S-adic systems associated with strongly convergent continued fractions

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Common work with Valérie Berthé, Milton Minervino, Wolfgang Steiner, Jörg Thuswaldner

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Caen, rencontre ANRDYNA3S

Continued fraction and rotations dynamic viewpoint

- The usual continued fraction of α is related to the dynamics of rotation R_α.
- It can be seen as induction of R_α on the smallest interval .
- (where R_α is an exchange of 2 intervals).
- Or as Euclid's algorithm on 1 and α.

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- Note that we use a second direction (natural extension).

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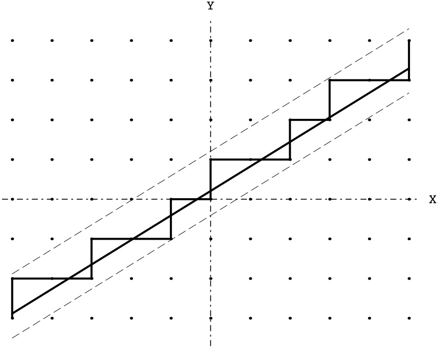

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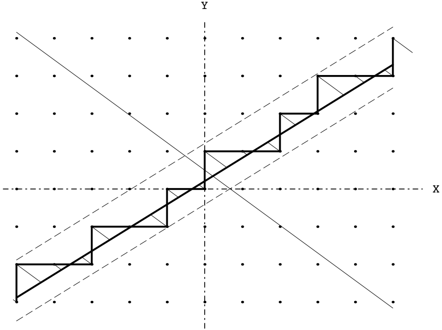

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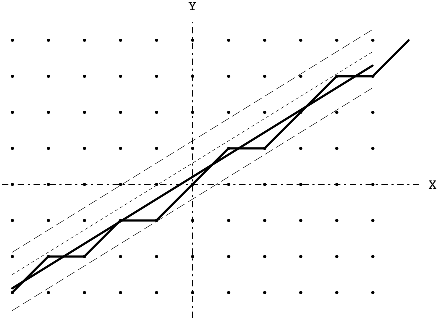

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- This is the geodesic flow on the modular surface $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$.

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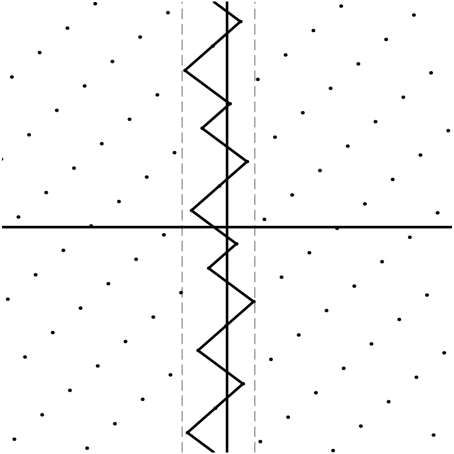

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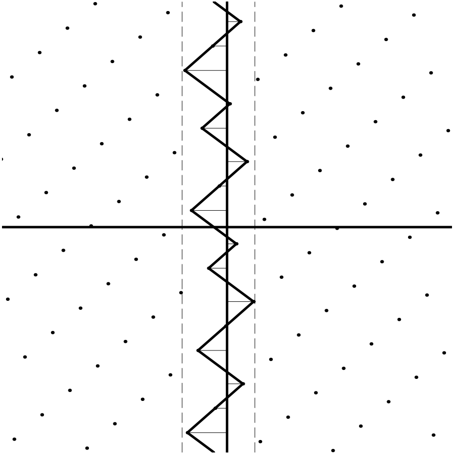

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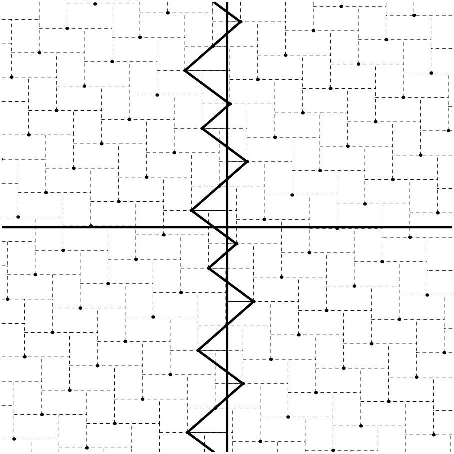

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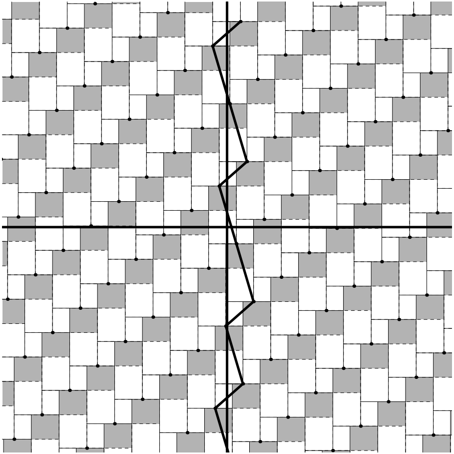

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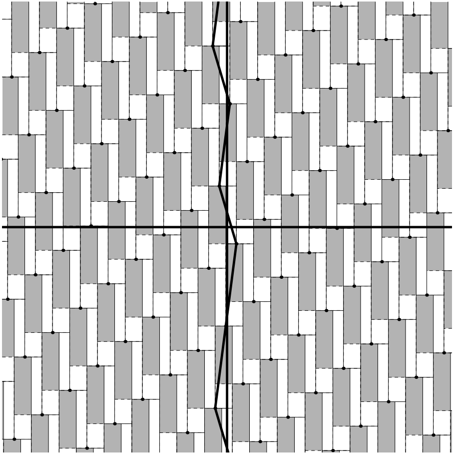

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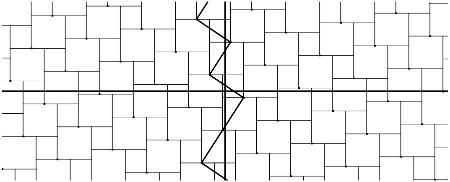

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- there is a dual viewpoint (replace $\mathbb{Z}^2$ by a lattice).
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Induction of a dynamical system

- Dynamical system $T : X \to X$.
- $A \subset X$.
- The induced map of T on A is
- $T_{|A} : A \to A$.
- $T_{|A}(x) = T^{n_x}(x)$, where $n_x = \inf\{n > 0 | T^n(x) \in A\}$.
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- Generalized unimodular continued fraction :
- A piecewise projective map T on a cone $\Lambda \subset \mathbb{R}^d$
- It is associated with a map $A : \Lambda \to GL(d, \mathbb{Z})$.
- $T(x) = A(x)^{-1}x$.
- In many cases, we can build a natural extension
- $\tilde{T}(x, y) = (A(x)^{-1}x, {}^tA(x)y)$.

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- We want to find a family of systems R_α.
- where α is in the domain of T.
- and a family of subsets A_α.
- such that $R_{\alpha|A_\alpha}$ is conjugate to $R_{T\alpha}$.
- this is difficult in dimension > 1.
- set A_α has special properties (bounded remainder set).
- It needs to have fractal boundaries.
- Just consider the periodic case.

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S-adic systems

- For each matrix A we choose a substitution σ_A in a countable set S, acting on the alphabet $\mathcal{A} = \{1, \ldots, d\}$.
- We consider infinite sequences $(\sigma_n) \in S^{\mathbb{N}}$.
- $\sigma_{[m,n)} = \sigma_m \ldots \sigma_{n-1}$.
- Basic property : the sequence is primitive is, for any m, there is n such that the corresponding matrix is positive.
- Limit point of the sequence : sequence of words $w_n \in \mathcal{A}^{\mathbb{Z}}$ such that $w_n = \sigma_n(w_{n+1})$.
- For a primitive sequence of substitutions, there is at most d^2 limit points and they all have the same language.
- We want to consider the limit point as a model set in $\mathbb{Z}^d \subset \mathbb{R}^d$, and find its window.
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S-adic systems : problems

- Does the limit word have a well-defined frequency ?
- Is it contained in a bounded window ?
- Is the projection space totally irrational ?
- Is it a model set, or only a part of the model set ?
- Is it associated to a translation of a torus, or is it a factor ?
- This imposes conditions on the sequence of substitutions, and on the continued fraction.
- There is a type of "Pisot" condition.

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S-adic system : recurrence

- We first suppose that the sequence σ_n is recurrent.
- It the continued fraction admits a Gauss measure, this is true almost everywhere.
- This implies that the limit words have a well-defined frequency.
- This a a generalisation of Perron Frobenius
- We have a projection direction for a model set.
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- We want the projection to be bounded.
- This is equivalent to a balance property :
- for two finite words U, V in the limit word
- $|U| = |V|$ implies $|U|_a - |V|_a < C$ for a fixed C
- C-balance property
- What condition ensures this ?

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S-adic systems : irrationality

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S-adic systems : Pisot condition

- We say that a sequence (A_n) of matrices in $GL(d, \mathbb{Z})$ satisfy the Pisot condition if :
- For any m there exists N such that, for all $n > N$:
- $A_{[m,n)}$ is of the Pisot type :
- All eigenvalues except one have modulus < 1.
- This implies that the characteristic polynomial is irreducible.
- Note that no eigenvalue is 0.
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Pisot condition and irrationality

- For a Pisot CF, almost any limit word has totally irrational frequency.
- Let w_n be a limit word with a rational relation.
- We can find an integer vector X such that, for any n, $< X, A_{[0,n)}e_i >$ is bounded.
- This implies that $< {}^tA_{[0,n)}X, e_i >$ is bounded.
- This implies that ${}^tA_{[0,n)}X$ takes infinitely often the same value.
- Hence 1 is eigenvalue of ${}^tA_{[0,n)}$ for infinitely many n.
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Pisot condition and Rauzy fractals

- Consider a Pisot algorithm with a Gauss map and a natural extension $\tilde{T}$.
- For almost every point (u, v) is the domain of $\tilde{T}$, there is an associate sequence of substitutions $(\sigma_n)_{n\in\mathbb{Z}}$
- which is primitive, recurrent, Pisot
- u is totally irrational, and is the frequency vector of the limit word of (σ_n).
- The limit word is balanced; we can project it on $v^\perp$ along u and take the closure.
- We obtain a set $\mathcal{R}$ which is compact, the closure of its interior, and cover the plane by translation along the projection of the diagonal group (Rauzy fractal) .

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Coincidence condition

- We suppose that the set S satisfy the coincidence condition : all limit words have a common letter, with prefixes of same abelianization.
- This condition is not known to be true in general.
- But easy to prove for some algorithms.
- For exemple if they all have the same first letter.
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- We suppose that the Rauzy fractal tiles the plane.
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Final word

- Under these conditions,
- realized by Brun's algortithm,
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- To almost any u we associate a rotation R_u on T^{d-1} seen as an exchange of d pieces
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A dual viewpoint

- take a basis $(u_1, \ldots, u_{d-1})$ of $v^\perp$, with $u = u_d$.
- Consider $\mathbb{Z}^d$ as a lattice in this base.
- We can associate to it a matrix B.
- We consider the group G which preserve u_d and the measure on $v^\perp$.
- The natural extension can be seen as a section of a flow acting on the homogeneous space
- $SL(d, \mathbb{Z}) \backslash SL(d, R) / G$

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