsto & {}^tM^{-1} \cdot \mathbf{x} \end{array}$$

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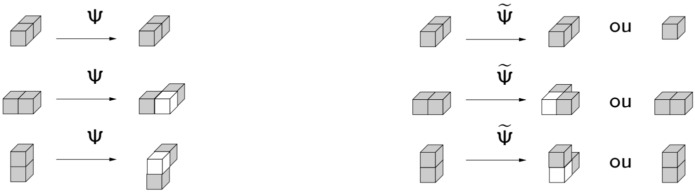

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$$\begin{array}{ccc} \mathbf{H}(M \cdot \mathbf{v}, \mu, \omega + v_1) & \longrightarrow & \mathbf{H}(\mathbf{v}, \mu, \omega) \\ \mathbf{x} & \longmapsto & \left| \begin{array}{l} {}^tM \cdot \mathbf{x} \text{ if } \langle \mathbf{xM} \cdot \mathbf{v} \rangle \in [0, \omega[\\ {}^tM \cdot (\mathbf{x} - \mathbf{e_1}) \text{ otherwise} \end{array} \right. \end{array}$$

If $\dim_{\mathbb{Q}}\{v_1, v_2, v_3\} = 1$, the one computes $\Omega(\mathbf{v}, , \mu)$ as follows:

Input A vector $\mathbf{v} \in \mathbb{R}^3_+$ and $\mu \in \mathbb{R}$.

Init.
- $\mathbf{v} \leftarrow \mathtt{sort}(\mathbf{v})$
- $\omega \leftarrow 0$

Loop `while` $\mathbf{v}$ has, at least, two non-zero components
- substract the smallest non-zero component $\min_{\mathbf{v}}$ of $\mathbf{v}$ the others and sort;
- $\omega \leftarrow \omega + min_{\mathbf{v}}$;

`end-of-while`

Output `return` $\omega + \mu \bmod v_3$.

The fully subtractive reduction

The rational case, i.e. $\dim_{\mathbb{Q}}\{v_1, v_2, v_3\} = 1$

If $\dim_{\mathbb{Q}}\{v_1, v_2, v_3\} = 1$, the one computes $\Omega(\mathbf{v}, , \mu)$ as follows:

Input A vector $\mathbf{v} \in \mathbb{R}^3_+$ and $\mu \in \mathbb{R}$.

Init.
- $\mathbf{v} \leftarrow \mathtt{sort}(\mathbf{v})$
- $\omega \leftarrow 0$

Loop `while` $\mathbf{v}$ has, at least, two non-zero components
- substract the smallest non-zero component $\min_{\mathbf{v}}$ of $\mathbf{v}$ the others and sort;
- $\omega \leftarrow \omega + min_{\mathbf{v}}$;

`end-of-while`

Output `return` $\omega + \mu \bmod v_3$.

Question

What about if $\dim_{\mathbb{Q}}\{v_1, v_2, v_3\} > 1$?

The fully subtractive reduction

Let $\beta = \frac{1}{2}(\sqrt{5} - 1) \simeq 0.618\ldots$, $\underline{K > \beta + 1}$ and $\mathbf{v} = (\beta, 1, K)$.

1. One has $1 - \beta = \beta^2$ and :

$$\begin{aligned}
\Omega\left((\beta, 1, K), \mu\right) &= \Omega\left((\beta^2, \beta, K - \beta), \mu\right) + \beta \\
&= \Omega\left((\beta^3, \beta^2, K - \beta - \beta^2), \mu\right) + \beta + \beta^2 \\
&\ldots \\
&= \Omega\left(\left(\beta^{n+1}, \beta^n, K - \beta\frac{1 - \beta^n}{1 - \beta}\right), \mu\right) + \beta\frac{1 - \beta^n}{1 - \beta}
\end{aligned}$$

2. At *limit*, one shall obtain:

$$\begin{aligned}
\Omega\left((\beta, 1, K), \mu\right) &= \Omega\left(\left(0, 0, K - \frac{\beta}{1 - \beta}\right), \mu\right) + \frac{\beta}{1 - \beta} \\
&= \mu \bmod \left(K - \frac{\beta}{1 - \beta}\right) + \frac{\beta}{1 - \beta} \\
&= \mu \bmod \left(K - (1 + \beta)\right) + 1 + \beta \\
&< K = \|\mathbf{v}\|_\infty \qquad \text{IMPOSSIBLE !}
\end{aligned}$$

Fact

The map $\Omega : \mathbb{R}_+^3 \longmapsto \mathbb{R}$ is not continuous.

If $\dim_{\mathbb{Q}}\{v_1, v_2, v_3\} > 1$ then $\Omega_2(\mathbf{v}, \mu) \neq \Omega_2(\mathbf{v}^{(\infty)}, \mu) + \omega^{(\infty)}$

Lemma

$$\begin{aligned} \omega \geq \|\mathbf{v}\|_\infty + \min\{v_1, v_2, v_3\} \quad &\Longrightarrow \quad \mathbf{H}(\mathbf{v}, \mu, \omega) \textit{ est 2-connexe} \\ \Omega(\mathbf{v}, \mu) \quad &\leq \quad \|\mathbf{v}\|_\infty + \min\{v_1, v_2, v_3\} \end{aligned}$$

Theorem (Domenjoud-J.-Toutant 2009)

$$\dim_{\mathbb{Q}}\{v_1, v_2, v_3\} > 1 \implies \Omega(\mathbf{v}) := \Omega(\mathbf{v}, \mu) = \|\mathbf{v}^{(\infty)}\|_\infty + \omega^{(\infty)}$$

Proof.

- For each $k \geq 0$, $\dim_{\mathbb{Q}}\{v_1, v_2, v_3\} = \dim_{\mathbb{Q}}\{v_1^{(k)}, v_2^{(k)}, v_3^{(k)}\} > 1$, and $\mathbf{v}^{(k)}$ always has, at least, two non-zero components
- Moreover $\|\mathbf{v^{(k)}}\|_\infty \leq \Omega(\mathbf{v^{(k)}}, \mu) \leq \|\mathbf{v^{(k)}}\|_\infty + \omega^{(k+1)} - \omega^{(k)}$
- $$\underbrace{\|\mathbf{v^{(k)}}\|_\infty + \omega^{(k)}}_{\substack{\downarrow \\ \|\mathbf{v}^{(\infty)}\|_\infty + \omega^{(\infty)}}} \leq \underbrace{\Omega(\mathbf{v^{(k)}}, \mu) + \omega^{(k)}}_{\substack{\| \\ \Omega(\mathbf{v}, \mu)}} \leq \underbrace{\|\mathbf{v^{(k)}}\|_\infty + \omega^{(k+1)}}_{\substack{\downarrow \\ \|\mathbf{v}^{(\infty)}\|_\infty + \omega^{(\infty)}}}$$

□

Corollary

$$v_1 + v_2 \leq v_3 \implies \Omega(\mathbf{v}) = v_3$$

Termination condition

Theorem (R. Fokkink, C. Kraaikamp, H. Nakada, 2011)

For almost every $\mathbf{v} \in \mathbb{R}^n$*, there exists* $k \in \mathbb{N}^\star$*, such that* $\mathbf{v}_1^{(k)} + \mathbf{v}_2^{(k)} < \mathbf{v}_3^{(k)}$

Corollary

- *For almost every* $\mathbf{v} \in \mathbb{R}^n$, $\Omega(\mathbf{v})$ *is computable:*

$$\Omega(\mathbf{v}, \mu) = \|\mathbf{v}^{(\mathbf{k})}\|_\infty + \sum_{i=1}^{k-1} \mathbf{v}_1^{(i)}, \textit{ with } k \in \mathbb{N} \textit{ s. t. } \mathbf{v}_1^{(k)} + \mathbf{v}_2^{(k)} < \mathbf{v}_3^{(k)}$$

- *In particular, if* $\mathbf{v}_1 + \mathbf{v}_2 < \mathbf{v}_3$*, then* $\Omega(\mathbf{v}) = \|\mathbf{v}\|_\infty$

R. Fokkink, C. Kraaikamp and H. Nakada (2011)
On Schweiger?s conjectures on fully subtractive algorithms,
In *Israel Journal of Mathematics 186*, 285-296 (2011).

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

A particular case

$$
\begin{array}{rcccl}
\phi & : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\
& & \mathbf{v} & \longmapsto & \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} v_2 - v_1 \\ v_3 - v_1 \\ v_1 \end{pmatrix}
\end{array}
$$

Let α be the real eigenvalue of ϕ :

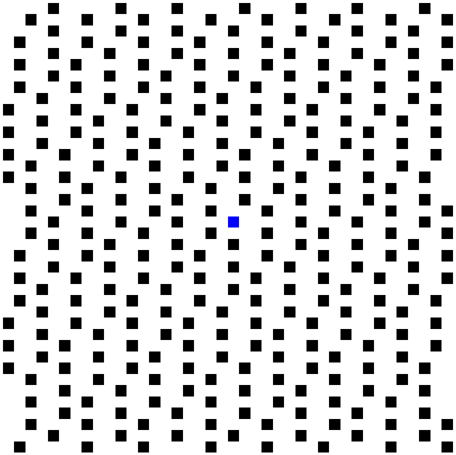
- $\alpha^3 + \alpha^2 + \alpha - 1 = 0$;
- $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$ is an eigenvector of ϕ associated to α;
- $v_1 + v_2 > v_3$ so $v_1^{(k)} + v_2^{(k)} > v_3^{(k)}$ with $\mathbf{v^{(k)}} = \phi^k(\mathbf{v})$.

$$\Omega_2(\mathbf{v}, \mu) = \Omega_2(\phi(\mathbf{v}), \mu) + \alpha = \Omega_2(\phi^2(\mathbf{v}), \mu) + \alpha^2 + \alpha = \cdots = \Omega_2(\underbrace{\phi^k(\mathbf{v})}_{\mathbf{v}^{(k)}}, \mu) + \underbrace{\alpha \frac{1 - \alpha^k}{1 - \alpha}}_{\omega^{(k)}}$$

Hence $\Omega_2(\mathbf{v}, \mu) = \|\phi^\infty(\mathbf{v})\|_\infty + \omega^{(\infty)} = \dfrac{\alpha}{1 - \alpha}$

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Projection of $\mathbf{H}\left(\mathbf{v}, 0, \alpha\frac{1-\alpha^n}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha+\alpha^2, 1)$ and $\alpha^3+\alpha^2+\alpha = 1$



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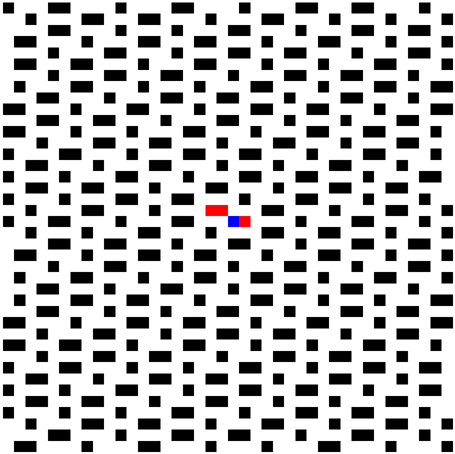

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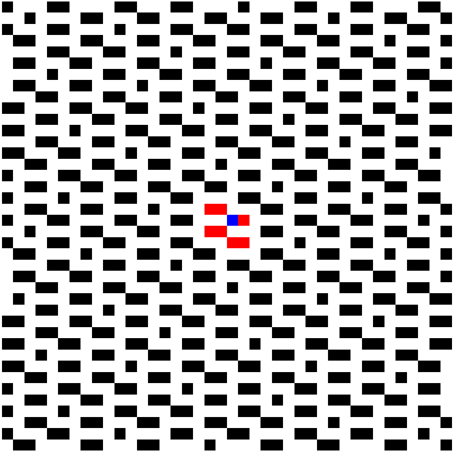

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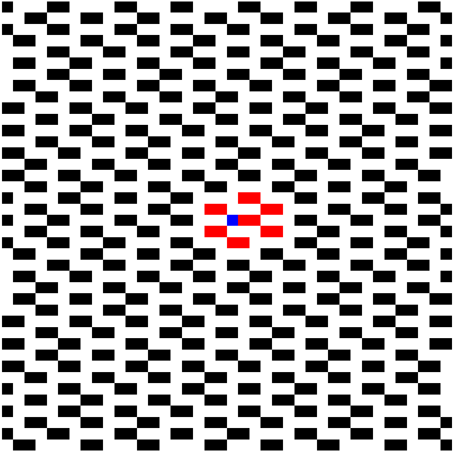

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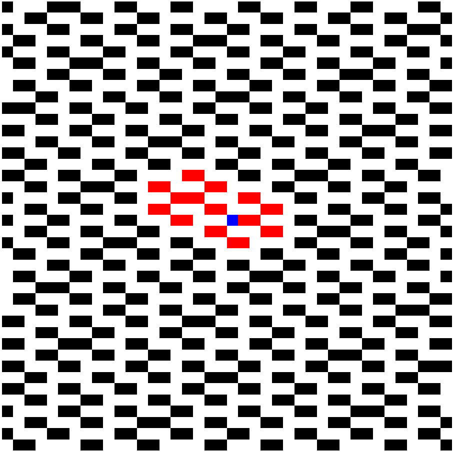

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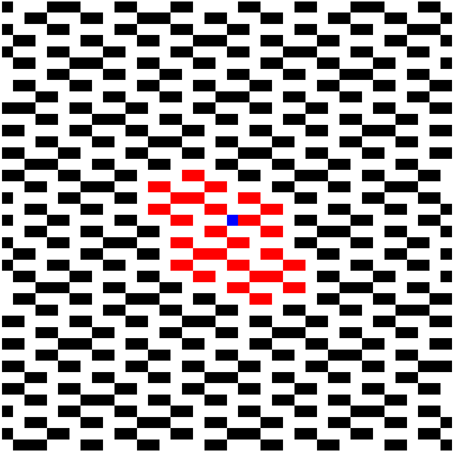

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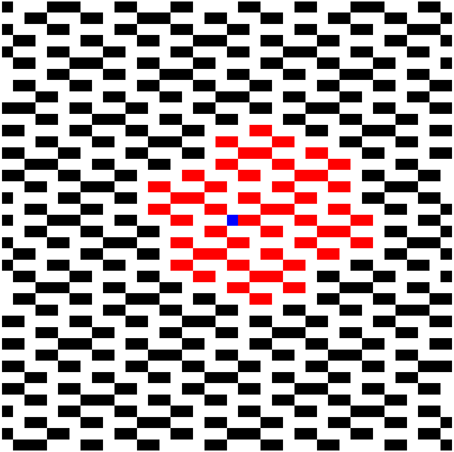

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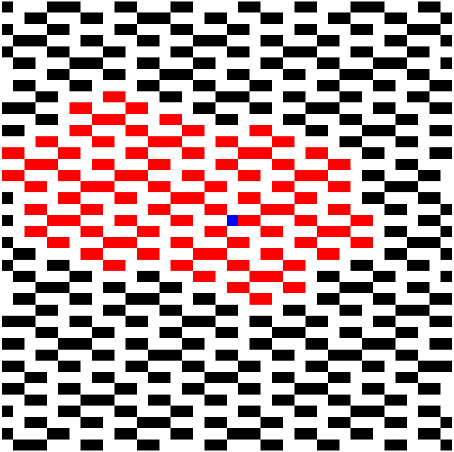


Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Iterative construction of $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha+\alpha^2, 1)$ and $\alpha^3+\alpha^2+\alpha = 1$

$$
\begin{array}{rcccl}
t & : & \mathbb{N}^\star & \longrightarrow & \mathbb{Z}^3 \\
 & & n & \longmapsto & \left| \begin{array}{ll} \mathbf{t_1} = \mathbf{e_1} & \\ \mathbf{t_2} = -\mathbf{e_1} + \mathbf{e_2} & \\ \mathbf{t_3} = -\mathbf{e_2} + \mathbf{e_3} & \\ \mathbf{t_n} = \mathbf{t_{n-3}} - \mathbf{t_{n-2}} - \mathbf{t_{n-1}} & \text{if } n > 3 \end{array} \right.
\end{array}
$$

$$
\begin{array}{rcccl}
\mathbf{P} & : & \mathbb{Z} & \longrightarrow & \mathfrak{P}\left(\mathbb{Z}^3\right) \\
 & & n & \longmapsto & \left| \begin{array}{ll} \mathbf{H}_n = \{(0,0,0)\} & \text{if } n = 0, \\ \mathbf{H}_n = \mathbf{H}_{n-1} \cup (\mathbf{H}_{n-1} + \mathbf{t}_n) & \text{if } n > 0 \end{array} \right.
\end{array}
$$

Since $\dim_{\mathbb{Q}}\{\alpha, \alpha+\alpha^2, 1\} = 3$, then $\mathbf{H}_n \simeq \{\langle \mathbf{x}, \mathbf{v} \rangle,\ \mathbf{x} \in \mathbf{H}_n\}$.

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Iterative construction of $\mathbf{H}\left(\mathbf{v}, 0, \dfrac{\alpha}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$ and $\alpha^3 + \alpha^2 + \alpha = 1$

$\mathbf{H}_0 \cong \{0\}$

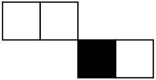
Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Iterative construction of $\mathbf{H}\left(\mathbf{v}, 0, \dfrac{\alpha}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$ and $\alpha^3 + \alpha^2 + \alpha = 1$

$$\mathbf{H}_1 \cong \{0, \alpha\}$$

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

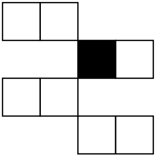
Iterative construction of $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$ and $\alpha^3 + \alpha^2 + \alpha = 1$



$$\mathbf{H}_2 \cong \{0, \alpha, \alpha^2, \alpha + \alpha^2\}$$

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

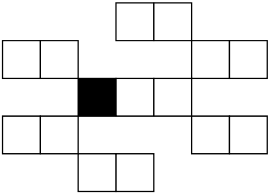
Iterative construction of $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$ and $\alpha^3 + \alpha^2 + \alpha = 1$



$$\mathbf{H}_3 \cong \{0, \alpha, \alpha^2, \alpha + \alpha^2, \alpha^3, \alpha, \alpha^2 + \alpha^3, \alpha + \alpha^2 + \alpha^3\}$$

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Iterative construction of $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$ and $\alpha^3 + \alpha^2 + \alpha = 1$



$$\mathbf{H}_4 \cong \{0\} \cup \left\{ \sum_{i=1}^{4} \varepsilon_i \alpha^i,\ \varepsilon \in \{0,1\}^4 \right\}$$

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

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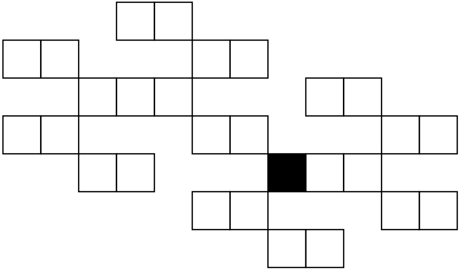
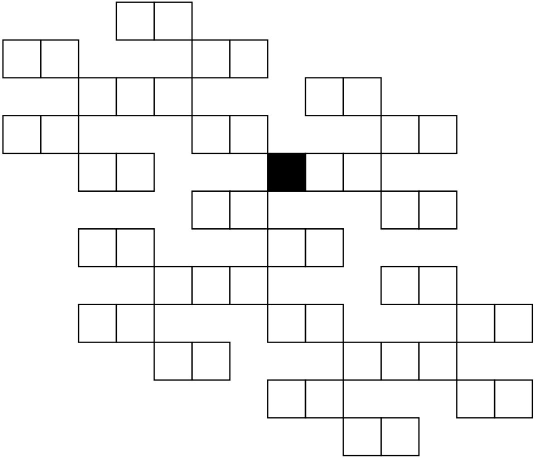


$$\mathbf{H}_5 \cong \{0\} \cup \left\{\sum_{i=1}^{5} \varepsilon_i \alpha^i,\ \varepsilon \in \{0,1\}^5\right\}$$

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Iterative construction of $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$ and $\alpha^3 + \alpha^2 + \alpha = 1$



$\mathbf{H}_6 \cong \{0\} \cup \left\{\sum_{i=1}^{6} \varepsilon_i \alpha^i,\ \varepsilon \in \{0,1\}^6\right\}$

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Iterative construction of $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$ and $\alpha^3 + \alpha^2 + \alpha = 1$

Lemma

The set $\mathbf{H}_n$ is connected.

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Iterative construction of $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$, with $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$ and $\alpha^3 + \alpha^2 + \alpha = 1$

Lemma

The set $\mathbf{H}_n$ is connected.

Proof.

- For each $n \in \mathbb{N}$, $\mathbf{H}_n \cong \{0\} \cup \left\{ \sum_{i=1}^{n} \varepsilon_i \alpha^i,\ \varepsilon \in \{0,1\}^n \right\}$
- For each $n \in \mathbb{N}^\star$ and each $k \leq n$, $\alpha^k \in \mathbf{H}_n$.
- For each $n \geq 4$,

$$\begin{aligned} \alpha^{n-3} &\in \mathbf{H}_{n-1} \\ \alpha^{n-3} &= \alpha^n + \alpha^{n-1} + \alpha^{n-2} \in \alpha^n + \mathbf{H}_{n-1} \end{aligned}$$

- Hence $\mathbf{H}_{n-1} \cap (\mathbf{H}_{n-1} + \alpha^n) \neq \emptyset$
- By induction...

□

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Theorem (Frougny & Solomyak, 1992)

- *Let $\beta > 1$ be an algebraic integer.*

$$\mathrm{Fin}(\beta) = \left\{ \sum_{i \in \mathbf{D}} x_i \beta^i,\ \mathbf{D} \subset \mathbb{Z}, |\mathbf{D}| < \infty,\ x_i \in \{0, \dots, \lfloor \beta \rfloor\} \right\}$$

- *Let $M_\beta(X) = X^d - a_{d-1}X^{d-1} - \cdots - a_1 X - a_0$ be the minimal polynomial of β.*

If $(a_{d-1}, a_{d-2}, \dots, a_1, a_0)$ is positive and non-increasing, then $\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]$.

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

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With $\beta = \alpha^{-1}$,

Corollary 1

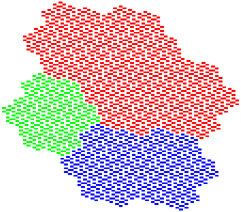
$$\lim_{n \to \infty} \mathbf{H}_n = \mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$$

Corollary 2

The discrete plane $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$ is connected.

Christiane Frougny and Boris Solomyak
Finite beta-expansions,
In *Ergodic Theory and Dynamical Systems* , 12 (4), 713-723 (1992).

More about the topology of $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$ ...



The connected components of $\mathbf{H}_{14} \setminus \{(0,0,0)\}$

Property

1. $\mathbf{H}_n \setminus \{(0,0,0)\}$ *(resp.* $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$*) has exactly 3 distinct connected components*
2. $\mathbf{H}_n$ *(resp.* $\mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right)$*) is a tree with rooted in* $(0,0,0)$

Is $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ connected ?

Theorem (Berthé-Jolivet-J.-Provençal 2013)

The discrete plane $\mathbf{H}(\mathbf{v}, 0, \Omega(\mathbf{v}))$ *is connected if and only if :*

- *either* $\mathbf{v} \in \mathcal{F}_3 = \left\{ \mathbf{v} \in \mathbb{R}_+^3 \mid v_1 \leq v_2 \leq v_3 \text{ and } v_1^{(n)} + v_2^{(n)} > v_3^{(n)}, \forall n \in \mathbb{N} \right\}$
- *or there exists* $n \in \mathbb{N}$ *such that* $v_1^{(n)} = 0$ *with* $\dim_{\mathbb{Q}}(v_2^{(n)}, v_3^{(n)}) = 2$.

Lemma (Berthé-Jolivet-J.-Provençal 2013)

If $\mathbf{v} \in \mathcal{F}_3$*, then* $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 3$.

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Lemma (Berthé-Jolivet-J.-Provençal 2013)

If $\mathbf{v} \in \mathcal{F}_3$, *then* $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 3$.

And if $\mu \neq 0$?

Question

Let $\mu \in \mathbb{R}$, is $\mathbf{H}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ connected ?

Is $\mathbf{H}(\mathbf{v}, \mu, \Omega(\mathbf{v}))$ connected ?

An example ?

Let $\mu = \dfrac{\alpha}{1-\alpha}$.

$$
\begin{aligned}
\mathbf{x} \in \mathbf{H}\left(\mathbf{v}, \mu, \frac{\alpha}{1-\alpha}\right) \quad &\iff \quad 0 \leq \langle \mathbf{x}, \mathbf{v} \rangle + \frac{\alpha}{1-\alpha} < \frac{\alpha}{1-\alpha} \\
&\iff \quad -\frac{\alpha}{1-\alpha} \leq \langle \mathbf{x}, \mathbf{v} \rangle < 0 \\
&\iff \quad 0 < \langle -\mathbf{x}, \mathbf{v} \rangle \leq \frac{\alpha}{1-\alpha} \\
&\iff \quad 0 < \langle -\mathbf{x}, \mathbf{v} \rangle < \frac{\alpha}{1-\alpha} \text{ car } \frac{\alpha}{1-\alpha} \notin \mathbb{Z}[\alpha] \\
&\iff \quad -\mathbf{x} \in \mathbf{H}\left(\mathbf{v}, 0, \frac{\alpha}{1-\alpha}\right) \setminus \{(0,0,0)\}
\end{aligned}
$$

and the discrete plane $\mathbf{H}\left(\mathbf{v}, \mu, \dfrac{\alpha}{1-\alpha}\right)$ is not connected

Open question

Any ideas for the general case ?

Outline

Matricial view

	Euclid algorithm	Approx.
n	v_n	a_n
0	$(\underline{7}, 9)$	$(1, 1)$
	$\downarrow$	$\downarrow$
1	$(7, \underline{2})$	$(1, 2)$
	$\downarrow$	$\downarrow$
2	$(5, \underline{2})$	$(2, 3)$
	$\downarrow$	$\downarrow$
3	$(3, \underline{2})$	$(3, 4)$
	$\downarrow$	$\downarrow$
4	$(\underline{1}, 2)$	$(4, 5)$
	$\downarrow$	$\downarrow$
5	$(1, 1)$	$(7, 9)$

Euclid algorithm

Given a vector (x, y), return

- $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ if $x < y$,
- $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ if $x > y$,
- **stop** if $x = y$.

Given a vector $v \in (\mathbb{N} \setminus \{0\})^2$, let :

- $v_0 = v$,
- For all $n \geq 1$: $\begin{cases} M_n = \textbf{Euclid}(v_{n-1}) \\ v_n = M_n v_{n-1}. \end{cases}$

Proposition

- $v_n = M_n M_{n-1} \cdots M_1 v$
- $a_n = M_1^{-1} M_2^{-1} \cdots M_n^{-1} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$

The Translation-Union Construction

Construction

[Domenjoud, Vuillon 12],
[Berthé, Jamet, Jolivet, P. 2013]

Let $v_0 = v$, $B_0 = \{\mathbf{0}\}$ and for all $n \geq 1$ let :

M_n : the matrix selected from v_{n-1},

$v_n = M_n v_{n-1}$

δ_n : the index of the coordinate of v_{n-1} that is subtracted,

$T_n = M_1^\top \cdots M_n^\top e_{\delta_n}$, *(translation)*

$B_n = B_{n-1} \cup (T_n + B_{n-1})$, *(body)*

$H_n = \sum_{i \in \{1,\dots,n\}} T_i$, *(highest point)*

$L_n = H_n + \{M_1^\top \cdots M_n^\top e_i\}$. *(legs)*

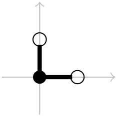
Note that:

$H_n \in B_n$,

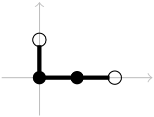
$L_n \cap B_n = \emptyset$.

● $\in B_n$, ○ $\in L_n$

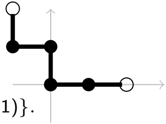
$v_0 = (2,3)$,
$a_0 = (1,1)$
$H_0 = (0,0)$,
$L_0 = \{(1,0),(0,1)\}$.



$v_1 = (2,1)$, $\delta_1 = 1$
$a_1 = (1,2)$
$T_1 = (1,0)$
$H_1 = (1,0)$,
$L_1 = \{(2,0),(0,1)\}$.



$v_2 = (1,1)$, $\delta_2 = 2$
$a_2 = (2,3)$
$T_2 = (-1,1)$
$H_2 = (0,1)$,
$L_2 = \{(2,-1),(-1,1)\}$.



3D continued fraction algorithms

Euclid algorithm

Given two numbers subtract the smallest to the largest.

$(7,9) \to (7,2) \to (5,2) \to (3,2) \to (1,2) \to (1,1) \to (1,0)\circlearrowleft$

Given three numbers

- **Selmer** : subtract the smallest to the largest.
  $(3,7,5) \to (3,4,5) \to (3,4,2) \to (3,2,2) \to (1,2,2) \to (1,2,0)\circlearrowleft$.
- **Brun** : subtract the second largest to the largest.
  $(3,7,5) \to (3,2,5) \to (3,2,2) \to (1,2,2) \to (1,2,0) \to (1,1,0) \to (1,0,0)\circlearrowleft$.
- **Fully subtractive** : subtract the smallest to the two others.
  $(3,7,5) \to (3,4,2) \to (1,2,2) \to (1,1,1) \to (1,0,0)\circlearrowleft$.
- **Poincaré** : subtract the smallest to the mid and the mid to the largest.
  $(3,7,5) \to (3,2,2) \to (1,2,0) \to (1,1,0) \to (1,0,0)\circlearrowleft$.
- **Arnoux-Rauzy** : subtract the sum of the two smallest to the largest (not always possible).
  $(3,7,5) \to \text{impossible.}$
- ...

Example : Fully Subtractive $v = (6, 8, 11)$

Construction

Let $v_0 = v$, $B_0 = \{\mathbf{0}\}$ and for all $n \geq 1$ let :

M_n : the matrix selected from v_{n-1},

$v_n = M_n v_{n-1}$

δ_n : the index of the coordinate of v_{n-1} that is subtracted,

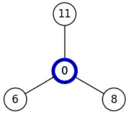
$T_n = M_1^\top \cdots M_n^\top e_{\delta_n}$, *(translation)*

$B_n = B_{n-1} \cup (T_n + B_{n-1})$, *(body)*

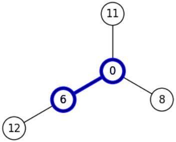
$H_n = \sum_{i \in \{1,\ldots,n\}} T_i$, *(highest point)*

$L_n = H_n + \{M_1^\top \cdots M_n^\top e_i\}$. *(legs)*

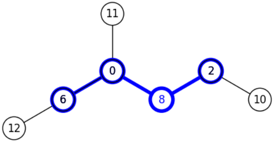
- Step 0 : $v_0 = (6, 8, 11)$, $a_0 = (1, 1, 1)$,



- Step 1 : $v_1 = (6, 2, 5)$, $a_1 = (1, 2, 2)$,

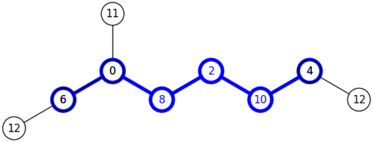


- Step 2 : $v_2 = (4, 2, 3)$, $a_2 = (2, 3, 4)$,

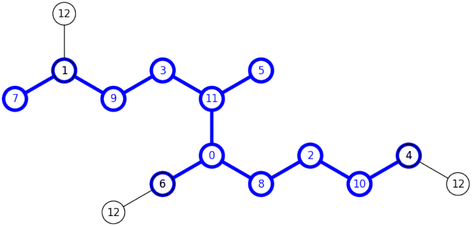


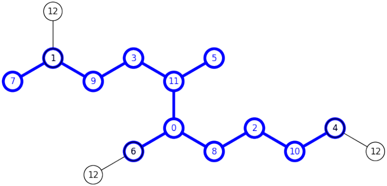
Example : Fully Subtractive $v = (6, 8, 11)$

- Step 3 : $v_3 = (2, 2, 1)$, $a_3 = (3, 4, 6)$,



- Step 4 : $v_4 = (1, 1, 1)$, $a_4 = (6, 8, 11)$,

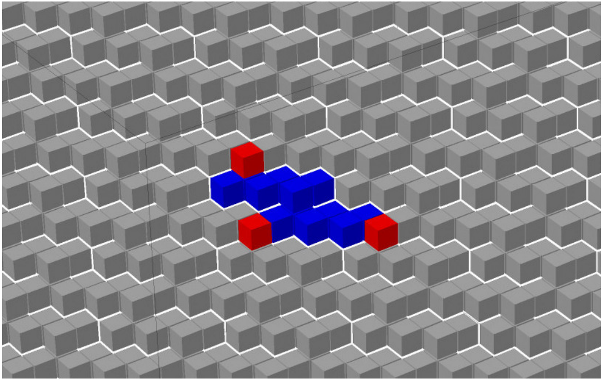




$\mathbf{H}((6, 8, 11), 13)$

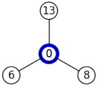
Expected properties of the pattern:

- Connected.
- Provides period vectors.
- Spans $\mathbf{H}(v, \omega)$ with these vectors.
- Should be as small as possible, to avoid redundancy.

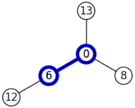


Example, Fully Subtractive $v = (6, 8, 13)$

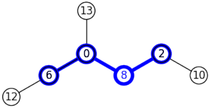
- Step 0 : $v_0 = (6, 8, 13)$, $a_0 = (1, 1, 1)$,



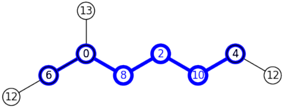
- Step 1 : $v_1 = (6, 2, 7)$, $a_1 = (1, 2, 2)$,



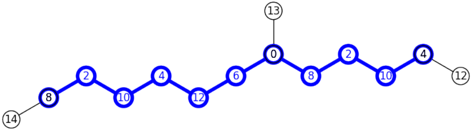
- Step 2 : $v_2 = (4, 2, 5)$, $a_2 = (2, 3, 4)$,



- Step 3 : $v_3 = (2, 2, 3)$, $a_3 = (3, 4, 6)$,



- Step 4 : $v_4 = (2, 0, 1)$, $a_4 = (5, 7, 11)$,



Fully Subtractive

Let $v \in (\mathbb{N} \setminus \{0\})^3$ with $\gcd(v) = 1$ and $(a, b, c) = \mathrm{sort}(v)$ (i.e. $a \leq b \leq c$) :

- If $a + b \leq c$ then let $(a', b', c') = \mathrm{sort}(\mathbf{FS}(v))$ then $a' + b' \leq c'$.
- If $a = b < c$, then one coordinate of $\mathbf{FS}(v)$ is 0.

Definition

Let $(a, b, c) = \mathrm{sort}(v)$, the vector v satisfies the condition **happy fully** if $a + b > c$ and $a \neq b$.

Definition

Let $\mathcal{K}$ be the set of vectors v such $\mathbf{FS}^N(v) = (1, 1, 1)$ for some $N \geq 1$.

Lemma

Let $v \in (\mathbb{N} \setminus \{0\})^3$, $v \in \mathcal{K}$ iff $\forall n$, $\mathbf{FS}^n(v)$ satisfies **happy fully**.

New generalized continued fraction algorithms

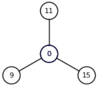
Let **X** denote algorithm **Brun** or **Selmer**.

Algorithm **FSX**
Input : $v \in \mathbb{N}^3$.

If v satisfies happy fully **then**
    Use **FS**.
else
    Use **X**.
end if

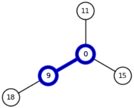
Example using **FSB**, $v = (9, 15, 11) \notin \mathcal{K}$

$v_0 = (9, 15, 11)$
$a_0 = (1, 1, 1)$



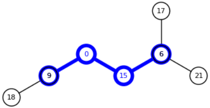
$\xrightarrow{\textbf{FS}}$

$v_1 = (9, 6, 2)$
$a_1 = (1, 2, 2)$



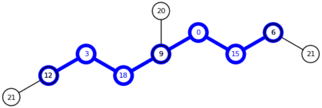
$\xrightarrow{\textbf{Brun}}$

$v_2 = (3, 6, 2)$
$a_2 = (2, 3, 3)$



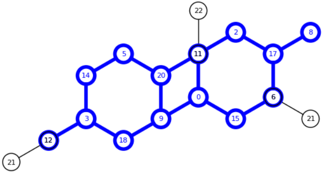
$\xrightarrow{\textbf{Brun}}$

$v_3 = (3, 3, 2)$
$a_3 = (3, 5, 4)$



$\xrightarrow{\textbf{FS}}$

$v_4 = (1, 1, 2)$
$a_4 = (6, 10, 7)$

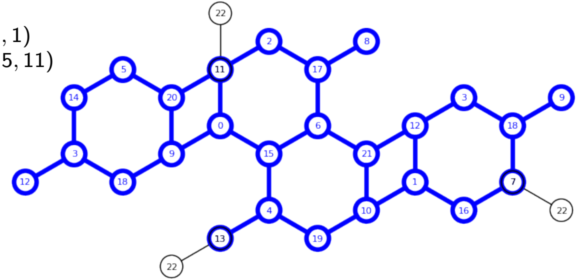


Theorem

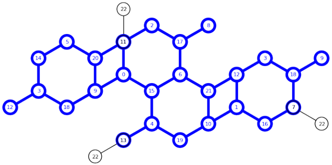
Using the algorithm **FSB** *or* **FSS***, for all vector* $v \in (\mathbb{N} \setminus \{0\})^3$ *with* $\gcd(v) = 1$*,*

1. $\exists N$ *such that* $v_N = (1, 1, 1)$.
2. *Vectors of* L_N *have same height, providing period vectors.*
3. $B_N \cup L_N$ *is connected.*
4. $B_N \cup L_N$ *spans* $\mathbf{P}(v, \omega)$ *with* $\frac{\|v\|_1}{2} \leq \omega < \|v\|_1$.

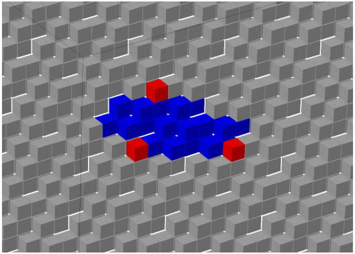
$\xrightarrow{\textbf{Brun}}$ $v_5 = (1, 1, 1)$
$a_5 = (9, 15, 11)$



Conclusion



$\mathbf{H}((9, 15, 11), 23)$



Good:

- Build a pattern that spans a digital plane for any rational normal vector.
- Construction is recursive and based on continued fractions algorithms.
- Generalizes Voss' *splitting formula* (equiv. *standard factorization* of Christoffel words) to higher dimensions.

Open questions :

- Find a GCF-algorithm that builds minimal patterns.
- Control the height of the pattern.
- Control the anisotropy of the patterns (avoid stretched forms in favor of *potato-likeness*).