Gaussian behavior of quadratics irrationals

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Alea EnAmSud and Dyna3s, June 2017

Joint work with Brigitte Vallée

Reduced quadratic irrationals: basic facts and notations

Every real number $x \in [0, 1[$ can be written as a continued fraction (CFE)

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We are interested in combinatorial properties of the set

$$\mathcal{P} = \{x \in \mathcal{I} \mid x \text{ is a rqi number}\} .$$

Additive costs

Definition

A digit-cost is a nonnegative and nonzero map on the naturals numbers:

$$c : \mathbb{Z}_{>0} \mapsto \mathbb{R}_{\geq 0}$$

A digit-cost defines a total additive cost C on $\mathcal{P}$.

For a rqi $x = [\overline{m_1, m_2, \ldots, m_p}]$,

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- $\ell(m) = \lfloor \log_2 m \rfloor + 1$;

  total cost $C(x)$: number of bits to store the CFE of x.

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Remark

ϵ is not additive and it is not multiplicative.

Our goals

The set $\mathcal{P}$ of rqi's is endowed with

- An additive cost C
- A notion of size ϵ.

The set of rqi with size at most N:

$$\mathcal{P}_N = \{x \in \mathcal{P} \mid \ \epsilon(x) \leq N\}$$

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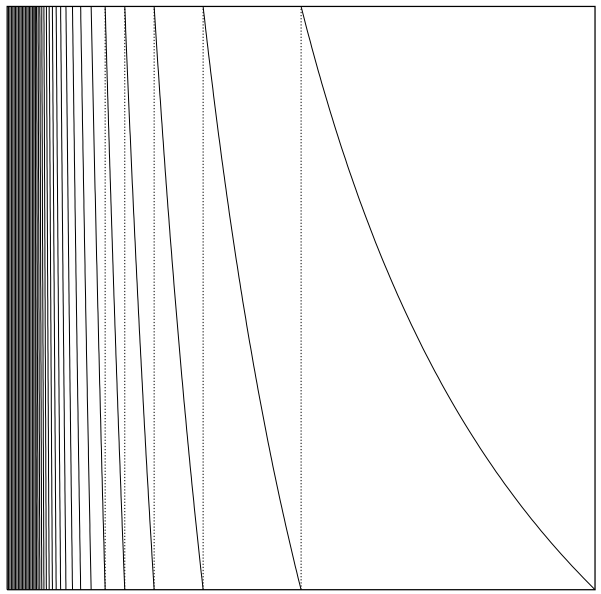
$$\mathbb{P}_N \left[x \mid \frac{C(x) - \mu(c) \log N}{\sqrt{\nu(c) \log N}} \leq t \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du + O\left(\frac{1}{\sqrt{\log N}} \right) \ .$$

The constants $\mu(c)$ and $\nu(c)$ are computable.

Our results: computation of constants $\mu(c)$ and $\nu(c)$

The Gauss map:

$$T : [0,1] \mapsto [0,1] \quad T(x) = \left\{ \frac{1}{x} \right\} \text{ if } x \neq 0 \quad \text{and} \quad T(0) = 0$$



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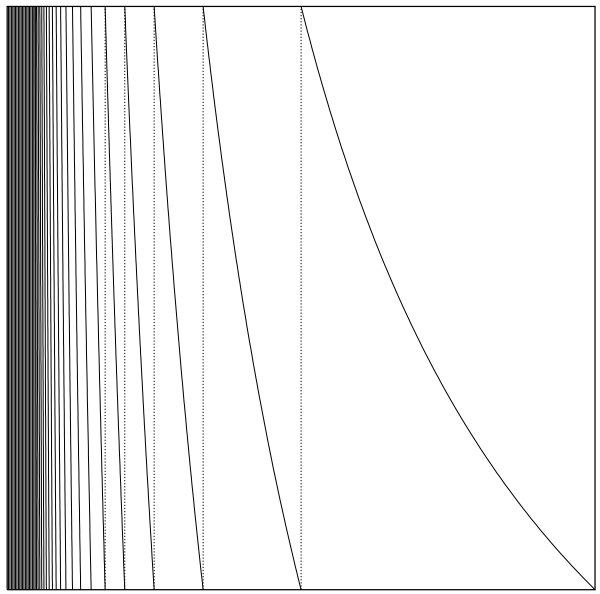

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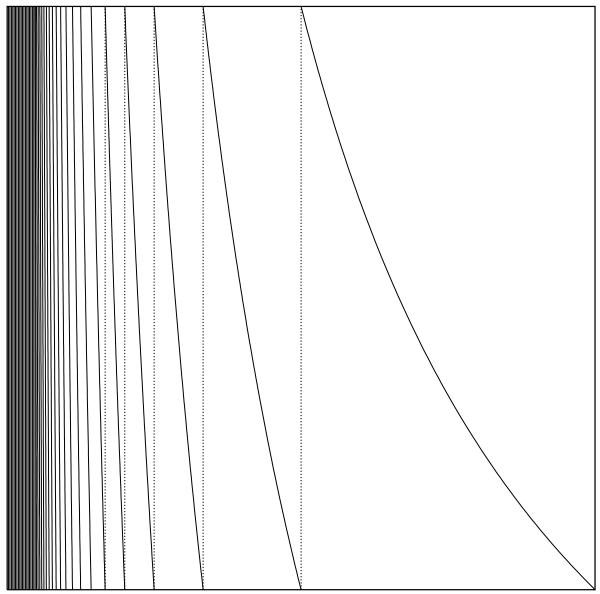

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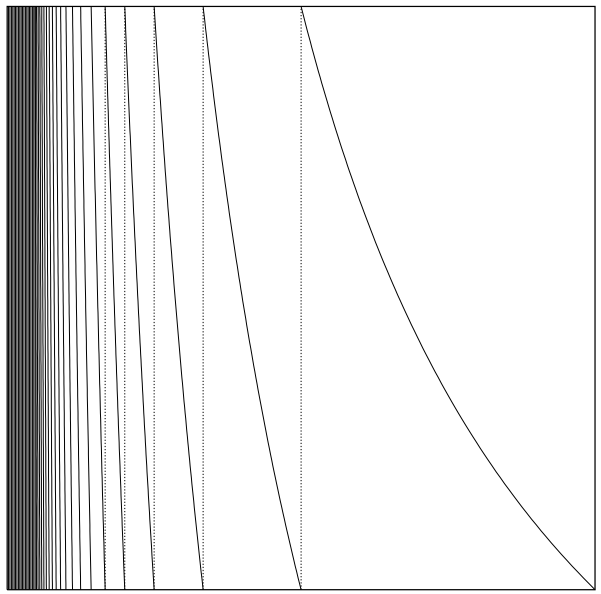

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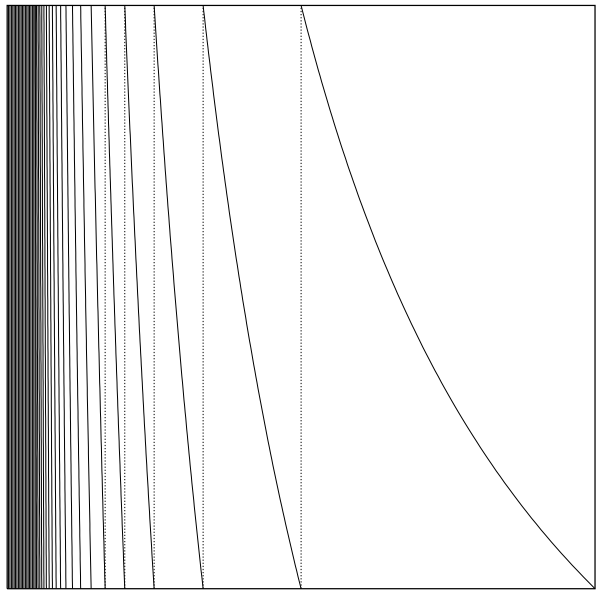

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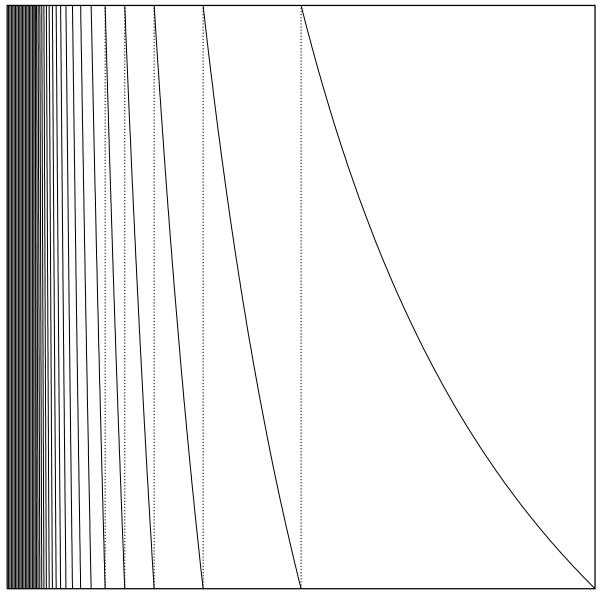

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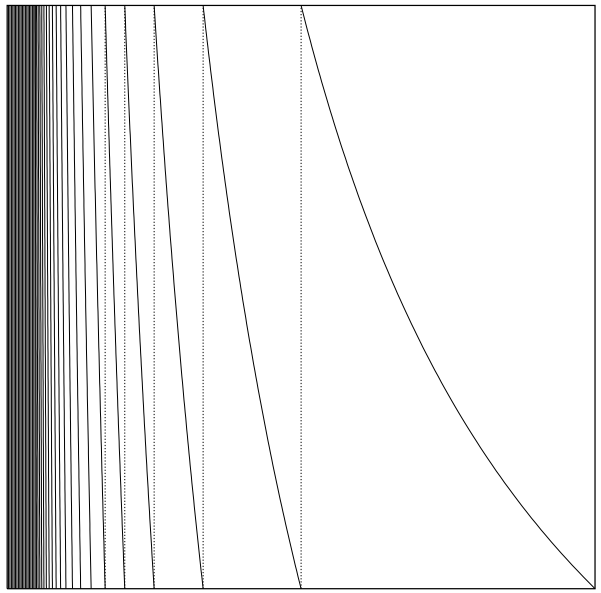

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Remark: $\mathbf{H} = \mathbf{H}_{1,0}$

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- $\lambda'_w(1,0)$ equals the weighted average $\mathbb{E}[c]$ of the digit-cost c with respect to the Gauss density, i.e.

$$\lambda'_w(1,0) = \sum_{m=1}^{\infty} c(m) \int_{1/(m+1)}^{1/m} \psi(x)dx.$$

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Let $\mathcal{E}$ be the entropy of the Gauss map and let $\mathbb{E}[c]$ be the weighted average:

$$\mu(c) = \frac{2\lambda'_w(1,0)}{|\lambda'_s(1,0)|} = \frac{2}{\mathcal{E}}\mathbb{E}[c] \quad \textit{and}$$

$$\mathbb{E}[c] = 1, \quad \mathbb{E}[\chi_n] = \frac{1}{\log 2} \log\left[\frac{(n+1)^2}{n(n+2)}\right], \quad \mathbb{E}[\ell] = \frac{1}{\log 2} \prod_{i=1}^{\infty} \log\left(1 + \frac{1}{2^k}\right) .$$

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Rqi framework: Pollicott, Faivre, Vallée

Previous works: rationals trajectories

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Remarks

- The constants $\mu(c)$ and $\nu(c)$ in the rational case are the same as for the rqi case.
- The constants hidden in the $O(1)$ are expressed in terms of the transfer operator in both cases.

They do not coincide !

- Our methods are inspired in those of Baladi and Vallée, 2005

Real trajectories

A classical result:

Theorem

The mean of a digit-cost c satisfies:

$$\frac{1}{N} \sum_{1 \leq k \leq N} c(m_k) \to \mathbb{E}[c] \qquad N \to \infty$$

with the exception of a set of zero measure.

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A "combinatorial" version

$$\frac{1}{p(x)} \sum_{1 \leq k \leq p(x)} c(m_k(x)) \to \mathbb{E}[c] \qquad N \to \infty.$$

Here $p(x)$ is the length of the smallest period of x.

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$$\frac{\mathbb{E}_N[C]}{\mathbb{E}_N[p]} = \sum_{x\in\mathcal{P}_N} \frac{1}{\sum_{x\in\mathcal{P}_N} p(x)} \sum_{1\le k\le p(x)} c(m_k(x)) \to \mathbb{E}[c] \qquad N\to\infty.$$

Here $p(x)$ is the length of the smallest period of x.

A comment about “moderate growth”

Remark

Our results hold under the assumption that the digit-cost c is of moderate growth: $c(m) = O(\log m)$.

Explanation

The weighted transfer operator depends of two complex parameters (s, t):

$$f \mapsto \mathbf{H}_{s,t}[f](x) = \sum_{m \geq 1} \frac{e^{t\, c(m)}}{(m+x)^{2s}} f\left(\frac{1}{m+x}\right)$$

For $c(m) = \Theta(\log^a(m))$ with $a > 1$, $\mathbf{H}_{1,t}$ is not convergent for $\Re t > 0$.

Spectral properties of the transfer operator are well-known around $(1, 0)$.

Our methods rely on the fact that $\mathbf{H}_{1,t}$ is convergent for t in a complex neighborhood of $t = 0$.

Methods: The role of Dirichlet series

"Combinatorial class"

The set $\mathcal{P}$ of rqi's is endowed with

- an additive cost C,
- a notion of size ϵ.

Our main object of analysis: the bivariate Dirichlet series $P(s,t)$

$$P(s,t) = \sum_{x \in \mathcal{P}} e^{tC(x)} \epsilon(x)^{-s}$$

where s and t are complex parameters.

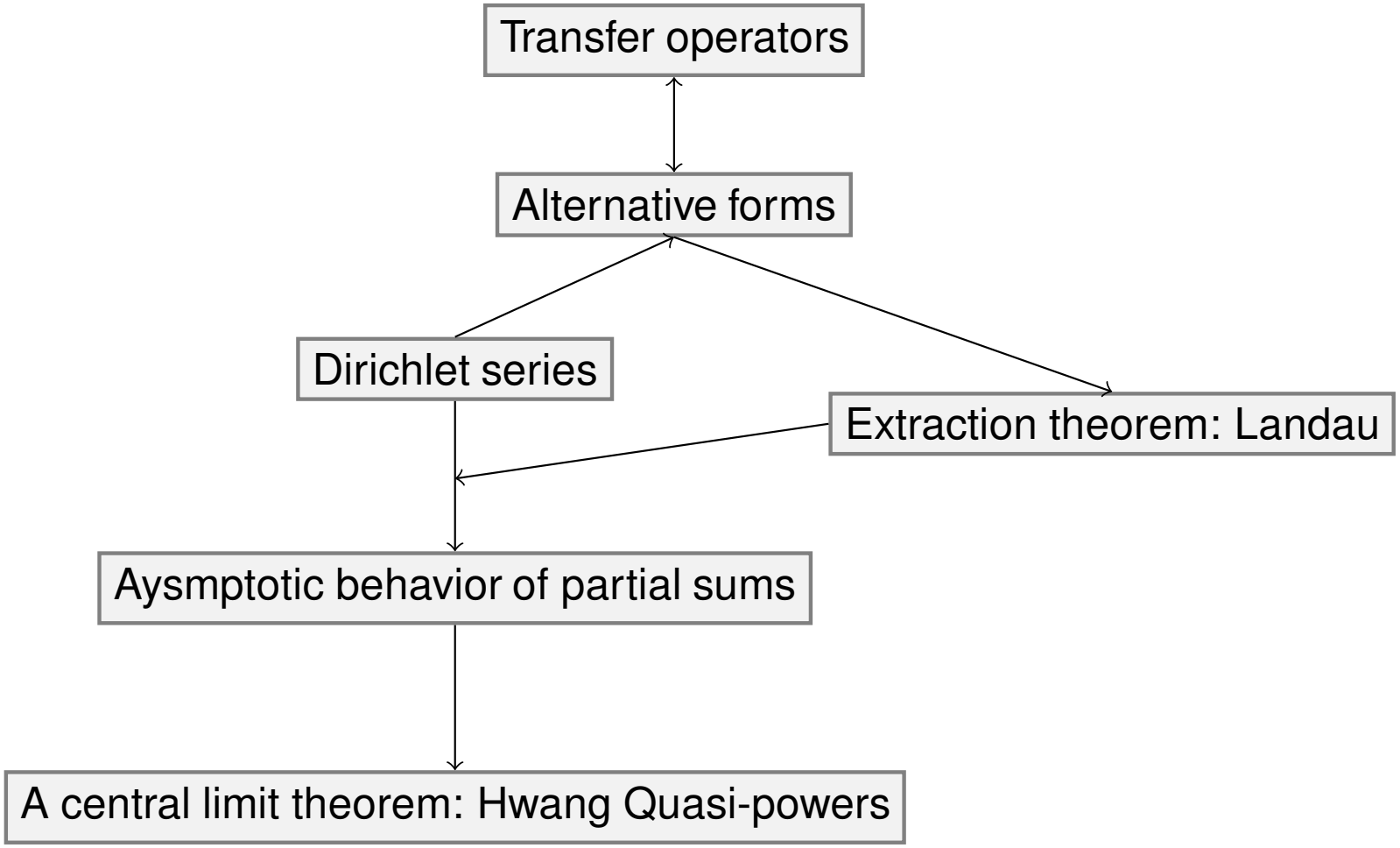
Extraction theorems

Extraction theorems gives information about the partial sums of coefficients

$$S_N^{[c]}(t) = \sum_{\epsilon(x) \leq N} e^{tC(x)}$$

if we have information about the analytic behavior of $s \mapsto P(s,t)$.

Interaction between the main elements of the proof



Alternative forms for $P(s,t)$: replace rqi's by homographies

Homographies

Recall the set of inverse branches of the Gauss map T:

$$\mathcal{H} := \{h_m; \quad h : x \mapsto \frac{1}{m+x}, \quad m \geq 1\}$$

Composition of inverse branches give rise to the two sets:

$$\mathcal{H}^k = \{h = h_{m_1} \circ h_{m_2} \circ \cdots \circ h_{m_k}\} \qquad \mathcal{H}^+ = \bigcup_{k \geq 1} \mathcal{H}^k$$

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- The number x is a rqi iff there exists $h \in \mathcal{H}^+$ so that x is its fixed point: $h(x) = x$.

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An homography is primitive iff there is not a $g \in \mathcal{H}^+$ so that $h = g \circ g \cdots \circ g$.

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Important relation

Each rqi x is the fixed point of only one primitive $h \in \mathcal{H}^+$.

Size: replace rqi's by homographies

Recall: $x \longrightarrow \mathbb{Q}(\sqrt{\Delta}) \longrightarrow \epsilon(x) =$ fundamental unit of $\mathbb{Q}(\sqrt{\Delta})$

Definition

Let x be a rqi number and define

$$\alpha(x) = \prod_{i=0}^{p(x)-1} T^i(x).$$

Here T is the Gauss map and $p(x)$ is the smallest period length of x.

Theorem

- *The size $\epsilon(x)$ and $\alpha(x)$ are related by*

$$\epsilon(x) = \alpha(x)^{-r(x)}, \quad r(x) = 1 \text{ for even } p(x), \qquad r(x) = 2 \text{ for odd } p(x).$$

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- *If x is the fixed point of the primitive homography h, then*

$$\alpha(x) = |h'(x)|^{1/2}$$

Ready to extend cost and size to $\mathcal{H}^k$. A new Dirichlet series

Definition

For $h \in \mathcal{H}^k$ (primitive or not) with a fixed point x_h, let

$$\alpha(h) := |h'(x_h)|^{1/2} .$$

The size of the homography is defined by

$$\epsilon(h) = \alpha(h)^{-r(h)}, \qquad r(h) = 1 \text{ for even } k, \qquad r(h) = 2 \text{ for odd } k.$$

Definition

The total cost C is naturally extended to $\mathcal{H}^+$.
If $h = h_{m_1} \circ h_{m_2} \circ \ldots h_{m_k}$, $C(h) = c(m_1) + c(m_2) + \cdots + c(m_k)$.

Definition

A new Dirichlet series: Replace the set $\mathcal{P}$ of rqi's by $\mathcal{H}^+$ and ϵ by α in the defition of $P(s,t)$:

$$Y(s,t) = \sum_{h \in \mathcal{H}^+} e^{tC(h)} \alpha(h)^{2s}$$

with $\alpha(h) = |h'(x_h)|^{1/2}$.

Alternative form for $Y(s,t)$ and transfer operators

Dirichlet series
$Y(s,t) = \sum_{h \in \mathcal{H}^+} e^{tC(h)} |h'(x_h)|^s$

Transfer operator
$\mathbf{H}_{s,t}(I - \mathbf{H}_{s,t})^{-1}[f](x) = \sum_{h \in \mathcal{H}^+} e^{tc(h)} |h'(x)|^s f(h(x))$

Relation
$Y(s,t) \approx \mathbf{H}_{s,t}(I - \mathbf{H}_{s,t})^{-1}$

Remark the difference between the evaluation points

Relating fixed points and traces of transfer operators

Theorem

$Y(s,t) = \mathrm{Trace}(\mathbf{H}_{s,t}(I - \mathbf{H}_{s,t})^{-1})$

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The statement is obtaining adding over all $h \in \mathcal{H}^+$.

Analytic properties of $Y(s,t)$

For s near the real axis, the transfer decomposes

$$\mathbf{H}_{s,t} = \lambda(s,t)\mathbf{P}_{s,t} + \mathbf{N}_{s,t}$$

where

- $\lambda(s,t)$ is the dominant eigenvalue,
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$\mathbf{H}_{s,t}^n = \lambda(s,t)^n\mathbf{P}_{s,t} + \mathbf{N}_{s,t}^n, \qquad \text{for any } n \geq 1\,.$

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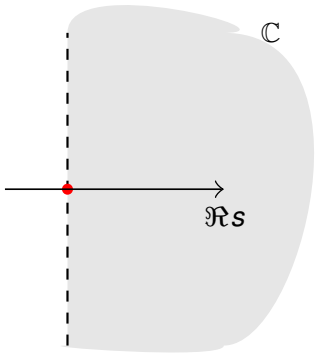

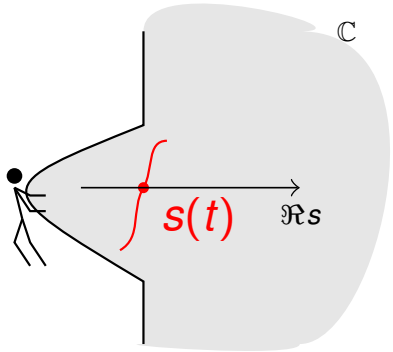

Theorem

$s \mapsto Y(s,t)$ extends meromorphically to a complex neighborhood of $s = 1$ (uniformly for $t \sim 0$):

$$Y(s,t) = \mathrm{Trace}((\mathbf{H}_{s,t}(I - \mathbf{H}_{s,t})^{-1})) = \frac{\lambda(s,t)}{1-\lambda(s,t)} + \mathrm{Trace}(\mathbf{N}_{s,t}(I - \mathbf{N}_{s,t})^{-1}) \, .$$

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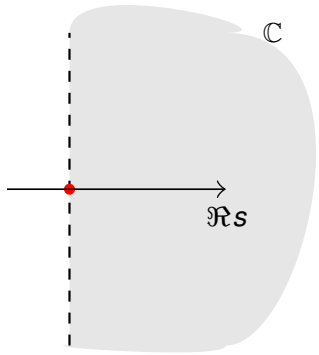
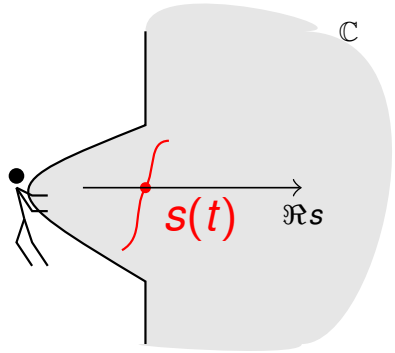
where

- $\lambda(s,t)$ is the dominant eigenvalue,
- $\mathbf{P}_{s,t}$ is the projector,
- $\mathbf{N}_{s,t}$ has spectral radius less than $\mathbf{H}_{s,t}$.

The powers of the transfer operator satisfy
$\mathbf{H}_{s,t}^n = \lambda(s,t)^n\mathbf{P}_{s,t} + \mathbf{N}_{s,t}^n, \qquad$ for any $n \geq 1$.





Theorem

$s \mapsto Y(s,t)$ extends meromorphically to a complex neighborhood of $s = 1$ (uniformly for $t \sim 0$):

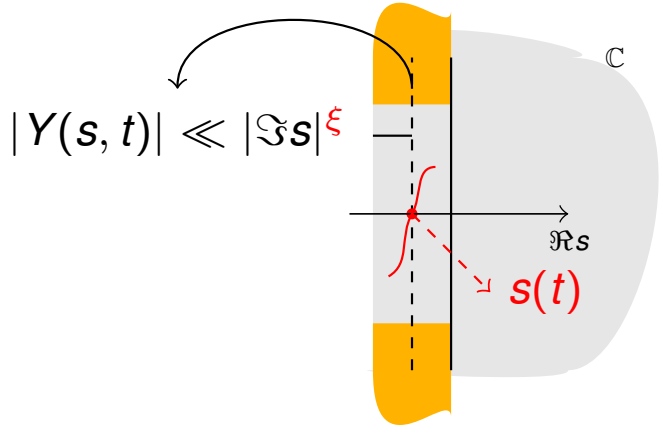
$$Y(s,t) = \mathrm{Trace}((\mathbf{H}_{s,t}(I - \mathbf{H}_{s,t})^{-1})) = \frac{\lambda(s,t)}{1-\lambda(s,t)} + \mathrm{Trace}(\mathbf{N}_{s,t}(I - \mathbf{N}_{s,t})^{-1}) \,.$$

The solutions $s(t)$ of $1 - \lambda(s,t) = 0$ are the only singularities of $s \mapsto Y(s,t)$.

Extensions of Dolgopyat-Baladi-Vallée results

- Dolgopyat, Baladi and Vallée for the "true quasi-inverse $(I - \mathbf{H}_{s,t})^{-1}$"

The following holds for $s \mapsto Y(s,t)$



Remark

The original series $P(s,t)$ and the series $Y(s,t)$ has the same analytic behavior.

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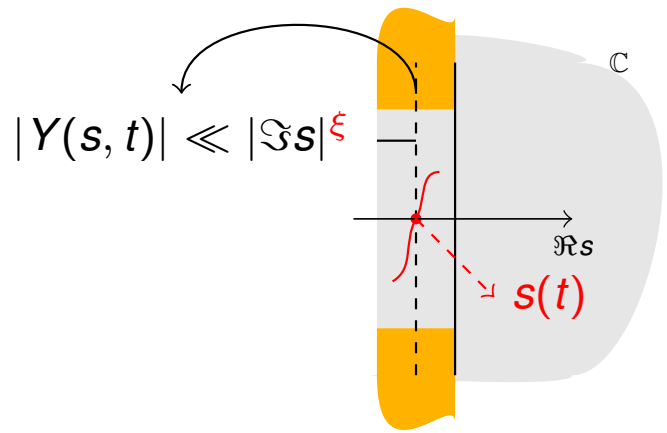

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- Dolgopyat, Baladi and Vallée for the "true quasi-inverse $(I - \mathbf{H}_{s,t})^{-1}$"
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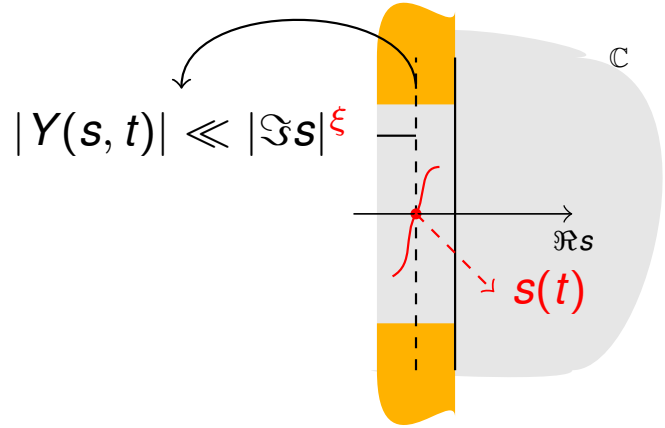
- This work: periodic points of the Gauss map (infinite number of branches).

The following holds for $s \mapsto Y(s,t)$



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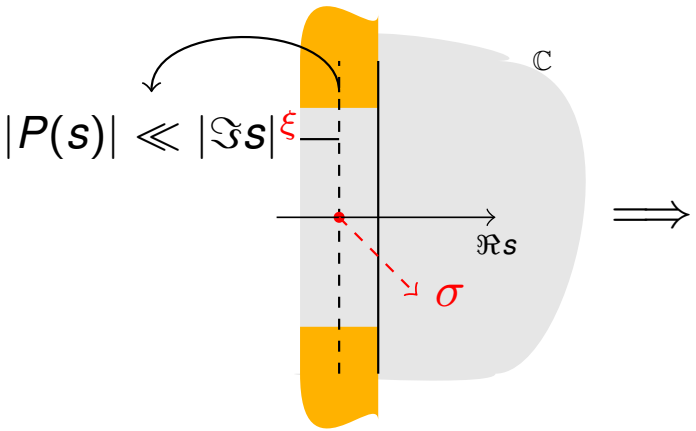
Dirichlet series and Landau Theorem

Dirichlet series: $P(s) = \sum_{x \in \mathcal{P}} \epsilon(x)^{-s}$

If



$$S_N = |\mathcal{P}_N| = \sum_{\epsilon(x) \leq N} 1$$

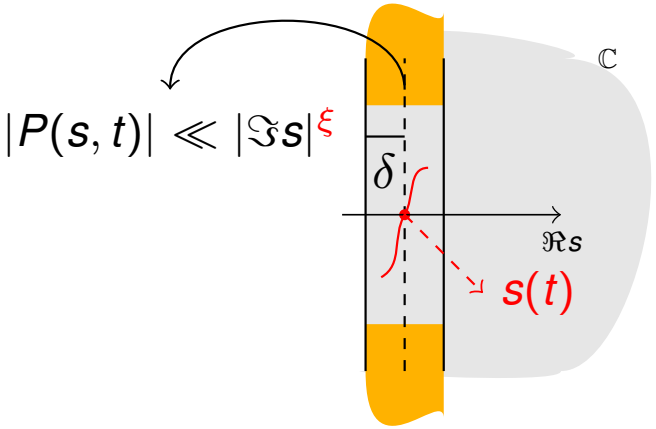
$$S_N = KN^{\sigma}\left(1 + O(N^{-\beta})\right)$$

$$\beta = \beta(\xi, \delta) > 0$$

Here δ is the distance between $s(t)$ and the left straight line.

Bivariate Dirichlet series: $P(s,t) = \sum_{x \in \mathcal{P}} e^{tC(x)} \epsilon(x)^{-s}$

If



$$S_N^{[c]}(t) = \sum_{\epsilon(x) \leq N} e^{tC(x)}$$

$$S_N^{[c]}(t) = v(t) N^{s(t)}\left(1 + O(N^{-\beta})\right)$$

$$\beta = \beta(\xi, \delta) > 0$$

A central limit theorem. The final element of the proof.

Definition

The moment generating function $\mathbb{E}_N[e^{tC}]$ satisfies

$$\mathbb{E}_N[e^{tC}] = \frac{S_N^{[c]}(t)}{|\mathcal{P}_N|} = \frac{S_N^{[c]}(t)}{S_N^{[c]}(0)}.$$

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Our moment generating function admits the form

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and then, $\mathbb{E}_N[C] = s'(0)\log N + O(1)$ *and* $\mathbb{V}_N[C] = s''(0)\log N + O(1)$.

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In addition, $s''(0) \neq 0$*, then the cost* C *is asymptotically Gaussian when* $N \to \infty$.

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Constants are computed from $s(t)$ using the relation $\lambda(s,t) = 1$.

Extensions and conclusions

- $x \mapsto \log \epsilon(x)$ is close to the Lévy constant. It is not an additive cost.
  - $\mathbb{E}_N[\log \epsilon] = 2 \log N + O(1)$.
  - The variance is of constant order.
- The set of $\mathcal{P}[M] = \{$rqi with partial quotients bounded by $M\}$
  - Similar Gaussian laws for additive parameters and M large enough.
  - Based on properties of the constrained transfer operator

$$\mathbf{H}_{M,s}[f](x) = \sum_{m \leq M} \frac{1}{(m+x)^{2s}} f\left(\frac{1}{m+x}\right) .$$

  - For small M, Dolgopyat like results are not proved to hold.