Modélisations de l'algorithme LLL et de ses entrées

Sur la base des travaux de:
B. Vallée, J. Clément, A. Vera, M. Georgieva, M. Madritsch, A. Akhavi,. . .
et bien d'autres encore!

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Projet ANR Dyna3S

Plan

Plan

GCD

Simultaneous Rational Approximation

Problem : Consider $\overrightarrow{y} \in \mathbb{R}^n$, find $q \in \mathbb{Z}$ with $q \leq M$ and $\overrightarrow{p} \in \mathbb{Z}^n$ such that $||q \cdot \overrightarrow{y} - \overrightarrow{p}||$ is small.

Continued Fraction Expansion

$$\frac{u}{v} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots \, \cfrac{}{m_{p-1} + \cfrac{1}{m_p + 0}}}}}$$

This morning ...

Generalizations of the Euclid algorithm

Lattice reduction

- Algorithms : LLL, HKZ, BKZ, . . .
- Models for the algorithms (sandpile, CFG, . . .)
- Models for the inputs (cryptography, factorization, . . .)

This afternoon. . .

GCD
Simultaneous Rational Approximation

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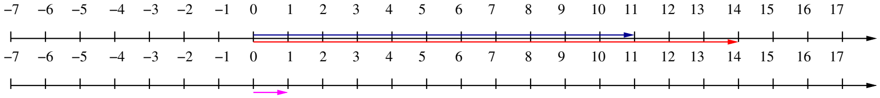
This morning . . .

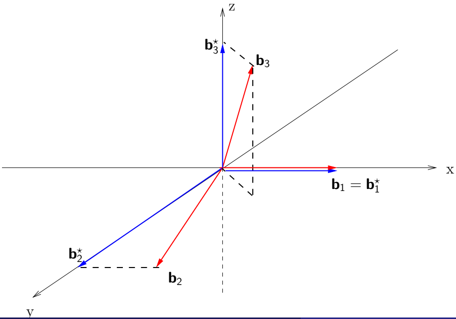


Plan

The LLL algorithm

Input : A lattice $\mathcal{L}$ given by a basis $(\mathbf{b}_1, \ldots, \mathbf{b}_m)$
$t > 1$

Output : A basis $(\mathbf{b}_1, \ldots, \mathbf{b}_m)$ of $\mathcal{L}$ such that the Gram-Schmidt Orthogonalization (GSO) satisfies ,



There exist a matrix M such that $\mathcal{B} = M\mathcal{B}^\star$ and

$$M = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \mu_{2,1} & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \mu_{m,1} & \cdots & \cdots & \mu_{m,m-1} & 1 \end{pmatrix}$$

The LLL algorithm

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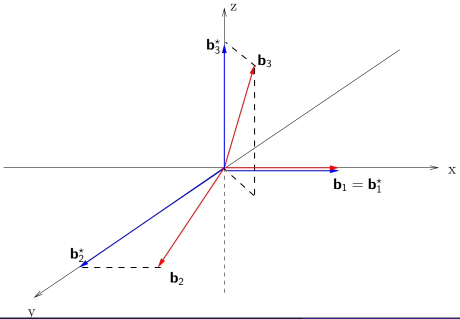
$t > 1$

Output : A basis $(\mathbf{b}_1, \ldots, \mathbf{b}_m)$ of $\mathcal{L}$ such that the Gram-Schmidt Orthogonalization (GSO) satisfies ,

1. for all $1 \leq j < i \leq m$, $|\mu_{i,j}| \leq \frac{1}{2}$ (size reduced)
2. for all $1 \leq i < m$, $||\mathbf{b}_{i+1}^\star||^2 + \mu_{i+1,i}^2||\mathbf{b}_i^\star||^2 \geq \frac{1}{t^2}||\mathbf{b}_i^\star||^2$ (Lovász conditions)



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The LLL algorithm

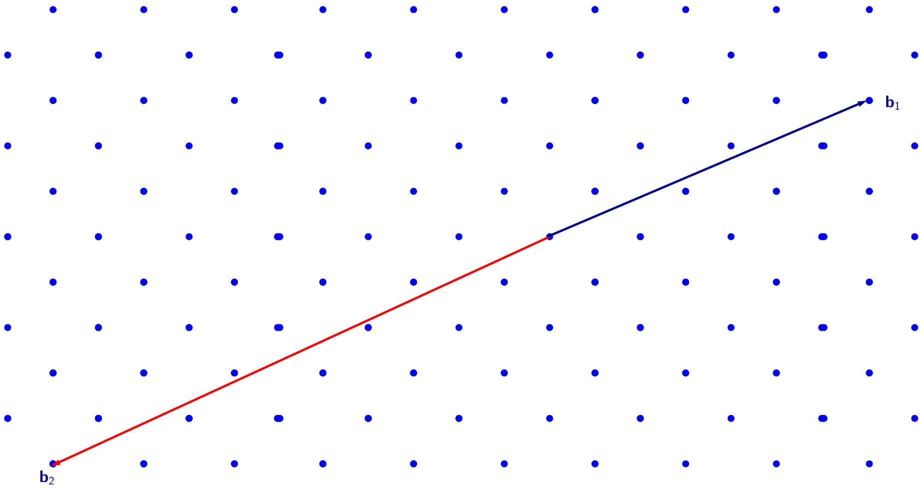
LLL performs translations and exchanges.

An exchange between two consecutive vectors is performed as soon as a Lovász condition is not satisfied.

The exchanges improve the orthogonality and globally the ratios $\mathbf{b}_i^\star/\mathbf{b}_{i+1}^\star$ decrease.

Translations are performed for shortening the vectors (size-reduction)

The LLL algorithm in dimension 2 (Gauss algorithm)



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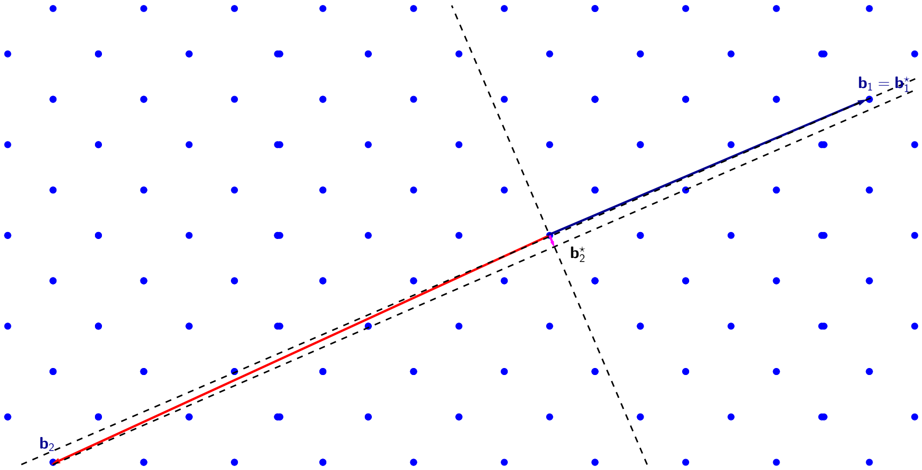

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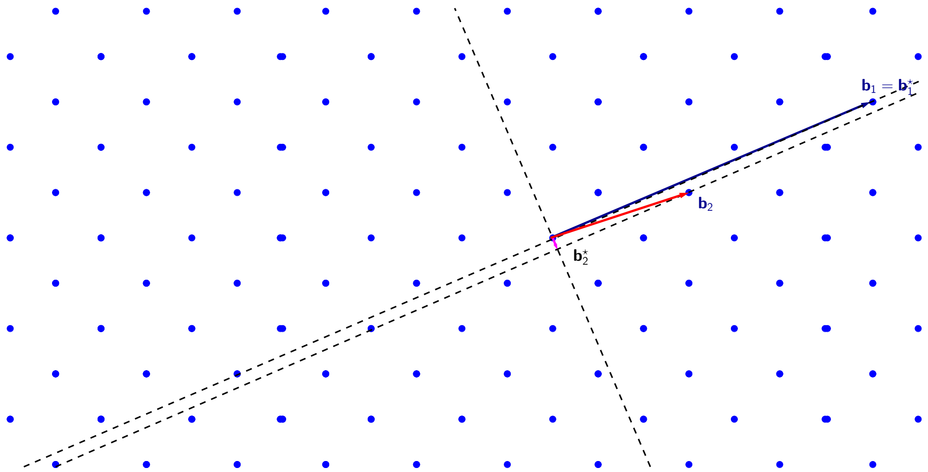

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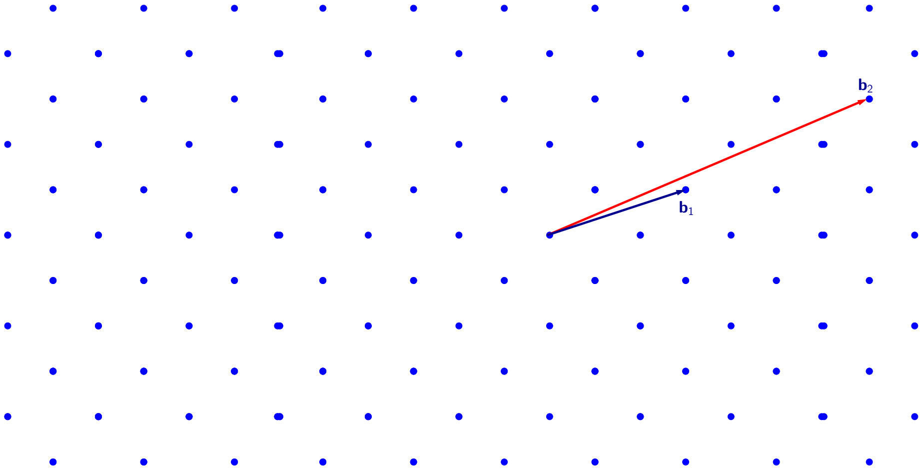

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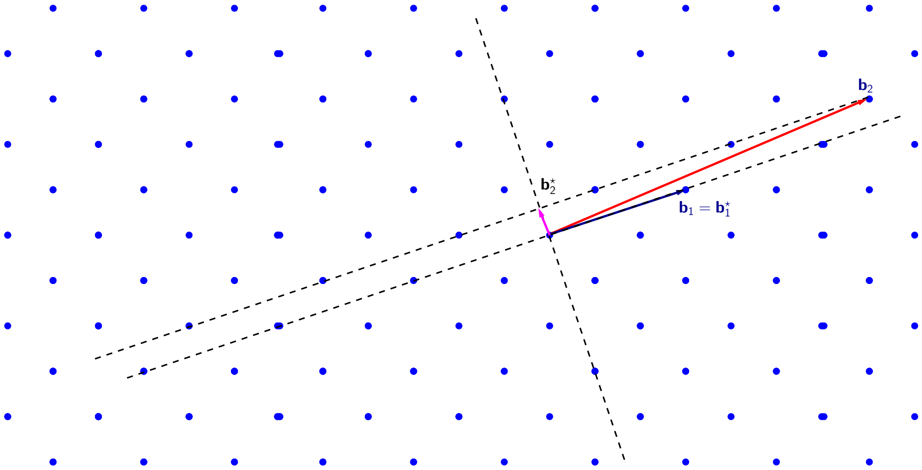

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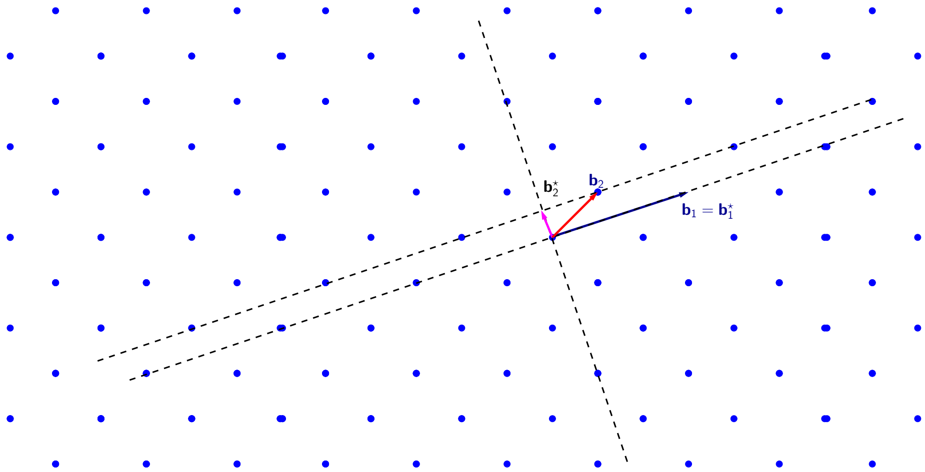

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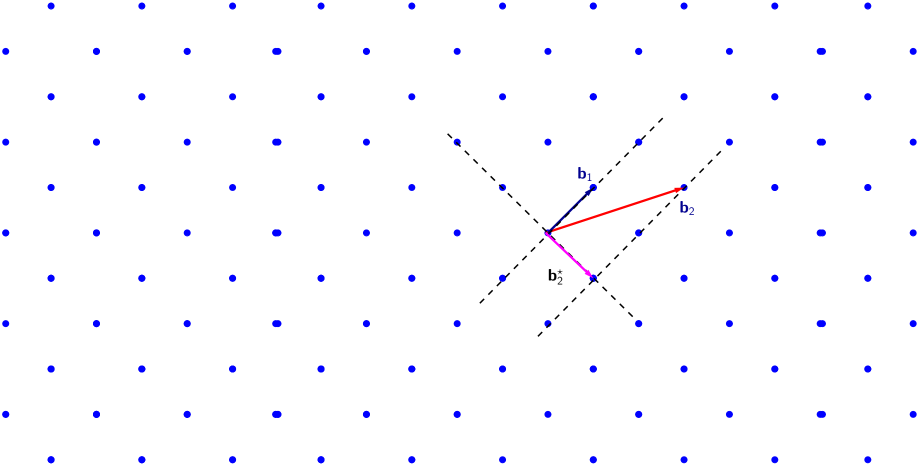

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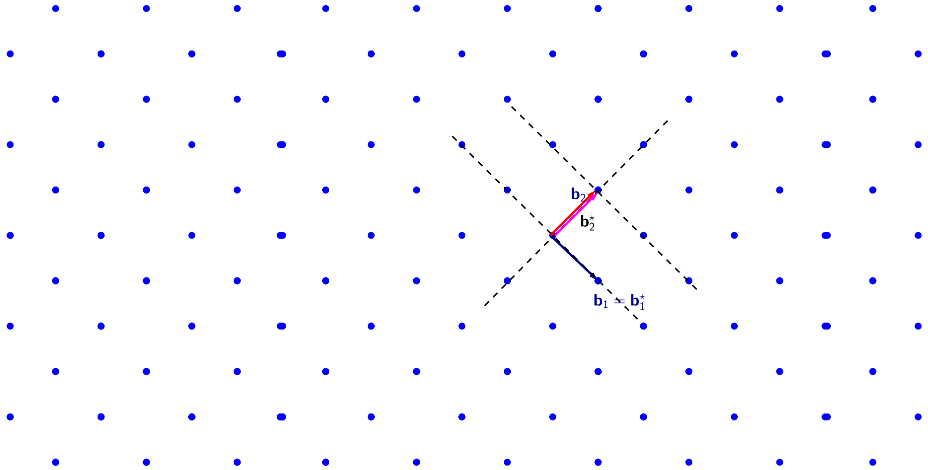
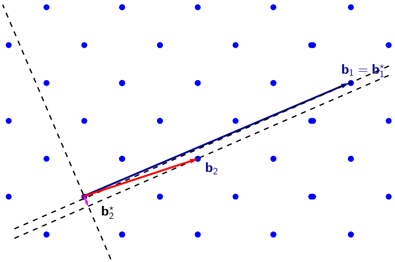


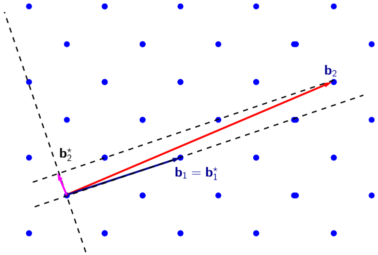
Translations and exchanges

Role of the translations (size-reduction):
The length of the vectors decreases with the translations.

Role of the exchanges:
Each exchange between $\mathbf{b}_i^\star$ and $\mathbf{b}_{i+1}^\star$ increases the length of $\mathbf{b}_{i+1}^\star$ and decreases the length of $\mathbf{b}_i^\star$.

The vectors are then more "orthogonal" and the ratio $||\mathbf{b}_i^\star||/||\mathbf{b}_{i+1}^\star||$ decreases.





The LLL algorithm

Input : A lattice $\mathcal{L} = \mathcal{L}(\mathbf{b}_1, \ldots, \mathbf{b}_m)$
$t \geq 1$
Output : A t-LLL reduced basis of $\mathcal{L}$

Algorithm

```
Compute the GSO
size-reduce the basis using only translations (|μ_{i,j}| < 1/2)
while the basis is not t-LLL reduced do
    choose i such that ||b*_{i+1}||^2 + μ^2_{i+1,i}||b*_i||^2 < (1/t^2)||b*_i||^2
                                          according to a strategy
    exchange b_{i-1} and b_i
    size-reduce the basis using only translations
end while
return (b_1, ..., b_m)
```

Remark : The LLL algorithm only uses the orthogonal basis to make decision

Quality of the output

- The norms $||\mathbf{b}_i^\star||$ do not decrease too quickly since $||\mathbf{b}_i^\star|| \geq s^{i-1}||\mathbf{b}_1^\star||$ with

$$\frac{1}{t^2} = \frac{1}{4} + \frac{1}{s^2}.$$

- b_1 is a "short enough" vector of the lattice since

$$||b_1|| \leq 2^{(m-1)/2}\lambda(\mathcal{L})$$

[$\lambda(\mathcal{L})$ is the length of a shortest non zero vector of $\mathcal{L}$.]

Complexity

Theorem

LLL performs $O(n^3 m \log B)$ arithmetic operations with integers of size $O(n \log B)$ where $B = \max_{i=1..n} ||\mathbf{b}_i||$.

The real D

$$D = \prod_{i=1}^{n} ||\mathbf{b}_i^\star||^{n-i} \leq B^{n^2}, \quad \text{with} \quad B = \max_{i=1..n} ||\mathbf{b}_i||,$$

decreases by a factor $\delta = (\frac{1}{4} + s^2)^{1/2}$ at each exchange. Then LLL performs $O(n^2 \log B)$ exchanges.

Between two exchanges, there are at most $O(n^2)$ arithmetic operations.

The size of the integer increases quickly and the computations need multiprecision even for low dimensions (≈ 20).

The LLL algorithm : a new point of vue

Input : the real vector $(\ell_1, \ldots, \ell_m)$ with $\ell_i = ||\mathbf{b}_i^\star||$
the subdiagonal coefficients $(\mu_1, \ldots, \mu_{m-1})$ with $|\mu_i| = |\mu_{i+1,i}| \leq \frac{1}{2}$
$t \geq 1$

Output : $(\hat{\ell}_1, \ldots, \hat{\ell}_m)$ and $(\hat{\mu}_1, \ldots, \hat{\mu}_{m-1})$ with for all $i = 1 \ldots m-1$,
$\hat{\ell}_{i+1}^2 \geq (\frac{1}{t^2} - \hat{\mu}_i^2)\hat{\ell}_i^2$

Algorithm

Compute the GSO
size-reduce the basis using only translations $(|\mu_{i,j}| < \frac{1}{2})$
while the basis is not t-LLL reduced **do**
    choose i such that $||\mathbf{b}_{i+1}^\star||^2 + \mu_{i+1,i}^2 ||\mathbf{b}_i^\star||^2 < \frac{1}{t^2}||\mathbf{b}_i^\star||^2$
    according to a strategy
    exchange $\mathbf{b}_{i-1}$ and $\mathbf{b}_i$
    size-reduce the basis using only translations
end while
return $(\mathbf{b}_1, \ldots, \mathbf{b}_m)$

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Algorithm

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2 : **while** there exists i such that $\boxed{\ell_{i+1}^2 < (\frac{1}{t^2} - \mu_i^2)\ell_i^2}$ **do**

3 : choose i such that $||\mathbf{b}_{i+1}^\star||^2 + \mu_{i+1,i}^2||\mathbf{b}_i^\star||^2 < \frac{1}{t^2}||\mathbf{b}_i^\star||^2$
according to a strategy

4 : exchange $\mathbf{b}_{i-1}$ and $\mathbf{b}_i$

5 : size-reduce the basis using only translations

6 : **end while**

7 : **return** $(\mathbf{b}_1, \dots, \mathbf{b}_m)$

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<u>Algorithm</u>

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according to a strategy
$\ell_i \leftarrow \rho\ell_i$ and $\ell_{i+1} \leftarrow (1/\rho)\ell_{i+1}$ with $\boxed{\rho^2 = \frac{\ell_{i+1}^2}{\ell_i^2} + \mu_i^2}$
size-reduce the basis using only translations
end while
return $(\mathbf{b}_1, \ldots, \mathbf{b}_m)$

The LLL algorithm : a new point of vue

Input : the real vector $(\ell_1, \ldots, \ell_m)$ with $\ell_i = ||\mathbf{b}_i^\star||$
the subdiagonal coefficients $(\mu_1, \ldots, \mu_{m-1})$ with $|\mu_i| = |\mu_{i+1,i}| \leq \frac{1}{2}$
$t \geq 1$

Output : $(\hat{\ell}_1, \ldots, \hat{\ell}_m)$ and $(\hat{\mu}_1, \ldots, \hat{\mu}_{m-1})$ with for all $i = 1 \ldots m-1$,
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Algorithm

while there exists i such that $\boxed{\ell_{i+1}^2 < (\frac{1}{t^2} - \mu_i^2)\ell_i^2}$ **do**
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  call an oracle that recomputes μ_i and μ_{i+1}
end while
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5 : call an oracle that recomputes μ_i and μ_{i+1}

6 : **end while**

7 : **return** $(\ell_1, \ldots, \ell_m)$ and $(\mu_1, \ldots, \mu_{m-1})$

Plan

Simplified versions of the LLL algorithm

Updates in the LLL algorithm:

During an exchange, the norms of the vectors in the orthogonal basis become

$$||\mathbf{b}_i^\star|| \leftarrow \rho||\mathbf{b}_i^\star|| \quad \text{and} \quad ||\mathbf{b}_{i+1}^\star|| \leftarrow \frac{1}{\rho}||\mathbf{b}_{i+1}^\star|| \quad \text{with} \quad \rho^2 = \frac{||\mathbf{b}_{i+1}^\star||^2}{||\mathbf{b}_i^\star||^2} + \mu_{i+1,i}^2$$

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	LLL	Model 3	Model 2	Model 1
Conditions on ρ		- $\mu_{i+1,i}$ follows a uniform law on $[-\frac{1}{2}, \frac{1}{2}]$ - ρ depends on the ratio $\frac{\lVert\mathbf{b}_{i+1}^\star\rVert^2}{\lVert\mathbf{b}_i^\star\rVert^2}$ and $\mu_{i+1,i}$	- the $\mu_{i+1,i}$ are supposed to be constant - ρ only depends on the ratio $\frac{\lVert\mathbf{b}_{i+1}^\star\rVert^2}{\lVert\mathbf{b}_i^\star\rVert^2}$	ρ is supposed to be constant
Remarks	Too complicated	Open problem	Work in progress	True Chip Firing Game [Madritsch, Vallée]

Plan

The LLL algorithm : additive point of vue

Input : the real vector $(\ell_1, \ldots, \ell_m)$ with $\ell_i = ||\mathbf{b}_i^\star||$
the subdiagonal coefficients $(\mu_1, \ldots, \mu_{m-1})$ with $|\mu_i| = |\mu_{i+1,i}| \leq \frac{1}{2}$
$t \geq 1$

Output : $(\hat{\ell}_1, \ldots, \hat{\ell}_m)$ and $(\hat{\mu}_1, \ldots, \hat{\mu}_{m-1})$ with for all $i = 1 \ldots m-1$,
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according to a strategy

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5 : call an oracle that recomputes μ_i and μ_{i+1}

6 : **end while**

7 : **return** $(\ell_1, \ldots, \ell_m)$ and $(\mu_1, \ldots, \mu_{m-1})$

The LLL algorithm : additive point of vue

Input : the real vector $(q_1, \ldots, q_m)$ with $q_i = \log \ell_i^2$
the subdiagonal coefficients $(\mu_1, \ldots, \mu_{m-1})$ with $|\mu_i| = |\mu_{i+1,i}| \leq \frac{1}{2}$
$t \geq 1$

Output : $(\hat{q}_1, \ldots, \hat{q}_m)$ and $(\hat{\mu}_1, \ldots, \hat{\mu}_{m-1})$ with for all $i = 1 \ldots m-1$,
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$t \geq 1$

Output : $(\hat{q}_1, \ldots, \hat{q}_m)$ and $(\hat{\mu}_1, \ldots, \hat{\mu}_{m-1})$ with for all $i = 1 \ldots m-1$,
$\hat{q}_{i+1} \geq \log(\frac{1}{t^2} - \hat{\mu}_i^2) + \hat{q}_i$

Algorithm

```
while there exists i such that [q_{i+1} < H + q_i] with H = log(1/t^2 − μ_i^2) do
    choose i such that ||b*_{i+1}||^2 + μ^2_{i+1,i}||b*_i||^2 < (1/t^2)||b*_i||^2
                                        according to a strategy
    ℓ_i ← ρℓ_i and ℓ_{i+1} ← (1/ρ)ℓ_{i+1} with [ρ^2 = ℓ^2_{i+1}/ℓ^2_i + μ_i^2]
    call an [oracle] that recomputes μ_i and μ_{i+1}
end while
return (ℓ_1, ..., ℓ_m) and (μ_1, ..., μ_{m−1})
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Algorithm

while there exists i such that $\boxed{q_{i+1} < H + q_i}$ with $H = \log(\frac{1}{t^2} - \mu_i^2)$ **do**
    choose i such that $q_{i+1} < H + q_i$ with $H = \log(\frac{1}{t^2} - \mu_i^2)$
    according to a strategy
    $\ell_i \leftarrow \rho \ell_i$ and $\ell_{i+1} \leftarrow (1/\rho)\ell_{i+1}$ with $\boxed{\rho^2 = \frac{\ell_{i+1}^2}{\ell_i^2} + \mu_i^2}$
    call an oracle that recomputes μ_i and μ_{i+1}
end while
return $(\ell_1, \ldots, \ell_m)$ and $(\mu_1, \ldots, \mu_{m-1})$

The LLL algorithm : additive point of vue

Input : the real vector $(q_1, \ldots, q_m)$ with $q_i = \log \ell_i^2$
the subdiagonal coefficients $(\mu_1, \ldots, \mu_{m-1})$ with $|\mu_i| = |\mu_{i+1,i}| \leq \frac{1}{2}$
$t \geq 1$

Output : $(\hat{q}_1, \ldots, \hat{q}_m)$ and $(\hat{\mu}_1, \ldots, \hat{\mu}_{m-1})$ with for all $i = 1 \ldots m-1$,
$\hat{q}_{i+1} \geq \log(\frac{1}{t^2} - \hat{\mu}_i^2) + \hat{q}_i$

Algorithm

```
while there exists i such that [q_{i+1} < H + q_i] with H = log(1/t^2 - mu_i^2) do
    choose i such that q_{i+1} < H + q_i with H = log(1/t^2 - mu_i^2)
                                          according to a strategy
    q_i <- q_i - h and q_{i+1} <- q_{i+1} + h with
            [h = -log rho^2 = -log e^{q_{i+1} - q_i} + mu_i^2]
    call an [oracle] that recomputes mu_i and mu_{i+1}
end while
return (l_1, ..., l_m) and (mu_1, ..., mu_{m-1})
```

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Hypotheses

Model 1

H and h are supposed to be constant.

General sandpiles and chip firing games with parameters (H, h).

In our context, $q_i = \log ||\mathbf{b}_i^\star||^2 \qquad r_i = q_i - q_{i+1} = \log \frac{||\mathbf{b}_i^\star||^2}{||\mathbf{b}_{i+1}^\star||^2}$

The equation

$$\text{If} \quad q_i > q_{i+1} + H, \quad \text{then} \quad [\check{q}_i = q_i - h, \quad \check{q_{i+1}} = q_{i+1} + h].$$

defines the sandpile model of parameters (H, h).

The equation

$$\text{If} \quad r_i > H, \quad \text{then} \quad [\check{r}_i = r_i - 2h, \quad \check{r_{i+1}} = r_{i+1} + h, \quad \check{r_{i-1}} = r_{i-1} + h,].$$

defines the chip firing game of parameters (H, h).

Classical instances studied : basic and decreasing.

– Basic instances : Initial integer q_i's and parameters H, h equal to 1.
– Basic (strictly) decreasing instances :
The sequence $i \mapsto q_i$ is (strictly) decreasing.

Here, we study general instances of sandpile models.

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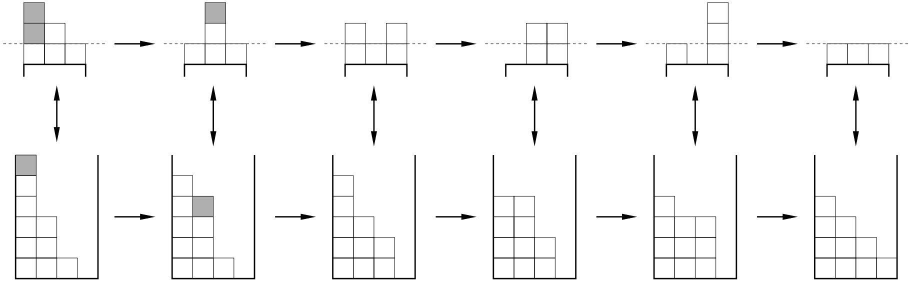
Here, we study general instances of sandpile models.

Evolution of a CFG and its sandpile

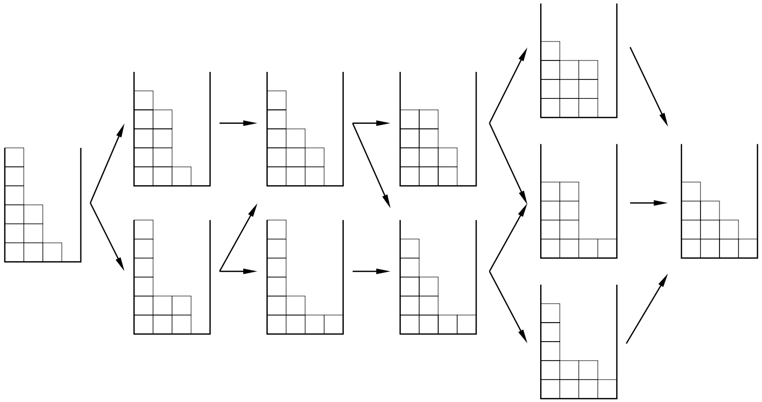


The evolution of a basic chip firing game (above),
and its associated sandpile (below).

above : $q_i = \log ||\mathbf{b}_i^\star||^2$

below : $r_i = q_i - q_{i+1} = \log \frac{||\mathbf{b}_i^\star||^2}{||\mathbf{b}_{i+1}^\star||^2}$

Evolutions of a basic sandpile



Possible evolutions of a basic sandpile.

For any sandpile of parameters (h, H)...

(i) There is a unique final $\hat{\mathbf{q}}$. The length of any path $\mathbf{q} \to \hat{\mathbf{q}}$ is

$$T(\mathbf{q}) = \frac{1}{2h} \sum_{i=1}^{n-1} i(n-i)\,(r_i - \hat{r}_i)$$

(ii) If the sandpile is decreasing, $H - 2h < \hat{r}_i \leq H$,

$$0 \leq T(\mathbf{q}) - \frac{1}{2h} \sum_{i=1}^{n-1} i(n-i)\,(r_i - H) \leq 2A(n) \qquad \text{with} \quad A(n) := n\frac{n^2-1}{12}$$

(iii) If the sandpile is strictly decreasing,

$$\exists! j \quad \forall i \neq j,\; H - h < \hat{r}_i \leq H, \qquad \text{and} \quad H - 2h < \hat{r}_j \leq H - h,$$

$$0 \leq T(\mathbf{q}) - \left[A(n) + \frac{1}{2h} \sum_{i=1}^{n-1} i(n-i)\,(r_i - H)\right] \leq \frac{1}{8}n^2$$

(iv) For a general sandpile,

$$H - 2h < \hat{r}_i \leq H \quad \text{if} \quad r_i > H - h, \qquad \hat{r}_i \geq r_i \quad \text{if} \quad r_i \leq H - h$$

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For any sandpile of parameters (h, H)...

(v) A sufficient condition for two adjacent strictly decreasing basic sandpiles

$$\mathbf{q}_- := (q_1, q_2, \ldots, q_p), \qquad \mathbf{q}_+ := (q_{p+1}, q_{p+2}, \ldots, q_{n+p})$$

to be independent is

$$\frac{1}{p}\left(\sum_{i=1}^{p} q_i\right) - \frac{1}{n}\left(\sum_{i=1}^{n} q_{p+i}\right) \leq \left(\frac{n+p}{2}\right) - 2.$$

In this case, the number of steps for the total sandpile $\mathbf{q}$ is (in parallel)

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Plan

Simplified versions of the LLL algorithm

Updates in the LLL algorithm:

During an exchange, the norms of the vectors in the orthogonal basis become

$$||\mathbf{b}_i^\star|| \leftarrow \rho||\mathbf{b}_i^\star|| \quad \text{and} \quad ||\mathbf{b}_{i+1}^\star|| \leftarrow \frac{1}{\rho}||\mathbf{b}_{i+1}^\star|| \quad \text{with} \quad \rho^2 = \frac{||\mathbf{b}_{i+1}^\star||^2}{||\mathbf{b}_i^\star||^2} + \mu_{i+1,i}^2$$

	LLL	Model 3	Model 2	Model 1
Conditions on ρ		- $\mu_{i+1,i}$ follows a uniform law on $[-\frac{1}{2}, \frac{1}{2}]$ - ρ depends on the ratio $\frac{\lVert\mathbf{b}_{i+1}^\star\rVert^2}{\lVert\mathbf{b}_i^\star\rVert^2}$ and $\mu_{i+1,i}$	- the $\mu_{i+1,i}$ are supposed to be constant - ρ only depends on the ratio $\frac{\lVert\mathbf{b}_{i+1}^\star\rVert^2}{\lVert\mathbf{b}_i^\star\rVert^2}$	ρ is supposed to be constant
Remarks	Too complicated	Open problem	Work in progress	True Chip Firing Game [Madritsch, Vallée]

Model M2(t, μ)

$$x_i = \frac{||\mathbf{b}^\star_{i+1}||^2}{||\mathbf{b}^\star_i||^2}, \qquad \rho^2 = x_i + \mu, \qquad \mathcal{O}_{t,\mu} = \left[\frac{1}{t^2} - \mu, +\infty\right[$$

Entrées: Le vecteur $\mathbf{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}_+^{d-1}$
Résultat: Le vecteur final $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_{d-1}) \in \mathcal{O}_{\mu,t}^{d-1}$

tant que $\mathbf{x} \notin \mathcal{O}_{\mu,t}^{d-1}$ **faire**
Choisir i tel que $x_i \notin \mathcal{O}_{\mu,t}$;
Calculer $\mathbf{x} := T_{i,\mu}(\mathbf{x})$ vérifiant

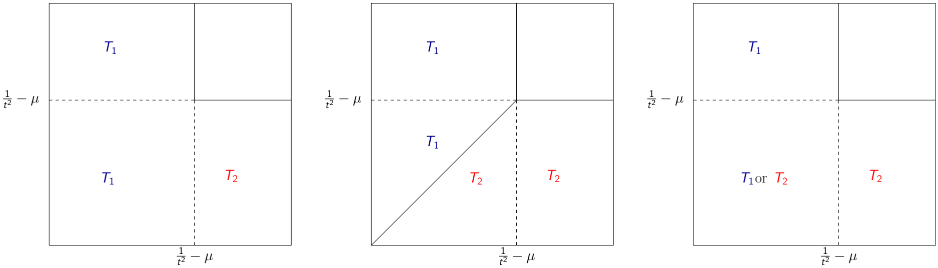
$$x_{i-1} := x_{i-1}(x_i + \mu), \qquad x_{i+1} := x_{i+1}(x_i + \mu), \qquad x_i = \frac{x_i}{(x_i + \mu)^2};$$

fintq

This is a general dynamical system in $\mathbb{R}^d$ with

- a hole $\mathcal{O}_{t,\mu}^{d-1}$
- and an attractive fixed point in $(1 - \mu, \ldots, 1 - \mu)$

Strategies



LLL strategy | Greedy strategy | Random strategy

First result when $t > 1$

Potential

$$P(\mathbf{x}) := \prod_{i=1}^{d-1} x_i^{i(d-i)}$$

With the greedy strategy, we have

$$P(T(\mathbf{x})) = \frac{P(\mathbf{x})}{\min_{i=1\ldots d-1} (x_i+\mu)^2} > P(\mathbf{x})$$

and if the basis is far from being reduced,

$$\min_{i=1\ldots d-1} (x_i+\mu)^2 \sim \mu^2, \qquad P(\mathbf{x}) \sim \mu P(T(\mathbf{x})), \qquad P(\mathbf{x}) \sim \mu^k P(T^k(\mathbf{x}))$$

But if the basis is close to be reduced,

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First result when $t > 1$

Number of iterations when $t > 1$

Consider the dynamical system $\mathrm{M2}(\mu, t)$ with $t > 1$, $\mu \in [0, \frac{1}{4}]$ and $\mathbf{x} \notin \mathcal{O}_{t,\mu}^{d-1}$. The number of iterations of $\mathrm{M2}(\mu, t)$ on $\mathbf{x}$, denoted by $K_{t,\mu}(\mathbf{x})$ satisfies

$$K_{t,\mu}(\mathbf{x}) = \frac{1}{2} \log_\mu P(\mathbf{x}) + O(d^3).$$

Remark :

- the known results on the complexity of LLL involve $\log_t P(\mathbf{x})$,
- nothing is known when $t = 1$.

Model M2(t, μ) with $t = 1$

We did not succeed in generalizing the result except for $d = 2$ (Gauss) and $d = 3$ (LLL in dimension 3).

Number of iterations when $t = 1$ and $d = 2, 3$

Consider the dynamical system M2$(\mu, 1)$, $\mu \in [0, \frac{1}{4}]$, $\mathbf{x} \notin \mathcal{O}_{t,\mu}^{d-1}$ and $d = 2, 3$. The number of iterations of M2$(\mu, 1)$ on $\mathbf{x}$, denoted by $K_{1,\mu}(\mathbf{x})$ satisfies

$$K_{1,\mu}(\mathbf{x}) = \frac{1}{2} \log_\mu P(\mathbf{x}) + O(1).$$

Conjecture

$$K_{1,\mu}(\mathbf{x}) = \frac{1}{2} \log_\mu P(\mathbf{x}) + O(d^3).$$

Model M2(t, μ) with $t = 1$

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Plan

Simplified versions of the LLL algorithm

Updates in the LLL algorithm:
During an exchange, the norms of the vectors in the orthogonal basis become

$$||\mathbf{b}_i^\star|| \leftarrow \rho||\mathbf{b}_i^\star|| \quad \text{and} \quad ||\mathbf{b}_{i+1}^\star|| \leftarrow \frac{1}{\rho}||\mathbf{b}_{i+1}^\star|| \quad \text{with} \quad \rho^2 = \frac{||\mathbf{b}_{i+1}^\star||^2}{||\mathbf{b}_i^\star||^2} + \mu_{i+1,i}^2$$

	LLL	Model 3	Model 2	Model 1
Conditions on ρ		- $\mu_{i+1,i}$ follows a uniform law on $[-\frac{1}{2}, \frac{1}{2}]$ - ρ depends on the ratio $\frac{\lVert\mathbf{b}_{i+1}^\star\rVert^2}{\lVert\mathbf{b}_i^\star\rVert^2}$ and $\mu_{i+1,i}$	- the $\mu_{i+1,i}$ are supposed to be constant - ρ only depends on the ratio $\frac{\lVert\mathbf{b}_{i+1}^\star\rVert^2}{\lVert\mathbf{b}_i^\star\rVert^2}$	ρ is supposed to be constant
Remarks	Too complicated	Open problem	Work in progress	True Chip Firing Game [Madritsch, Vallée]

Model M3(t, μ)

Entrées: Le vecteur $\mathbf{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}_+^{d-1}$
Le vecteur $\mu = (\mu_1, \ldots, \mu_{d-1}) \in [0, \frac{1}{4}]^{d-1}$
Résultat: Les vecteurs $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_{d-1})$ et $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_{d-1})$ tels que
$\hat{x}_i \geq \frac{1}{t^2} - \hat{\mu}_i$

tant que *il existe* i *tel que* $x_i < \frac{1}{t^2} - \mu_i$ **faire**
Choisir i tel que $x_i < \frac{1}{t^2} - \mu_i$
Calculer $\mathbf{x} := T_{i,\mu_i}(\mathbf{x})$ vérifiant

$$x_{i-1} := x_{i-1}(x_i + \mu_i), \qquad x_{i+1} := x_{i+1}(x_i + \mu_i), \qquad x_i = \frac{x_i}{(x_i + \mu_i)^2};$$

Générer aléatoirement un nouveau μ_i
fintq

This is a probabilistic dynamical system

Results

NONE !

Some ideas

$K(\mathbf{x})$ denotes the number of iterations on the input $\mathbf{x}$

Consider P_0 the potential such that $P(\mathbf{x}) \geq P_0 \Rightarrow \mathbf{x}$ is "reduced"

Consider a sequence of i.i.d. random variables $(\mu_i)_i$ that follow the same uniform law over $[0, \frac{1}{4}]$.

Consider the stopping time $T(\mathbf{x})$ defined as the minimum k such that

$$P_0 \prod_{i=1}^{k} \mu_i < P(\mathbf{x}).$$

my feeling : $K(\mathbf{x}) = T(\mathbf{x}) + O(d^3)$.

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Plan

Plan

Various notions of a random basis of a lattice.

(a) "Useful" lattice bases arise in applications : variations around knapsack bases and their transposes with bordered identity matrices.

$$\begin{pmatrix} A \mid I_p \end{pmatrix} \quad \left(\begin{array}{c|c} y & 0 \\ \hline x & qI_p \end{array}\right) \quad \left(\begin{array}{c|c} I_p & H_p \\ \hline 0_p & qI_p \end{array}\right) \quad \left(\begin{array}{c|c} q & 0 \\ \hline x & I_{p-1} \end{array}\right)$$

(b) Ajtai "bad" bases $B_p := (b_{i,p})$ associated to a sequence $a_{i,p}$

$$b_{i,p} \in \mathbb{Z}^p, \quad b_{i,p} = a_{i,p}\, e_i + \sum_{j=1}^{i-1} a_{i,j,p}\, e_j \qquad (\Rightarrow ||b^\star_{i,p}|| = a_{i,p})$$

with $$\alpha_{i,j,p} = \frac{a^{(p)}_{i,j}}{a^{(p)}_j} = \text{rand}\left(-\frac{1}{2}, \frac{1}{2}\right) \qquad \text{[size-reduced]}$$

and $$\frac{||b^\star_{i+1,p}||}{||b^\star_{i,p}||} = \frac{a^{(p)}_{i+1}}{a^{(p)}_i} \to 0 \qquad \text{when } p \to \infty \qquad \text{[bad ratios - non reduced]}$$

Experimental mean values .... versus proven upper bounds [Nguyen and Stehlé]

Main parameters.	$\|\|b^\star_{i+1,p}\|\|/\|\|b^\star_{i,p}\|\|$	approx. factor	Nb. steps
Worst-case (Proven upper bounds)	$1/s$	s^{p-1}	$\Theta(Mp^2)$
"Bad" lattice bases Random Ajtai bases (Experimental mean values)	$1/\beta$	β^{p-1}	$\Theta(Mp^2)$
"Useful " lattice bases Random knapsack–shape bases (Experimental mean values)	$1/\beta$	β^{p-1}	$\Theta(Mp)$

The execution parameters depend on the type of the lattice basis.

The output configuration does not depend strongly neither on index i nor on the type of bases.

"experimental" value : $\beta \approx 1.04$
the

Other notions of a random basis of a lattice – reference models.

(c) Spherical model.
Choose independently each one of the p vectors in the ambient space $\mathbb{R}^n$, under a common distribution that is invariant by rotation.
Classical instances :

- uniform distribution in the ball, on the sphere
- gaussian distribution on coordinates

(d) Random lattices.
The space of (full-rank) lattices in $\mathbb{R}^n$ (modulo scale) is $X_n = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$.
It possesses a unique probability measure
which is invariant under the action of $SL_n(\mathbb{R})$.
This gives rise to a natural notion of random lattices.

Probabilistic analyses of lattice reduction

- Akhavi, Marckert, Rouault (2005) [spherical model]
  - All the local bases are reduced except the last few ones
  - For the last few local bases, the length of the $\mathbf{b}_i^\star$ follows an explicit distribution

- Daudé and Vallée (1994) [random ball model]
  - The mean number of steps K satisfies

$$\mathrm{E}_n[K] \leq n^2 \left(\frac{1}{\log t} \right) \left[\frac{1}{2} \log n + 2 \right]$$

  - The mean size of the smallest nonzero vector of the lattice satisfies

$$\mathrm{E}_n[\lambda] \geq \frac{1}{4\sqrt{n}}$$

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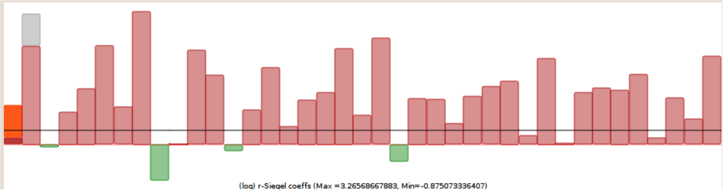
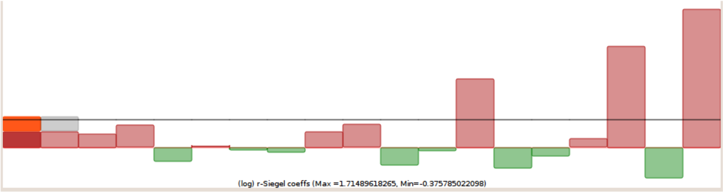
Some instances of cfg related to natural inputs



Ajtai type input

Knapsack type input

Some instances of cfg related to natural inputs



uniform distribution in the unit ball

Coppersmith's method

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General model of Ajtai inputs

This is a general model of random cfg. Once a cfg is given, it is easy to compute a random lattice associated to the cfg.

The general model is based on 3 parameters

- Υ : the total mass of the cfg
- d : the dimension of the cfg
- g : density function over $[0, 1]$

The cfg $(c_1, \ldots, c_{d-1})$ satisfies

- for all i, c_i follows an exponential law
- $\mathbb{E}[c_i] = \dfrac{1}{d-1} \Upsilon g\left(\dfrac{i}{d}\right)$
- the total mass : $\mathbb{E}[\mathcal{M}] = \sum_{i=1}^{d-1} \mathbb{E}[c_i] \underset{d\to\infty}{\sim} \Upsilon$
- the energy (potential) :
  $$\mathbb{E}[\mathcal{E}] = \sum_{i=1}^{d-1} i(d-i)\mathbb{E}[c_i] \underset{d\to\infty}{\sim} d^2 \Upsilon \int_0^1 x(1-x)g(x)dx\Upsilon$$

model $\mathcal{A}(\Upsilon, d, g)$

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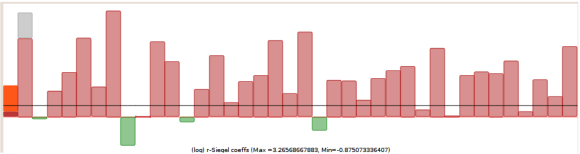


In Ajtai work,

$$g(x) = \frac{a+1}{2^{a+1}-1}(2-x)^a.$$

"Uni-tas" general model : $\mathcal{U}(\Upsilon, d, \beta)$

The general model is based on 3 parameters

- Υ : the total mass of the cfg
- d : the dimension of the cfg
- $\beta \in [0, 1]$ related to the position of the unique "pile"

$$i = 1 + \lfloor \beta(d-2) \rfloor$$

The cfg $(c_1, \ldots, c_{d-1})$ satisfies

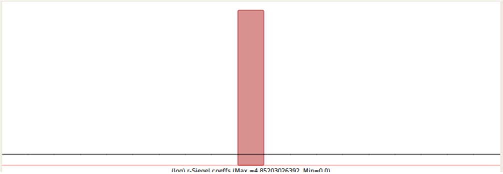
- $\mathbb{E}[c_i] = \Upsilon, \quad \mathbb{E}[c_j] = 0$ for $j \neq i$
- the total mass : $\mathbb{E}[\mathcal{M}] = \Upsilon$
- the energy (potential) :

$$\mathbb{E}[\mathcal{E}] \sim \begin{cases} d^2 \beta(1-\beta)\,\Upsilon & \text{si } \beta \in]0,1[\\ d\,\Upsilon & \text{si } \beta \in \{0,1\} \end{cases}.$$

“Uni-tas” general model : $\mathcal{U}(\Upsilon, d, \beta)$

Applications

- Knapsack problem : $\beta = 0$ (model $\mathcal{K}(\Upsilon, d)$)
- Schnorr factorization : $\beta = 0$
- protocol NTRU : $\beta = 1/2$ (model $\mathcal{N}(\Upsilon, d)$)
- ...



Coppersmith general model

A Coppersmith cfg can also be represented as the concatenation of several cfg

Results with the models M1/M2

Modèles	K **pire des cas**	K **pour** $\texttt{M1}(\alpha)$	K **pour** $\texttt{M2}(\mu)$	K **expérimental**
$\mathcal{A}(\Upsilon, d)$	$\frac{1}{12\log t} d(d+1)\tilde{\Upsilon}$	$\frac{1}{12\alpha} d(d+1)\tilde{\Upsilon}$	$\frac{1}{6\|\log\mu\|} d(d+1)\tilde{\Upsilon}$	$\Theta(d^2\tilde{\Upsilon})$
$\mathcal{N}(\Upsilon, d)$	$\frac{1}{8\log t} d^2\tilde{\Upsilon}$	$\frac{1}{8\alpha} d^2\tilde{\Upsilon}$	$\frac{1}{4\|\log\mu\|} d^2\tilde{\Upsilon}$	$\Theta(d^2\tilde{\Upsilon})$
$\mathcal{K}(\Upsilon, d)$	$\frac{1}{2\log t} d\tilde{\Upsilon}$	$\frac{1}{2\alpha} d\tilde{\Upsilon}$	$\frac{1}{\|\log\mu\|} d\tilde{\Upsilon}$	$\Theta(d\tilde{\Upsilon})$

- $\tilde{\Upsilon} = \alpha \cdot \Upsilon$ with α a known constant

Plan

Conclusion

- Simplified models,
  very useful for explaining, making experiments, finding conjectures......
- Only qualitative similarities with the actual LLL algorithm.
- Possible (easy) proofs.
- Some results in dimension $d \geq 3$ that do not exist for the LLL algorithm
- New challenges : model M3 which is a probabilistic dynamical system with a hole