Gaussian laws in Euclidean Context,

Vertical strip with polynomial growth

and Condition UNI

Brigitte Vallée, Laboratoire GREYC

(CNRS et Université de Caen)

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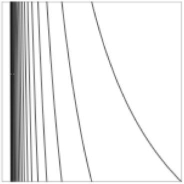
The transfer operator (I).



Density Transformer:
For a density f on $[0,1]$, $\mathbf{H}[f]$ is the density on $[0,1]$ after one iteration of the shift

$$\mathbf{H}[f](x) = \sum_{h\in\mathcal{H}} |h'(x)|\, f\circ h(x) = \sum_{m\in\mathbb{N}} \frac{1}{(m+x)^2} f(\frac{1}{m+x}).$$

Transfer operator (Ruelle):

$$\mathbf{H}_s[f](x) = \sum_{h\in\mathcal{H}} |h'(x)|^s\, f\circ h(x).$$

The k-th iterate satisfies:

$$\mathbf{H}_s^k[f](x) = \sum_{h\in\mathcal{H}^k} |h'(x)|^s\, f\circ h(x)$$

The transfer operator (II)

The density transformer $\mathbf{H}$ expresses the new density f_1 as a function of the old density f_0, as $f_1 = \mathbf{H}[f_0]$. It involves the set $\mathcal{H}$

$$\mathbf{H}: \qquad \mathbf{H}[f](x) := \sum_{h \in \mathcal{H}} |h'(x)| \cdot f \circ h(x)$$

With a cost $c : \mathcal{H} \to \mathbf{R}^+$ extended to $\mathcal{H}^\star$ by additivity,

it gives rise to the weighted transfer operator

$$\mathbf{H}_{s,w}: \qquad \mathbf{H}_{s,w}[f](x) := \sum_{h \in \mathcal{H}} \exp[wc(h)] \cdot |h'(x)|^s \cdot f \circ h(x)$$

$$\left\{ \begin{array}{c} \text{Multiplicative properties of the derivative} \\ \text{Additive properties of the cost} \end{array} \right\} \Longrightarrow$$

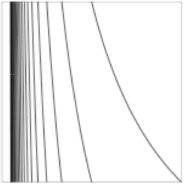
$$\mathbf{H}^n_{s,w}[f](x) := \sum_{h \in \mathcal{H}^n} \exp[wc(h)] \cdot |h'(x)|^s \cdot f \circ h(x)$$

The n–th iterate of $\mathbf{H}_{s,w}$ generates the CFs of depth n.

The quasi inverse $(I - \mathbf{H}_{s,w})^{-1} = \sum_{n \geq 0} \mathbf{H}^n_{s,w}$ generates all the finite CFs.

Properties of the dynamical system: the Good Class

$$M_h := \sup\{|h'(x)|, \quad x \in X\}$$



(1) Uniform contraction.

$$\forall h \in \mathcal{H}, \quad M_h \leq 1$$

$$\exists \rho < 1, n_0 \geq 1 \quad M_h \leq \rho \quad \forall h \in \mathcal{H}^{n_0}$$

(2) Bounded distortion.

$$\exists K > 0, \forall h \in \mathcal{H}, \forall x \in X, \quad |h''(x)| \leq K\, |h'(x)|.$$

(3) Convergence on the left of $\Re s = 1$.

$$\exists \sigma_0 < 1, \forall \sigma > \sigma_0, \quad \sum_{h \in \mathcal{H}} M_h^{\sigma} < \infty$$

Properties of the cost

A cost $c : \mathcal{H} \to \mathbf{R}^+$ first defined on $\mathcal{H}$,
then extended to $\mathcal{H}^\star$ by additivity $c(h \circ k) := c(h) + c(k)$.

A cost is of moderate growth if $c(h) = O(|\log M_h|)$

What is needed on the operator $\mathbf{H}_{s,w}$ for the analysis of the algorithm?

For the average case,

only properties on $(I - \mathbf{H}_s)^{-1}$ for $\Re s \geq 1$ and near $s = 1$

For the distributional analysis,

additional properties on $(I - \mathbf{H}_{s,w})^{-1}$ on the left of $\Re s = 1$.

Quasi-Compactness

For an operator $\mathbf{L}$,

– the spectrum $\mathrm{Sp}(\mathbf{L}) := \{\lambda \in \mathbb{C}; \quad L - \lambda I \quad \text{non inversible}\}$

– the spectral radius $R(\mathbf{L}) := \sup\{|\lambda|, \quad \lambda \in \mathrm{Sp}(\mathbf{L})\}$

– the essential spectral radius $R_e(\mathbf{L}) =$ the smallest $r > 0$ s.t
any $\lambda \in \mathrm{Sp}(\mathbf{L})$ with $|\lambda| > r$ is an isolated eigenvalue of finite multiplicity.

– For compact operators, the essential radius equals 0.

– $\mathbf{L}$ is quasi-compact if the inequality $R_e(\mathbf{L}) < R(\mathbf{L})$ holds.

Then, outside the closed disk of radius $R_e(\mathbf{L})$, the spectrum of the operator consists of isolated eigenvalues of finite multiplicity.

Sufficient conditions for quasi-compactness

A theorem, due to Hennion:

Suppose that the Banach space $\mathcal{F}$
– is endowed with two norms, a weak norm $|.|$ and a strong norm $||.||$,
– and the unit ball of $(\mathcal{F}, ||.||)$ is precompact in $(\mathcal{F}, |.|)$.

If $\mathbf{L}$ is a bounded operator on $(\mathcal{F}, ||.||)$ for which there exist two sequences $\{r_n \geq 0\}$ and $\{t_n \geq 0\}$ s.t.

$$||\mathbf{L}^n[f]|| \leq r_n \cdot ||f|| + t_n \cdot |f| \qquad \forall n \geq 1, \forall f \in \mathcal{F},$$

Then: $\quad R_e(\mathbf{L}) \leq r := \lim_{n\to\infty} \inf\, (r_n)^{1/n}$.

If $R(\mathbf{L}) > r$, then the operator $\mathbf{L}$ is quasi-compact on $(\mathcal{F}, ||.||)$.

For systems of the Good Class, $\mathcal{F} := \mathcal{C}^1(X)$,
– the weak norm is the sup-norm $||f||_0 := \sup|f(t)|$,
– the strong norm is the norm $||f||_1 := \sup|f(t)| + \sup|f'(t)|$.
– the density transformer satisfies the hypotheses of Hennion's Theorem.

Main Analytical Properties of $\mathbf{H}_{s,w}$ for a dynamical system of the Good Class and a digit-cost c of moderate growth.

$\mathbf{H}_{s,w}$ acts on $\mathcal{C}^1(\mathcal{I})$ for $\Re s > \sigma_0$ and $\Re w$ small enough
The map $(s,w) \mapsto \mathbf{H}_{s,w}$ is analytic near the reference point $(1,0)$

For s and w real, the operator is quasi–compact. Thus:

Property *UDE* : Unique dominant eigenvalue $\lambda(s,w)$,

Property *SG* : Existence of a spectral gap.

With perturbation theory, this remains true for (s,w) near $(1,0)$,

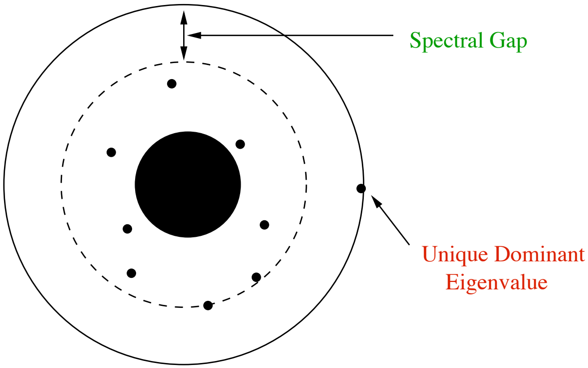
$(s,w) \mapsto \lambda(s,w)$ is analytic.

A spectral decomposition $\quad \mathbf{H}_{s,w} = \lambda(s,w) \cdot \mathbf{P}_{s,w} + \mathbf{N}_{s,w}$.

$\mathbf{P}_{s,w}$ is the projector on the dominant eigensubspace.

$\mathbf{N}_{s,w}$ is the operator relative to the remainder of the spectrum, whose spectral radius $\rho_{s,w}$ satisfies $\rho_{s,w} \le \theta\lambda(s,w)$ with $\theta < 1$.

.....which extends to all $n \geq 1$, $\quad \mathbf{H}^n_{s,w} = \lambda^n(s,w) \cdot \mathbf{P}_{s,w} + \mathbf{N}^n_{s,w}$.


Spectral Gap
Unique Dominant
Eigenvalue

Then, a Quasi–Power Property

$$\mathbf{H}_{s,w}^{n}[f] = \lambda^{n}(s,w) \cdot \mathbf{P}_{s,w}[f] \cdot [1 + O(\theta^{n})]$$

and, a decomposition for the quasi–inverse

$$(I - \mathbf{H}_{s,w})^{-1} = \lambda(s,w) \frac{\mathbf{P}_{s,w}}{1 - \lambda(s,w)} + (I - \mathbf{N}_{s,w})^{-1}$$

Since $\mathbf{H}_{1,0}$ is a density transformer, one has

$$\lambda(1,0) = 1, \qquad \mathbf{P}_{1,0}[f](x) = \Psi(x) \cdot \int_{I} f(t)dt$$

“Dominant” (polar) singularities of $(I - \mathbf{H}_{s,w})^{-1}$ near the point $(1,0)$: along a curve $s = \sigma(w)$ on which the dominant eigenvalue satisfies

$$\lambda(\sigma(w), w) = 1$$

Another important condition: the Aperiodicity condition:

On the line $\Re s = 1$, $1 \notin \mathrm{Sp}\mathbf{H}_s$.

The triple $UDE + SG + \mathrm{Aperiodicity}$ entails good properties for $(I - \mathbf{H}_s)^{-1}$, sufficient for applying Tauberian Theorems

$s = 1$ is the only pole
on the line $\Re s = 1$

Expansion near the pole $s = 1$

$$(I - \mathbf{H}_s)^{-1} \sim \frac{a}{s-1}$$

s=1

Half–plane of convergence $\Re s > 1$

No hypothesis needed
on the half–plane $\Re s < 1$.

Property $US(s,w)$: Uniformity on Vertical Strips

There exist $\alpha > 0, \beta > 0$ such that,

on the vertical strip $\mathcal{S} := \{s; |\Re(s) - 1| < \alpha\}$,

and uniformly when $w \in \mathcal{W} := \{w; |\Re w] < \beta\}$,

(i) [Strong aperiodicity] $s \mapsto (I - \mathbf{H}_{s,w})^{-1}$ has a unique pole inside $\mathcal{S}$;
it is located at $s = \sigma(w)$ defined by $\lambda(\sigma(w), w) = 1$.

(ii) [Uniform polynomial estimates] For any $\gamma > 0$, there exists $\xi > 0$ s.t,

$$(I - \mathbf{H}_{s,w})^{-1}[1] = O(|\Im s|^{\xi}) \qquad \forall s \in \mathcal{S}, \quad |t| > \gamma, \quad w \in \mathcal{W}$$

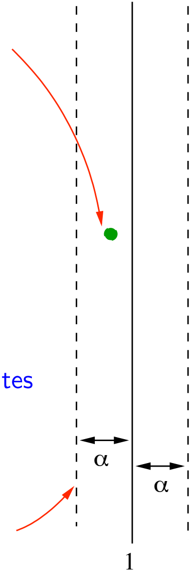
With the Property *US*,

it is easy to deform the contour of the Perron Formula

and use Cauchy's Theorem . . .

Near $w = 0$, the function σ is defined by $\lambda(\sigma(w), w) = 1$

$s = \sigma(w)$ is the only pole
on the strip $|\Re s - 1| \leq \alpha$



Expansion
near the pole $s = \sigma(w)$
$(I - \mathbf{H}_{s,w})^{-1} \sim \dfrac{a}{s - \sigma(w)}$

Half–plane of
convergence $\Re s > \sigma(w)$

Uniform polynomial estimates
needed on the left domain
$1 - \alpha \leq \Re s \leq 1, |\Im s| \geq \gamma$.

Property $US(s)$ is not always true

Item (i) is always false for Dynamical Systems with affine branches.

Example: Location of poles of $(I - \mathbf{H}_s)^{-1}$ near $\Re s = 1$

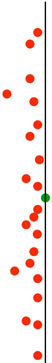
in the case of affine branches of slopes $1/p$ and $1/q$ with $p + q = 1$.

Two main cases

If $\quad \dfrac{\log p}{\log q} \in \mathbf{Q}$

Regularly spaced poles

on $\Re s = 1$



If $\quad \dfrac{\log p}{\log q} \notin \mathbf{Q}$

Only one pole at $s = 1$

on $\Re s = 1$

but accumulation of poles

on the left of $\Re s = 1$

Three main facts.

(a) There exist various conditions, (introduced by Dolgopyat),

the Conditions *UNI* that express that

"the dynamical system is quite different from a system

with piecewise affine branches"

(b) For a good Dynamical system

[complete, strongly expansive, with bounded distortion],

Conditions *UNI* imply the Uniform Property $US(s, w)$.

(c) Conditions *UNI* are true in the Euclid context.

Dolgopyat (98) proves the Item (b) but

– only for Dynamical Systems with a finite number of branches

– He considers only the $US(s)$ Property

Baladi-Vallée adapt his arguments to generalize this result:

For a Dynamical System
with a denumerable number of branches (possibly infinite),
Conditions UNI [Strong or Weak] imply $US(s, w)$.

Precisions about the UNI Conditions

Distance Δ. $\quad \Delta(h,k) := \inf_{x\in\mathcal{I}} \Psi'_{h,k}(x), \quad$ with $\quad \Psi_{h,k}(x) := \log \frac{|h'(x)]}{|k'(x)|}$

Contraction ratio ρ. $\quad \rho := \limsup \left(\{\max |h'(x)|; h \in \mathcal{H}^n, x \in \mathcal{I}\}\right)^{1/n}.$

Probability $\Pr_n$ on $\mathcal{H}^n \times \mathcal{H}^n$. $\quad \Pr_n(h,k) := |h(\mathcal{I})| \cdot |k(\mathcal{I})|$

For a system $\mathcal{C}^2$–conjugated with a piecewise-affine system :
For any $\hat\rho$ with $\rho < \hat\rho < 1$, for any n, $\quad \Pr_n[\Delta < \hat\rho^n] = 1$

Strong Condition UNI.

For any $\hat\rho$ with $\rho < \hat\rho < 1$, for any n, $\quad \Pr_n[\Delta < \hat\rho^n] \ll \hat\rho^n$

Weak Condition UNI.

$\exists D > 0, \exists n_0 \geq 1$, $\forall n \geq n_0, \quad \Pr_n[\Delta \leq D] < 1.$

The main ideas to prove UNI Condition $\Longrightarrow$ Vertical Strip with Polynomial Growth. (I) Using the L^2 norm.

Study $$\int_{\mathcal{I}} |\mathbf{H}^n_{s,w}[f](x)|^2 dx, \qquad s = \sigma + it,\ w = \nu + i\tau$$

There are two parts in the double sum

$$|\mathbf{H}^n_{s,w}[f]|^2 = \sum_{(h,k)\in\mathcal{H}^n\times\mathcal{H}^n} \exp[wc(h) + \bar{w}c(k)] \cdot \exp[it\Psi_{h,k}(x)] \cdot R_{h,k}(x)\,,$$

with $$\Psi_{h,k}(x) := \log\frac{|h'(x)|}{|k'(x)|}\,, \qquad \Delta(h,k) := \inf_{x\in\mathcal{I}} \Psi'_{h,k}(x)\,.$$

– the part brought by the "close" pairs (h,k) (with $\Delta(h,k)$ small)

With the UNI Condition, there are "few" close pairs

– the part brought by the other pairs, (with $\Delta(h,k)$ large and a lower bound for $\Psi'_{k,k}$). Then, there are oscillatory integrals

$$I_{(h,k)} = \int_{\mathcal{I}} \exp[it\Psi_{h,k}(x)]\ R_{h,k}(x)\ dx$$

Van der Corput Lemma [Stein]
Oscillatory integrals.

$$I(t) = \int_{\mathcal{I}} \exp[it\Psi(x)]\; r(x)\; dx$$

with $t \in \mathbb{R}$ and

(i) $\Psi \in \mathcal{C}^2(\mathcal{I})$, $|\Psi''(x)| \leq Q$, $|\Psi'(x)| \geq \Delta$ with $|t|^{-1} \leq \Delta \leq 1$

(ii) $r \in \mathcal{C}^\infty(\mathcal{I})$ with $||r||_0 \leq R\,, \quad ||r||_{1,1} \leq RD$

Then $\qquad |I(t)| \leq R\, C(Q) \left[\frac{D+1}{|t|\Delta} + \frac{1}{|t|\Delta^2} \right] .$

wth a uniform bound $C(Q)$

The main ideas to prove

UNI Condition $\Longrightarrow$ Vertical Strip with Polynomial Growth.

(II) From the L^2 norm to other norms.

More standard. Seems easy to extend...

The `UNI` Condition in the Euclidean context.

For two LFT's h_1 and h_2, with $h_i(x) = (a_i x + b_i)/(c_i x + d_i)$, we have

$$\Psi'_{h_1,h_2}(x) = \left| \frac{h_1''}{h_1'}(x) - \frac{h_2''}{h_2'}(x) \right| = \frac{|c_1 d_2 - c_2 d_1|}{|(c_1 x + d_1)(c_2 x + d_2)|},$$

$$\Delta(h_1, h_2) := \left| \frac{c_1}{d_1} - \frac{c_2}{d_2} \right| \cdot \inf_{x \in \mathcal{I}} \left| \frac{h_1'(x) h_2'(x)}{h_1'(0) h_2'(0)} \right|^{1/2} \geq \frac{1}{L} \left| \frac{c_1}{d_1} - \frac{c_2}{d_2} \right|$$

h^* the mirror of h : $\qquad h^*(x) = \dfrac{ax + c}{bx + d} \qquad$ if $\qquad h(x) = \dfrac{ax + b}{cx + d}$.

$$\Delta(h_1, h_2) \geq \frac{1}{L} \left| h_1^*(0) - h_2^*(0) \right|.$$

For the Classical Euclidean algorithm,

the two systems $(\mathcal{I}^*, T^*)$ and $(\mathcal{I}, T)$ coincide.

One has $\rho = \rho^\star$, and $L^\star$ is the distortion constant.

The $\texttt{UNI}$ Condition in the Euclidean context. Using the mirror system.

Compare

$$J^*(h,\eta) = \bigcup_{\substack{k\in\mathcal{H}^n \\ \Delta(h,k)\le\eta}} |k^*(\mathcal{I})|, \quad J(h,\eta) = \bigcup_{\substack{k\in\mathcal{H}^n \\ \Delta(h,k)\le\eta}} |k(\mathcal{I})|$$

$$\Delta(h,k) \ge \frac{1}{L}\,|h^*(0) - k^*(0)| \Rightarrow |J^*(h,\eta)| \ll 2L\eta + 2\hat{\rho}^n$$

$$\frac{1}{(LL^*)^{1/2}} \le \frac{|k(\mathcal{I})|}{|k^*(\mathcal{I}^*)|} = \frac{|k(\mathcal{I})|}{|k'(0)|}\,\frac{|(k^*)'(0)|}{|k^*(\mathcal{I}^*)|} \le (LL^*)^{1/2}\,.$$

$$|J(h,\eta)| \le (LL^*)^{1/2}|J^*(h,\eta)| \ll (LL^*)^{1/2}(2L\eta + \rho^n)$$

And finally

$$\eta \le \hat{\rho}^n \Rightarrow |J(h,\eta)| \ll \hat{\rho}^n$$