MULTIPLE GCDs:
Probabilistic analysis of the plain algorithm
The number case.

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Work in progress...

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Computing GCDs of d inputs

For $d = 2$: the "classical" Euclid algorithm,

For $d \geq 3$, there are various strategies.

The plain algorithm performs a sequence of computations on two entries;

On the input $(x_1, x_2, \ldots, x_d)$, it computes

- first: $y_2 := \gcd(x_1, x_2)$
- then, for $k \in [3..d]$: $y_k := \gcd(x_k, y_{k-1}) = \gcd(x_1, x_2, \ldots, x_k)$.

The "total" gcd $y_d := \gcd(x_1, x_2, \ldots, x_d)$ is obtained after $d - 1$ phases.

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A very natural scheme, proposed in Knuth's book, but not yet analyzed.

In this talk, – we focus on the analysis of the number case,

– we explain the similarities/differences between the two cases.

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The same phenomena occur for the size of the partial gcd.

Probabilistic analysis of the plain d–GCD algorithm on $\mathbb{Z}$.

On the input $(x_1, x_2, \ldots, x_d)$,

- the algorithm computes the total gcd $y_d := \gcd(x_1, x_2, \ldots, x_d)$
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The set of inputs is $\Omega = \{\underline{x} := (x_1, \ldots, x_d); \quad x_i \in \mathbb{N}\}$

The size of an input: $\mathrm{d}(\underline{x}) := \mathrm{d}(x_1 x_2 \ldots x_d)$ with $\mathrm{d}(x) := \lceil \log x \rceil$

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Main parameters of interest

- the number L_k of divisions during the k–th phase

i.e. on the input (x_k, y_{k-1})

- the size D_k of the k–th gcd

(at the beginning of the k-th phase).

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$$\gcd(u_1, u_2) = 1, \gcd(u_2, u_3) = 1$$

$$\frac{u_2}{u_1} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cfrac{1}{\ddots + \cfrac{1}{m_r}}}}}, \qquad \frac{u_3}{u_2} = \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cfrac{1}{\ddots + \cfrac{1}{m_r}}}},$$

$$\frac{u_2}{u_1} = h_{m_1} \circ h_{m_2} \circ \ldots \circ h_{m_r}(0), \qquad \frac{u_3}{u_2} = h_{m_2} \circ \ldots \circ h_{m_r}(0)$$

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where the Riemann zeta function $\zeta(s)$ is the gen. function of the set $\mathbb{N}$.

With the $\gcd y$

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With the previous decomposition with the LFT's

$$2 \sum_{(u_1,u_2)} \frac{1}{u_1^{s_1}} \frac{1}{u_2^{s_2}} = (I - \mathbf{G}_{s_1+s_2})^{-1} \circ (\mathbf{G}_{s_1} + \mathbf{G}_{s_2})[1](0)$$

where $\mathbf{G}_s$ is the generating operator for the quotients,

$$\mathbf{G}_s[f](x) = \sum_{m\geq 1} |h_m'(x)|^{s/2} f\circ h_m(x) = \sum_{m\geq 1} \left(\frac{1}{m+t}\right)^s f\left(\frac{1}{m+x}\right)$$

The Euclid algorithm ($d = 2$) on integers

translates as a product of Dirichlet generating functions

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This is a functional operator which depends on a complex parameter s,

$$\mathbf{G}_s[f](t) = \sum_{m \geq 1} \left(\frac{1}{m+t} \right)^s f\left(\frac{1}{m+t} \right)$$

This is the transfer operator of the Euclidean underlying dynamical system,

The analysis is more involved than the previous one,

but provides the same type of results.

Similarities and differences between the two analyses

	Polynomials	Integers
GF	Power GF	Dirichlet GF and operators
Basic tool	$G(z) = \sum_m z^{\mathrm{d}(m)}$	$\mathbf{G}_s[f](t) = \sum_{m\geq 1} \left(\frac{1}{m+t}\right)^s f\left(\frac{1}{m+t}\right)$
Phase GF	$\frac{U(z^k)+U(z)-1}{1-G(z^{k+1})}$	$(I-\mathbf{G}_{(k+1)s})^{-1} \circ [\mathbf{G}_{ks}+\mathbf{G}_s]$
Singularities	z s.t. $G(z^{k+1})=1$	s s.t. $\lambda((k+1)s)=1$
Extraction	Cauchy Formula	Perron Formula
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$\lambda(s)$ is the dominant eigenvalue of $\mathbf{G}_s$

$\lambda(2)=1$; $\lambda'(2)$ closely related to the entropy

Generating function relative to the d-Euclid Algorithm

We have shown that the Euclid algorithm ($d = 2$) translates as a product

$$\zeta(s_1)\,\zeta(s_2) = T(s_1, s_2)\;\zeta(s_1 + s_2),$$

with $\quad 2T(s,t) = (1 - \mathbf{G}_{s+t})^{-1} \circ (\mathbf{G}_s + \mathbf{G}_t)\,[1](0)$

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Then, for any $d \geq 2$, the d–Euclid algorithm translates as the product

$$\zeta(s_1) \cdot \ldots \cdot \zeta(s_d) = \zeta(t_d) \prod_{k=1}^{d-1} T(s_{k+1}, t_k) \qquad [t_k := s_1 + s_2 + \ldots + s_k,]$$

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$$\zeta(s_1) \cdot \ldots \cdot \zeta(s_d) = \zeta(t_d) \prod_{k=1}^{d-1} T(s_{k+1}, t_k) \qquad [t_k := s_1 + s_2 + \ldots + s_k,]$$

Now, with $s = s_1 = \ldots = s_d$,
the (plain) generating function $S(s)$ of $\mathbb{N}^d$ has an alternative expression

$$S(s) = \zeta(s)^d = \zeta(ds) \prod_{k=1}^{d-1} T(s, ks)$$

which is an exact translation of the d-Euclid algorithm.

T is the "phase generating function".

Generating functions relative to the d-Euclid Algorithm

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For the expectations, the cumulative generating functions are useful:

$$\widehat{L}_k(s) := \frac{\partial}{\partial u} L_k(s,u)|_{u=1} = \frac{\zeta(s)^d}{T(s,ks)} \frac{\partial}{\partial u} T(s, ks, u)|_{u=1}$$

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Towards the distributional analysis of L_k and D_k.

The generating functions of the events $[L_k > m]$ and $[D_k > m]$ are

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The asymptotics depends on the value $a := A_k(1)$ at the pole $s = 1$

For $k = 1$, one has $a = 1$ – For $k \geq 2$, one has $a < 1$.

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We need extractors. For a Dirichlet generating function,

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$$\Psi_n[S] := \sum_{p;\mathrm{d}(p)=n} a_p = \sum_{p=e^{n-1}+1}^{e^n} a_p \qquad \mathrm{d}(p) := \lceil \log p \rceil$$

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We need theorems which relate

– the asymptotic properties of $\Psi_n[S]$ (for $n \to \infty$)

– and analytic properties of $S(s)$ near its dominant singularity,

Dominant singularity = singularity with the largest real part = here $s = 1$.

We are interested in the case when S contains a large m-th power.

A general result.

Consider the sequence of Dirichlet series

$$F^{[m]}(s) = B(s)\ A(s)^m = \sum_{p \geq 1} \frac{a_p^{[m]}}{p^s}, \qquad a_p^{[m]} \geq 0$$

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(i) $B(s)$ has an only pole at $s = 1$ of order $d \geq 2$;

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Then, when $m/n \in [0, c_0]$ with $c_0 < a/|b|$, one has:

$$\Psi_n\left(F^{[m]}\right) := \sum_{p;\mathrm{d}(p)=n} a_p^{[m]} = e^n(e-1)\frac{n^{d-1}}{(d-1)!}\ a^m \left(1 - \frac{|b|}{a}\frac{m}{n}\right)^{d-1} \left[1 + O\left(\frac{1}{n}\right)\right].$$

Result based on the Landau Theorem: it uses the Perron formula

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We have to adapt the previous result, as $A_m(s)$ is not a true m–th power.

When $|\tau|$ is small: $A_m(s)=a^m(s)f(s)[1+\theta_m(s)]$, $|\theta_m(s)|\le\theta^m$, $(\theta<1)$

$$\left[e^{ns}e^{\log A_m(s)}\right]_{s=1}^{(d-1)} \approx \left[n+m\frac{a'(1)}{a(1)}\right]^{d-1}[e^nA_m(1)]$$

When $|\tau|\to\infty$: $|A_m(s)|\le|\tau|^\xi|A_m(\sigma)|\ll|\tau|^\xi|a^m(\sigma)|$

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$$\mathbb{P}_n[L_k > m] = \mathbf{G}_{k+1}^m[1](0) \left(1 - \frac{|b|}{a}\frac{m}{n}\right)^{d-1} \left[1 + O\left(\frac{1}{n}\right)\right]$$

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For the D-case, with the constants $a = e^{1-k}, b = -ke^{1-k}$.

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For the first phase: $a = 1 \implies$ A quasi –beta behavior $(1, d-1)$

For the subsequent phases: $a < 1 \implies$ A quasi–geometric behavior

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The number of divisions L_1 performed by the d-Euclid algorithm during the first phase has a mean value of linear order

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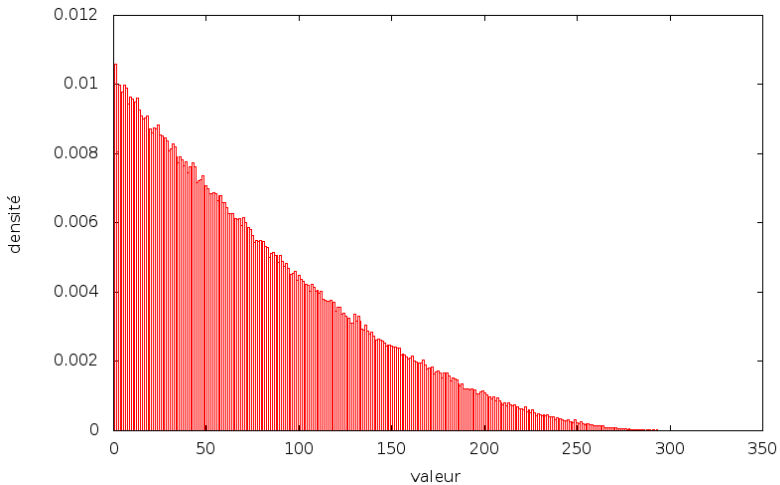

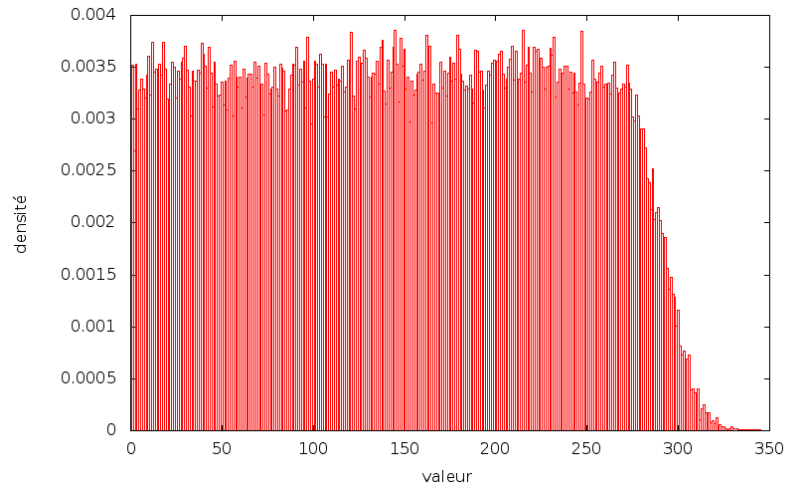

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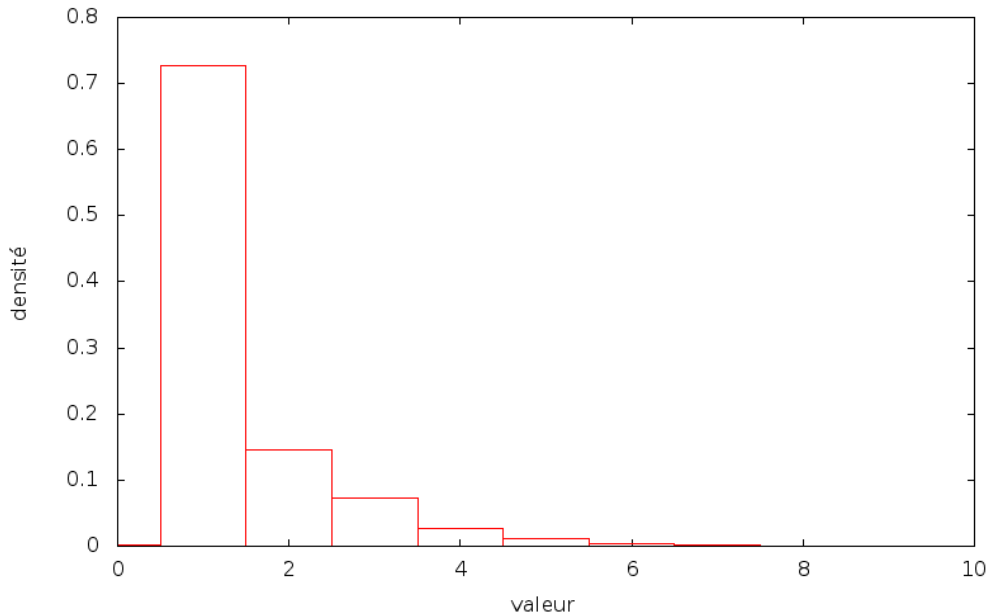

Main result for the size D_k of the $\gcd$ – First phase ($k = 1$)

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