Asymptotic behaviour of the Brun Dynamical System in high dimensions.

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The Brun dynamical system in d dimensions $\texttt{BrunSD}(d)$.

It is defined on the simplex $\mathcal{J}_{(d)} \subset \mathbb{R}^d$

$$\mathcal{J}_{(d)} := \{\boldsymbol{x} = (x_1, \ldots, x_d) \mid 1 \geq x_1 \geq \ldots \geq x_d \geq 0\}\,.$$

with the transformation $T_{(d)}$

$$T_{(d)}(\mathbf{0}^{\boldsymbol{d}}) = \mathbf{0}^{\boldsymbol{d}}, \qquad T_{(d)}(\boldsymbol{x}) = \texttt{Ins}\left(\left\{\frac{1}{x_1}\right\}, \frac{1}{x_1}\texttt{End}\,\boldsymbol{x}\right) \quad \text{for } \boldsymbol{x} \neq \mathbf{0}^{\boldsymbol{d}}\,,$$

The map $\texttt{Ins}(y_0, \boldsymbol{y})$ is the insertion "in front of", with two cases:

(G) if y_0 is not present in the list $\boldsymbol{y}$, this is an usual insertion;

(P) if y_0 is already present in the list $\boldsymbol{y}$, we insert y_0 in front of the block of components equal to y_0.

Branches and inverse branches (I).

For any $\boldsymbol{x} \in \mathcal{J}_{(d)}$, the map $T_{(d)}$ uses a *digit* $(m, j) \in \mathcal{A}_{(d)} := \mathbb{N}^* \times [1..d]$ with a quotient $m(\boldsymbol{x}) \geq 1$ and an insertion index $j(\boldsymbol{x}) \in [1..d]$.

Associate with (m, j) $\quad \mathcal{K}_{(d,m,j)} := \{\boldsymbol{x} \in \mathcal{J}_{(d)} \mid m(\boldsymbol{x}) = m, \quad j(\boldsymbol{x}) = j\}$.

When (m, j) varies in $\mathcal{A}_{(d)}$,

– the subsets $\mathcal{K}_{(d,m,j)}$ form a topological partition of $\mathcal{J}_{(d)}$

– the restriction $T_{(d,m,j)}$ of $T_{(d)}$ to $\mathcal{K}_{(d,m,j)}$ is a bijection $\mathcal{K}_{(d,m,j)} \to \mathcal{J}_{(d)}$,

$$T_{(d,m,j)}(x_1, x_2, \ldots, x_d) = \left(\frac{x_2}{x_1}, \ldots, \frac{x_{j-1}}{x_1}, \frac{1}{x_1} - m, \frac{x_{j+1}}{x_1}, \ldots, \frac{x_d}{x_1}\right).$$

Its inverse is a bijection from $\mathcal{J}_{(d)}$ onto $\mathcal{K}_{(d,m,j)}$

$$h_{(d,m,j)}(y_1, \ldots, y_d) = \left(\frac{1}{m + y_j}, \frac{y_1}{m + y_j}, \ldots, \frac{y_{j-1}}{m + y_j}, \frac{y_{j+1}}{m + y_j}, \ldots, \frac{y_d}{m + y_j}\right).$$

The set of the inverse branches of the map $T_{(d)}$ is

$$\mathcal{H}_{(d)} := \left\{h_{(d,m,j)} \mid (m, j) \in \mathcal{A}_{(d)}\right\}.$$

Branches and inverse branches (II).

The set of the inverse branches of the map $T_{(d)}^k$ is $\mathcal{H}_{(d)}^k$.
The total set of inverse branches is $\bigoplus_{k\geq 0} \mathcal{H}_{(d)}^k$.

An important property: the inverse branches are LFT's

Any $h \in \mathcal{H}_{(d)}^\star$ is written as $\quad h = \frac{1}{D[h]}\Big(N_1[h], N_2[h], \ldots, N_d[h]\Big)$,

The denominator $D[h]$ and the numerators $N_i[h]$ are co-prime affine functions.
The Jacobian $J[h]$ and the denominator $D[h]$ are related,

$$J[h] = \left|\frac{1}{D[h]}\right|^{d+1}.$$

Density transformer of $\mathtt{BrunSD}(d)$.

The density transformer $\mathbf{H}_{(d)}$ is defined as

$$\mathbf{H}_{(d)}[f](\boldsymbol{x}) = \sum_{h\in\mathcal{H}_{(d)}} |J[h](\boldsymbol{x})| \, f \circ h(\boldsymbol{x})$$

A convenient functional space is $C^1(\mathcal{J}_{(d)})$ endowed with the norm $||\cdot||_1$,

$$||f||_1 = \sup_{\boldsymbol{x}\in\mathcal{J}_{(d)}} |f(\boldsymbol{x})| + \sup_{\boldsymbol{x}\in\mathcal{J}_{(d)}} ||\boldsymbol{D}f(\boldsymbol{x})||_{(d)} \, .$$

Here $\boldsymbol{D}f(\boldsymbol{x})$= the differential of f at $\boldsymbol{x}$ and $||\cdot||_{(d)}$= a norm on $\mathbb{R}^d$

$\mathbf{H}_{(d)}$ acts on $\left(C^1(\mathcal{J}_{(d)}), \|\cdot\|_1\right)$ and is quasi-compact.

The quasi-compacity is due to two properties:

(a) A contraction ratio $\quad \tau_d := \limsup_{n\to\infty} \sup_{h\in\mathcal{H}^n_{(d)}} \sup_{\boldsymbol{x}\in\mathcal{J}_{(d)}} ||\boldsymbol{D}h(\boldsymbol{x})||^{1/n}_{(d)} \, , < 1$

[an algebraic number, the smallest real root of $z^{d+1} + z - 1 = 0$].

(b) A distortion constant :

$$\exists L > 0, \quad ||\boldsymbol{D}J[h](\boldsymbol{x})||_{(d)} \le L \, |J[h](\boldsymbol{x})|, \qquad \forall h \in \mathcal{H}^\star_{(d)}, \forall \boldsymbol{x} \in \mathcal{J}_{(d)} \, .$$

Spectral properties of $\mathbf{H}_{(d)}$ acting on $C^1(\mathcal{J}_{(d)})$.

(a) As $\mathbf{H}_{(d)}$ is a density transformer, quasi-compact, with an ergodic DS, $\Longrightarrow \lambda = 1$ is the unique simple dominant eigenvalue of maximum modulus, isolated from the remainder of the spectrum by a spectral gap.

(b) The dominant eigenfunction is explicit,

$$\psi_d(\boldsymbol{x}) = \sum_{\sigma \in \mathfrak{S}_d} \prod_{i=1}^{k} \frac{1}{1 + x_{\sigma(1)} + x_{\sigma(2)} + \ldots + x_{\sigma(i)}} \,.$$

(c) Except for small d, there is no explicit expression known for the integral

$$\kappa_d := \int_{\mathcal{J}_{(d)}} \psi_d(\boldsymbol{x})\, d\boldsymbol{x}$$

The invariant density Ψ_d and the invariant measure ν_d are not explicit.

Distribution of quotients wrt to the invariant measure. Role of $\mu_d(y) := \nu_d[y\mathcal{J}_{(d)}]$.

The variable $m : \mathcal{J}_{(d)} \to \mathbb{N}$ associates with $\boldsymbol{x} \in \mathcal{J}_{(d)}$ the quotient $m(\boldsymbol{x})$. For $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$, the quotient $m(\boldsymbol{x})$ is equal to $\lfloor 1/x_1 \rfloor$, and

$$[m(\boldsymbol{x}) \geq m] = \left\{ \boldsymbol{x} \in \mathcal{J}_{(d)} \mid x_1 \leq \frac{1}{m} \right\} = \left(\frac{1}{m} \right) \mathcal{J}_{(d)} \,.$$

The distribution of digits is defined via the function μ_d

$$\mu_d : y \in [0,1] \mapsto \nu_d[y\mathcal{J}_{(d)}].$$

$$\text{one has:} \qquad \mu_d(y) = \frac{\kappa_d(y)}{\kappa_d(1)} \qquad \text{with} \quad \kappa_d(y) := \int_{y\mathcal{J}_{(d)}} \psi_d(\boldsymbol{x}) d\boldsymbol{x},$$

$$\text{and} \quad \psi_d(\boldsymbol{x}) = \sum_{\sigma \in \mathfrak{S}_d} \prod_{i=1}^{k} \frac{1}{1 + x_{\sigma(1)} + x_{\sigma(2)} + \ldots + x_{\sigma(i)}} \,.$$

The study of $\kappa_d : y \mapsto \kappa_d(y)$ is the main step in the study of μ_d.

Expression of $\kappa_d(y)$ and entropy $\mathcal{E}_d$ as one-dimensional integrals.

The following holds

$$\kappa_d(y) = \frac{1}{d!}\int_0^\infty e^{-u}\beta(uy)^d du, \qquad \mathcal{E}_d = (d+1)\int_0^1 \frac{1}{y}\frac{\kappa_d(y)}{\kappa_d}dy\,.$$

and involves the function β

$$\beta(u) := \int_0^u \alpha(v)dv \quad \text{with} \quad \alpha(u) := \int_0^1 e^{-tu}dt = \frac{1-e^{-u}}{u}\,.$$

Main tool for $\kappa_d(y)$: with the Laplace transform, $\dfrac{1}{1+x} = \displaystyle\int_0^\infty e^{-t}e^{-tx}dt$

Main tool for the entropy: The Rohlin formula.

Asymptotic estimates for $\kappa_d(y)$ (for $d \to \infty$).

Easy bounds for $\kappa_d(y)$, but not of great use when $d \to \infty$.

$$\frac{1}{d!} y^d \le \kappa_d(y) \le y^d \,,$$

We perform a finer asymptotic study for $\kappa_d(y)$, mainly based on Laplace estimates. We deduce for $\mu_d(y)$:

The following estimates hold for the function $y \mapsto \mu_d(y)$, when $d \to \infty$, and exhibit two different regimes:

$$\begin{cases} \mu_d(y) \in \left[\left(\dfrac{\log dy}{\log d}\right)^{d(1+\epsilon(d))-1/2}, \left(\dfrac{\log dy}{\log d}\right)^{d(1-\epsilon(d))-1/2} \right] & y \in [y(d), 1]\,, \\ \mu_d(y) \le (\log d)^{1/2} \left(\dfrac{dy}{\log d}\right)^{d} & y \in [0, y(d)]\,. \end{cases}$$

defined with the two functions $\quad y(d) := (\log d)/d, \quad \epsilon(d) = (\log \log d)^{-1}.$

Remark. When $y = \Theta(1)$, one has $\mu_d(y) = y^{\Theta(1) d / \log d}$

Laplace estimates for an integral $I_d = \int_0^\infty \exp\left[F_d(v)\right]\, dv$

Consider a function $F_d \in L^1[0,\infty]$ of class $\mathcal{C}^\infty$ which satisfies the following

(a) it admits a unique maximum on $[0,\infty[$, attained at $v = v_d \in]0,+\infty[$,

(b) its second derivative F_d'' is not zero at v_d

(c) the quotient $\gamma_d := \left|F_d'''(v_d)/(F_d''(v_d))^{3/2}\right|$ tends to zero

Then, the following estimate holds for $d \to \infty$

$$I_d \sim \sqrt{2\pi} \cdot \left[\left|F_d''(v_d)\right|^{-1/2} \cdot \exp\left[F_d(v_d)\right]\right] \Big(1 + O(\gamma_d)\Big).$$

Here, simple application to $\kappa_d(y)$ (when y not too small)

with $F_{d,y}(v) := d \log \beta(vy) - v$

Estimate for the entropy,
Distribution for the quotients in the $\texttt{BrunSD}(d)$ for $d \to \infty$.

Consider the main characteristics of the dynamical system $\texttt{BrunSD}(d)$:

$\mathcal{E}_d =$ the entropy of the system.

$\rho_d = \nu_d[m = 1] =$ probability that $m(\boldsymbol{x})$ be equal to 1.

$\sigma_d := \mathbb{E}_d[m] =$ the expectation of the variable m

They admit asymptotic estimates for $d \to \infty$

$$\log d \le \mathcal{E}_d \le \log d + (\log d)^{1/2},$$

$$\rho_d \sim 1 - (1/2)^{(d-1/2)/\log d},$$

$$\frac{d}{\log d} \le \sigma_d \le \frac{d}{\log d}\left[2 + (\log d)^{-1/2}\right].$$