Anatomy of an extensional collapse

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Abstract

We prove that, in the hierarchy of simple types, the set-theoretic coherence semantics is the extensional collapse of the multiset-theoretic coherence semantics.

Introduction

The notion of *stable* function has been introduced by Berry for the purpose of modelling functional programming languages like PCF [Ber78]. In the framework of *dilators* (functors acting on ordinals), Girard discovered independently stability as a condition allowing for a finitary representation of these functors. He applied the same idea to the denotational semantics of system F (see [Gir86]) and this led him to the crucial observation that this semantics (which is an extension of Berry's semantics of PCF) can be described in the very simple framework of *coherence spaces*. Berry actually developped his semantics in the framework of dI-domains (Scott domains satisfying some further properties). Coherence spaces are very particular dI-domains which define a sub-cartesian-closed category of the category of dI-domains and stable functions.

A coherence space is a symmetric and reflexive unlabelled graph (its *web* is the set of vertices; two vertices which are related are said to be *coherent*). The cliques of this graph are the elements of the corresponding dI-domain (singletons correspond to prime elements, finite cliques to compact elements).

The space of stable functions from a coherence space X to a coherence space Y can in turn be described as a coherence space Z through traces: if f is a stable function from X to Y, the trace of f is the set of all couples (x_0, b) where b is a vertex of Y and x_0 is a finite clique of X minimal such that $b \in f(x_0)$. This leads to the idea that the function space operation (which corresponds to implication through the Curry-Howard isomorphism) is not atomic. It can be decomposed in two operations: (set-theoretic) exponential and linear functions space.

The exponential of a coherence space X has as web the set of all finite cliques of X, and the linear function space of two coherence spaces X' and Y has as web the cartesian product of the webs of X' and Y.

These two operations have logical counterparts which are made explicit as logical connectives in linear logic ([Gir87, GLT89, Gir95] describe the coherence space semantics of linear logic).

Van de Wiele observed that alternative definitions of the exponential operation on coherence spaces are available. More specifically, from a categorical viewpoint, the exponential is an endofunctor on the category of coherence spaces and linear maps, and this functor has an additional structure of *comonad* satisfying some further requirements (the image of a coherence space by this functor has a canonical structure of commutative comonoid, see [Bie95]). These properties do not characterize the exponential in a unique way. Van de Wiele proposed in particular a version of this operation

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where the web of the exponential of X is the set of all finite *multisets* of the web of X whose support is a clique: the multiset-theoretic exponential. From a categorical viewpoint, this exponential is extremely natural: the image of a coherence space by this functor is the *free* commutative comonoid on this coherence space.

This multiset-theoretic exponential gives rise to a semantics of functional languages where the number of times a function uses a given value of its argument is taken into account. For instance, the two PCF terms

$$t_1 = \lambda x.\mathtt{if}\ x\ \mathtt{then\ true\ else}\ \Omega$$

and

$$t_2 = \lambda x.\mathtt{if}\ x\ \mathtt{then}\ x\ \mathtt{else}\ \Omega$$

(where Ω is any forever looping programme) have the same set-theoretic semantics but different (and incompatible) multiset-theoretic semantics. This semantics is not *extensional*: when applied to the same arguments, the morphisms interpreting the two terms t_1 and t_2 give the same results, but these morphisms are different. In other term, the multiset-theoretic morphisms are not characterized by their applicative beheaviour, in sharp contrast with the set-theoretic semantics which is extensional.

The purpose of this paper is to give an account of the connection between the set-theoretic and the multiset-theoretic coherence semantics of simply typed functional languages. We restrict our attention to the hierarchy of simple types based on the type of natural numbers. This unique ground type is interpreted in both semantics as the discrete coherence space which has the set of natural numbers as web (the flat domain of natural numbers).

One can define a logical relation[1] (called *heterogeneous relation* in the sequel) between the cliques of the multiset-theoretic interpretation of a type and the cliques of the set-theoretic interpretation of the same type: at ground type, this relation is the equality, and at a functional type, two morphisms are related if, applied to related arguments, they yield related results. This relation induces on the multiset-theoretic interpretation of a type a partial equivalence relation (PER): two cliques are equivalent if they are related to some common clique in the set-theoretic interpretation of the same type. The main result of the paper is that this PER is *logical*, that is: at a functional type, two multiset-theoretic cliques are related by this PER iff, when applied to arguments which are related by this PER, they yield results which are related by this PER.

As the PER previously described is obviously the equality at ground type, the fact that it is logical can be interpreted as follows:

In the hierarchy of simple types, the set-theoretic coherent semantics is the extensional collapse of the multiset-theoretic semantics[2].

The proof of this intuitively very natural result follows a pattern which can be found in other recent proofs of similar results (see [Buc96, Ehr96]): it is based on the fact that the heterogeneous relation is *onto* for finite cliques[3]. This means that, for any type, any finite clique of the set-theoretic interpretation of this type is related (by the heterogeneous relation) to some clique of the multiset-theoretic semantics of the same type.

Proving directly that the heterogeneous relation is onto for finite cliques seems to be rather difficult. We follow a roundabout method, associating by induction to each type a structure that we call the "heterogeneous structure". This structure is intended to give a fine grain description of the heterogeneous logical relation: it induces, at each type, a relation between multiset-theoretical cliques and set-theoretical cliques. As it is defined, this new relation is not logical at first sight, but it is not very hard to prove that it is onto for finite cliques, and then, using this fact, that it is logical. As this new relation is the equality at ground type, it is identical to the heterogeneous relation.

Beyond the fact that is allows for proving the theorem above on the extensional collapse of the multiset-theoretical semantics, the real outcome of this approach is that it gives an insight on the internal structure of the heterogeneous relation (compare with [Ehr96] where a similar theorem is

[1] See [Mit90] for a detailed introduction to logical relations

[2] This is also due to the fact that the heterogeneous relation is a partial function from the cliques of the multiset-theoretic interpretation of a type to the cliques of the set-theoretic interpretation of the same type.

[3] From this, one easily deduces that it is a partial function.

proven, but the detailed structure of the heterogeneous relation remains quite hidden). We may hope to find similar structures in various situations where extensional and non-extensional models are compared by logical relations. So this paper should hopefully be considered as a first step towards a theory of the fine grain structure of extensional collapses.

1 Preliminaries

Let S be a set. A multiset μ of S is a function mapping each element a of S to a natural number, the *multiplicity* of a in μ. We denote by $|\mu|$ the set $\{a \,/\, \mu(a) \neq 0\}$, which we call *support* of μ. We denote a multiset by an enumeration (delimited by square brackets) of the elements of its support, each as many times as its multiplicity in the multiset. Observe that, since multisets are functions to natural numbers, the sum of multisets is well-defined.

Definition 1 *A* coherence space *is a couple* $X = (|X|, \mathrel{\overset{\frown}{\smile}}_X)$ *where* $|X|$ *is a set (the* web *of* X *whose elements are called* points*) and* $\mathrel{\overset{\frown}{\smile}}_X$ *is a symmetric and reflexive binary relation on* $|X|$*. Two elements of* $|X|$ *that are in this relation are said to be* coherent*. Otherwise they are said to be* incoherent*.*

A clique *of* X *is a subset* x *of* $|X|$ *such that, for any* $a_1, a_2 \in x$, $a_1 \mathrel{\overset{\frown}{\smile}}_X a_2$*.*

A multiclique *of* X *is a multiset* μ *of* X *such that* $|\mu|$ *is a clique of* X*.*

X *is said to be* discrete *when, for any* $a_1, a_2 \in |X|$ *such that* $a_1 \neq a_2$, a_1 *and* a_2 *are incoherent.*

A coherence space Y *is a* coherence subspace *of* X *if* $|Y| \subseteq |X|$ *and* $\mathrel{\overset{\frown}{\smile}}_Y$ *is the restriction of* $\mathrel{\overset{\frown}{\smile}}_X$ *to* $|Y|$*.*

We denote by $\frown_X$ and call *strict coherence relation* of X the relation obtained from $\mathrel{\overset{\frown}{\smile}}_X$ by removing the diagonal. We denote by $\asymp_X$ the complementary relation of $\frown_X$ and by $\smile_X$ the one of $\mathrel{\overset{\frown}{\smile}}_X$, which are the *incoherence relations.*

We recall how linear negation, linear implication and the exponentials (both set-theoretic and multiset-theoretic, which we denote by $\underset{s}{!}$ and $\underset{m}{!}$ respectively) are defined.

Let X and Y be coherence spaces.

- *Linear negation.* $X^\perp$ is defined by $|X^\perp| = |X|$ and $a_1 \mathrel{\overset{\frown}{\smile}}_{X^\perp} a_2$ if $a_1 \asymp_X a_2$.
- *Linear implication.* $X \multimap Y$ is defined by $|X \multimap Y| = |X| \times |Y|$ and
$$\begin{array}{rl} (a_1, b_1) \mathrel{\overset{\frown}{\smile}}_{X \multimap Y} (a_2, b_2) & \\ \text{if} & a_1 \mathrel{\overset{\frown}{\smile}}_X a_2 \Rightarrow b_1 \mathrel{\overset{\frown}{\smile}}_Y b_2 \\ \text{and} & a_1 \frown_X a_2 \Rightarrow b_1 \frown_Y b_2. \end{array}$$
- *Set-theoretic exponential.* The points of $|\underset{s}{!}X|$ are the finite cliques of X. Two finite cliques x_1 and x_2 are coherent in $\underset{s}{!}X$ if, for any $a_1 \in x_1$ and $a_2 \in x_2$, $a_1 \mathrel{\overset{\frown}{\smile}}_X a_2$.
- *Multiset-theoretic exponential.* The points of $|\underset{m}{!}X|$ are the finite multicliques of X. Two finite multicliques μ_1 and μ_2 are coherent in $\underset{m}{!}X$ if $|\mu_1| \mathrel{\overset{\frown}{\smile}}_{\underset{s}{!}X} |\mu_2|$.

Let f be a function from the cliques of X to the cliques of Y. We say that f is *stable* when f is monotone, continuous (commutation to directed unions) and, for any cliques x_1 and x_2 of X such that $x_1 \cup x_2$ is a clique,
$$f(x_1 \cap x_2) = f(x_1) \cap f(x_2).$$

There is a biunique correspondance between the cliques of $\underset{s}{!}X \multimap Y$ and the stable functions sending cliques of X to cliques of Y. If f is a clique of $\underset{s}{!}X \multimap Y$ then the corresponding stable function, which we shall denote by the same symbol, is defined, for any clique x of X, by:
$$f(x) = \{b \,/\, \exists x_0 \in |\underset{s}{!}X| \;\; (x_0, b) \in f \text{ and } x_0 \subseteq x\}.$$

Let φ be a clique of $\underset{m}{!}X \multimap Y$. The co-Kleisli category of the comonad $\underset{m}{!}$ in the category of coherent spaces is a model of intuitionistic linear logic. Since it is a cartesian closed category, there is a canonical notion of application of φ to a clique x of X, which yields the following clique of Y:
$$\varphi(x) = \{b \,/\, \exists \mu_0 \in |\underset{m}{!}X| \;\; (\mu_0, b) \in \varphi \text{ and } |\mu_0| \subseteq x\}.$$

For more information on the coherence spaces denotational semantics of linear logic, we refer to [Gir87, GLT89, Gir95].

We call $\mathbf{N}$ the discrete coherence space that has, as points of its web, the natural numbers. Observe that, since $\mathbf{N}$ is discrete, its cliques are the singletons and the empty set.

The *full hierarchy of simples types* is defined inductively by $\sigma := \iota \mid \sigma \to \sigma$, where ι is the type of natural numbers. We inductively interpret a type σ of this hierarchy by σ_s in the set-theoretical semantics and by σ_m in the multiset-theoretical semantics as follows:

$$\iota_{s,m} = \mathbf{N} \quad \text{and} \quad (\sigma \to \tau)_s = \underset{s}{!}\sigma_s \multimap \tau_s$$
$$(\sigma \to \tau)_m = \underset{m}{!}\sigma_m \multimap \tau_m.$$

We write $\frown_\sigma$ to denote both coherence relations $\frown_{\sigma_m}$ and $\frown_{\sigma_s}$. The same applies to strict coherence and incoherence relations.

Until Section 5 (in which we shall extend our results to the full hierarchy of simple types) we restrict ourselves to the *hierarchy of simple types* defined inductively by $\sigma := \iota \mid \sigma \to \iota$. The interpretation of the types in this hierarchy is the same as for the full hierarchy.

If X and Y are coherence spaces: we denote by $\rho \subseteq X \times Y$ the fact that ρ is a relation between the cliques of X and those of Y; if x is a subset of $|X|$, we denote by x the coherence subspace of X whose web is x.

Let f be a function with domain D. We define f^{-1} on the subsets y of the codomain of f as follows:

$$f^{-1}(y) = \{a \,/\, a \in D \text{ and } f(a) \in y\}.$$

If b is in the codomain of f we shall write $f^{-1}(b)$ to denote $f^{-1}(\{b\})$.

2 The heterogeneous structure

We shall start by defining the *heterogeneous relation* between the cliques of the multiset-theoretic interpretation of a type and the cliques of the set-theoretic intepretation of the same type.

Definition 2 *Let σ be a simple type, we inductively define the* heterogeneous relation $\mathcal{R}_\sigma \subseteq \sigma_m \times \sigma_s$ *as follows:*

- *$x \,\mathcal{R}_\iota\, y$, if $x = y$;*
- *$\varphi \,\mathcal{R}_{\sigma\to\iota}\, f$ if, for every ξ and x cliques of σ_m and σ_s respectively, $\xi \,\mathcal{R}_\sigma\, x \Rightarrow \varphi(\xi) \,\mathcal{R}_\iota\, f(x)$.*

Observe that the heterogeneous relation is a logical relation.

To each type σ we also associate the relations $\mathcal{I}_\sigma \subseteq \sigma_m \times \sigma_s$ and $\mathcal{I}_\sigma^\perp \subseteq {\sigma_m}^\perp \times {\sigma_s}^\perp$ by means of the *heterogeneous structure*. Once some remarkable properties of both relations are established, we shall prove that $\mathcal{I}_\sigma$ is nothing but the heterogeneous relation.

Let σ be a type. The *heterogeneous structure* $\mathcal{H}_\sigma$ for σ is a tuple $(D_\sigma, \pi_\sigma, N_\sigma, \mathrm{I}_\sigma)$ where D_σ (the domain of π_σ) and N_σ (the set of *neutral points*) are subsets of $|\sigma_m|$, π_σ (the *forgetful map*) is a map from D_σ to $|\sigma_s|$, and, for each $a \in |\sigma_s|$, $\mathrm{I}_\sigma(a)$ is a set of cliques of $\pi_\sigma^{-1}(a)$ (the *locally equivariant cliques above a*). But before defining $\mathcal{H}_\sigma$ we shall introduce $\mathcal{I}_\sigma$ and $\mathcal{I}_\sigma^\perp$, which may be viewed as notations.

We denote by P_σ the set $|\sigma_m| \setminus (D_\sigma \cup N_\sigma)$ whose elements are called *pathological points*. We denote by $\mathrm{I}_\sigma(a)^\perp$ the set of cliques ξ' of $\pi_\sigma^{-1}(a)^\perp$ such that, for every $\xi \in \mathrm{I}_\sigma(a)$, $\xi \cap \xi' \neq \emptyset$. Such a ξ' is a *locally equivariant anti-clique above a*.

Definition 3 *Let σ be a simple type and $\mathcal{H}_\sigma$ the corresponding heterogeneous structure. Given ξ a clique of σ_m and x a clique of σ_s, $\xi \,\mathcal{I}_\sigma\, x$ if the following properties are satisfied:*

- *$\xi \subseteq D_\sigma \cup N_\sigma$;*
- *$\pi_\sigma(\xi \cap D_\sigma) = x$;*
- *for any $a \in x$, $\xi \cap \pi_\sigma^{-1}(a) \in \mathrm{I}_\sigma(a)$.*

Such a ξ is called an equivariant clique above x.

Given ξ' a clique of ${\sigma_m}^\perp$ and x' a clique of ${\sigma_s}^\perp$, $\xi' \,\mathcal{I}_\sigma^\perp\, x'$ if the following properties are satisfied:

- *$\xi' \subseteq D_\sigma \cup P_\sigma$;*
- *$\pi_\sigma(\xi' \cap D_\sigma) = x'$;*
- *for any $a \in x'$, $\xi' \cap \pi_\sigma^{-1}(a) \in \mathrm{I}_\sigma(a)^\perp$.*

Such a ξ' is called an equivariant anti-clique above *x'.*

Let us now give some intuitions about the heterogeneous structure $\mathcal{H}_\sigma$.

What we have in mind is a projection sending multiset-theoretic points to set-theoretic points in the most obvious way: by forgetting hereditarily the multiplicities. And this is indeed what π_σ does.

Nevertheless, due to some coherence problems (at least), the projection is partial (with domain D_σ).

The meaning of neutral and pathological points is the following:

$\alpha \in N_\sigma$ means that $\{\alpha\}\ \mathcal{R}_\sigma\ \emptyset$;
$\alpha \in P_\sigma$ means that, if $\xi\ \mathcal{R}_\sigma\ x$, then $\alpha \notin \xi$.

I_σ is supposed to be a local version of $\mathcal{R}_\sigma$. Actually, as on may easily grasp by inspection of Definition 3, I_σ is the local version of $\mathcal{I}_\sigma$. But, since we shall prove that $\mathcal{R}_\sigma$ and $\mathcal{I}_\sigma$ are the same relation, I_σ has the intended meaning.

We define $\mathcal{H}_\sigma = (D_\sigma, \pi_\sigma, N_\sigma, \mathrm{I}_\sigma)$ by a mutual induction on the type. We start by

$$\mathcal{H}_\iota = (\mathbf{N}, \pi_\iota, \emptyset, \mathrm{I}_\iota)$$

where, for any $a \in \mathbf{N}$, $\pi_\iota(a) = a$, and $\mathrm{I}_\iota(a) = \{\{a\}\}$. Now, assuming that $\mathcal{H}_\sigma$ is well-defined, we define $\mathcal{H}_{\sigma\to\iota}$ as follows:

- $(\mu_0, b) \in D_{\sigma\to\iota}$ if $|\mu_0| \subseteq_{\text{fin}} D_\sigma$ and there are ξ and x such that $\xi\ \mathcal{I}_\sigma\ x$ and $|\mu_0| \subseteq \xi$;
- $\pi_{\sigma\to\iota}(\mu_0, b) = (\pi_\sigma(|\mu_0|), b)$;
- $(\mu_0, b) \in N_{\sigma\to\iota}$ if $(\mu_0, b) \in |(\sigma \to \iota)_m|$ and, for any ξ and x such that $\xi\ \mathcal{I}_\sigma\ x$, it holds that $|\mu_0| \not\subseteq \xi$;
- $\varphi \in \mathrm{I}_{\sigma\to\iota}(x, b)$ if φ is a clique of $\pi^{-1}_{\sigma\to\iota}(x, b)$ and, for any clique ξ such that $\xi\ \mathcal{I}_\sigma\ x$, there is a μ_0 such that $(\mu_0, b) \in \varphi$ and $|\mu_0| \subseteq \xi$.

Let us check that $\mathcal{H}_{\sigma\to\iota}$ is well-defined. The fact that $N_{\sigma\to\iota}$ and $\mathrm{I}_{\sigma\to\iota}$ are well-defined is explicitly in the definition. Checking that $D_{\sigma\to\iota}$ is a subset of $|(\sigma \to \iota)_m|$ is immediate. And we just have to show that the codomain of the forgetful map is a subset of $|(\sigma \to \iota)_s|$:

> Let $(\mu_0, b) \in D_{\sigma\to\iota}$. By definition of $D_{\sigma\to\iota}$, $|\mu_0| \subseteq_{\text{fin}} D_\sigma$ and there are ξ and x, cliques of σ_m and σ_s respectively, such that $\xi\ \mathcal{I}_\sigma\ x$ and $|\mu_0| \subseteq \xi$. Then we have
>
> $$\pi_\sigma(|\mu_0|) \subseteq \pi_\sigma(\xi \cap D_\sigma) = x,$$
>
> which entails that $\pi_\sigma(|\mu_0|)$ is a clique of σ_s. Since μ_0 is finite, $\pi_\sigma(|\mu_0|)$ is also finite. Hence
>
> $$\pi_{\sigma\to\iota}(\mu_0, b) = (\pi_\sigma(|\mu_0|), b) \in |(\sigma \to \iota)_s|.$$

Observe that at types ι, $\iota \to \iota$ and $(\iota \to \iota) \to \iota$ there are neither neutral nor pathological points. At type $((\iota \to \iota) \to \iota) \to \iota$ there are still no pathological points, but $N_{((\iota\to\iota)\to\iota)\to\iota}$ and $P_{(((\iota\to\iota)\to\iota)\to\iota)\to\iota}$ are non-empty:

> Let $([\alpha, \beta], 0)$ be an element of the web of $(((\iota \to \iota) \to \iota) \to \iota)_m$ with:
>
> $$\begin{aligned} \alpha &= ([\,([0], 0)\,],\ 1) \\ \beta &= ([\,([0, 0], 0)\,],\ 2) \end{aligned}$$
>
> It is easy to check that both α and β are in $D_{(\iota\to\iota)\to\iota}$ and are coherent in $((\iota \to \iota) \to \iota)_m$. Their images by $\pi_{(\iota\to\iota)\to\iota}$ are, respectively:
>
> $$(\{\,(\{0\}, 0)\,\},\ 1) \text{ and } (\{\,(\{0\}, 0)\,\},\ 2),$$
>
> which are incoherent in $((\iota \to \iota) \to \iota)_s$. Then, for any ξ and x, cliques of $((\iota \to \iota) \to \iota)_m$ and $((\iota \to \iota) \to \iota)_s$ respectively, such that $\xi\ \mathcal{I}_{(\iota\to\iota)\to\iota}\ x$, we have, since x is a clique,
>
> $$\pi_{(\iota\to\iota)\to\iota}(\{\alpha, \beta\}) \not\subseteq x.$$
>
> Hence, given that
>
> $$\pi_{(\iota\to\iota)\to\iota}(\xi \cap D_{(\iota\to\iota)\to\iota}) = x,$$
>
> we get $\{\alpha, \beta\} \not\subseteq \xi$. Which means that
>
> $$([\alpha, \beta], 0) \in N_{((\iota\to\iota)\to\iota)\to\iota}.$$
>
> It is then easy to check that
>
> $$([\,([\alpha, \beta], 0)\,],\ 0) \in P_{(((\iota\to\iota)\to\iota)\to\iota)\to\iota}.$$

We shall now state a technical lemma and then procede to the proof of the fact that $\mathcal{I}_\sigma$ and $\mathcal{I}_\sigma^\perp$ are *onto* for the finite cliques of σ_s.

Lemma 1 *Let σ be a simple type. Let ξ and x be cliques of σ_m and σ_s respectively, such that $\xi\ \mathcal{I}_\sigma\ x$, and let $A \subseteq N_\sigma$.*

If $y \subseteq x$ then $\xi \cap (\pi_\sigma^{-1}(y) \cup A)\ \mathcal{I}_\sigma\ y$.

This lemma is a direct consequence of Definition 3.

Proposition 1 *For every simple type σ, $\mathcal{H}_\sigma$ has the two following properties:*

i) for any finite *clique x of σ_s, there is a clique ξ of σ_m such that $\xi\ \mathcal{I}_\sigma\ x$;*

ii) for any finite *anti-clique x' of σ_s, there is an anti-clique ξ' of σ_m such that $\xi'\ \mathcal{I}_\sigma^\perp\ x'$.*

Proof: We show this result by induction on the type. Checking that it is true for $\mathcal{H}_\iota$ is trivial, so let us assume, as inductive hypothesis, that properties i) and ii) hold for $\mathcal{H}_\sigma$.

We shall start by proving that property i) holds for $\mathcal{H}_{\sigma\to\iota}$. Let

$$f = \{(x_1, b_1), \ldots, (x_n, b_n)\}$$

be a finite clique of $(\sigma \to \iota)_s$. Then, for any $i, j \in \{1, \ldots, n\}$ such that $i < j$, there is a $a_{ij} \in x_i$ and a $a_{ji} \in x_j$ such that $a_{ij} \smile_\sigma a_{ji}$. Hence, for each i,

$$x_i = \{a_{i1}, \ldots, a_{ii-1}, a_{ii+1}, \ldots, a_{in}, a_i^1, \ldots, a_i^{k_i}\},$$

where the a_i^l are the remaining elements of x_i. Observe that k_i may be equal to zero, which means that all the elements of x_i are of the form a_{ij}. By property ii) of $\mathcal{H}_\sigma$ we have:

for any $j \in \{1, \ldots, n\}$ such that $j \neq i$, there are $\xi'_{ij} \in \mathrm{I}_\sigma(a_{ij})^\perp$ and $\xi'_{ji} \in \mathrm{I}_\sigma(a_{ji})^\perp$ such that $\xi'_{ij} \cup \xi'_{ji}$ is an anti-clique of σ_m;

for any integer l such that $0 < l \leq k_i$, there is $\xi''^l_i \in \mathrm{I}_\sigma(a_i^l)^\perp$.

Let, for any $i \in \{1, \ldots, n\}$,

$$\varphi_i = \{\ (\ \sum_{\substack{j=1\\ j\neq i}}^{n} [\alpha_{ij}] + \sum_{k=1}^{k_i} [\alpha_i^k]\ ,\ b_i\ ) \in |(\sigma \to \iota)_m| \quad /\ \alpha_{ij} \in \xi'_{ij} \text{ and } \alpha_i^l \in \xi''^l_i\ \}.$$

By construction, φ_i is a clique[4] of $\pi^{-1}_{\sigma\to\iota}(x_i, b_i)$. Given $i \in \{1, \ldots, n\}$, let ξ be a clique of σ_m such that $\xi\ \mathcal{I}_\sigma\ x_i$. Since any of the sets $\xi \cap \pi_\sigma^{-1}(a)$, with $a \in x_i$, is a locally equivariant clique above a that must intersect any locally equivariant anti-clique above a, it holds that, for any $j \in \{1, \ldots, n\}$ such that $j \neq i$ and any integer l such that $0 < l \leq k_i$,

$$\xi \cap \xi'_{ij} \neq \emptyset \quad , \quad \xi \cap \xi'_{ji} \neq \emptyset \quad \text{and} \quad \xi \cap \xi''^l_i \neq \emptyset.$$

[4] This requires the finiteness of f, to ensure that the left-hand side of the couples in φ_i are finite.

Hence, it is easy to check that there is a $\mu_0 \in |{}_m^!\sigma_m|$ such that $|\mu_0| \subseteq \xi$ and $(\mu_0, b_i) \in \varphi_i$. This entails $\varphi_i \in \mathrm{I}_{\sigma\to\iota}(x_i, b_i)$. Observe also that, since property i) of $\mathcal{H}_\sigma$ ensures the existence of such a ξ, φ_i is non-empty.

Let

$$\varphi = \bigcup_{i=1}^{n} \varphi_i$$

which is a clique of $(\sigma \to \iota)_m$, since, given any $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, we know that $\xi'_{ij} \cup \xi'_{ji}$ is an anti-clique of σ_m. Furthermore φ has the following properties:

$\varphi \subseteq D_{\sigma\to\iota}$, since, for any $i \in \{1, \ldots, n\}$, we have $\varphi_i \subseteq \pi^{-1}_{\sigma\to\iota}(x_i, b_i) \subseteq D_{\sigma\to\iota}$;

$\pi_{\sigma\to\iota}(\varphi) = f$, since, for any $i \in \{1, \ldots, n\}$, φ_i being a *non-empty* clique of $\pi^{-1}_{\sigma\to\iota}(x_i, b_i)$, it holds that $\pi_{\sigma\to\iota}(\varphi_i) = \{(x_i, b_i)\}$;

for every $(x_i, b_i) \in f$, $\varphi \cap \pi^{-1}_{\sigma\to\iota}(x_i, b_i)$ is a locally equivariant clique above (x_i, b_i), since $\varphi \cap \pi^{-1}_{\sigma\to\iota}(x_i, b_i) = \varphi_i$.

This means that φ is an equivariant clique above f, in other terms $\varphi\ \mathcal{I}_{\sigma\to\iota}\ f$.

Now we shall prove that property ii) holds for $\mathcal{H}_{\sigma\to\iota}$. Let

$$f' = \{(x_1, b_1), \ldots, (x_n, b_n)\}$$

be a finite clique of $(\sigma \to \iota)_s{}^\perp$. All the x_i are pairwise coherent and, thus, $x = x_1 \cup \ldots \cup x_n$ is a clique of σ_s. Hence, by property i) of $\mathcal{H}_\sigma$, there is a clique ξ of σ_m such that $\xi\ \mathcal{I}_\sigma\ x$. For each $i \in \{1, \ldots, n\}$ we define

$$\xi_i = \xi \cap \pi_\sigma^{-1}(x_i)$$

which, by Lemma 1, obeys $\xi_i\ \mathcal{I}_\sigma\ x_i$.

For each $i \in \{1, \ldots, n\}$, let

$$\varphi'_i = \{(\mu_0, b_i)\ /\ |\mu_0| \subseteq_{\text{fin}} \xi_i \text{ and } \pi_\sigma(|\mu_0|) = x_i\}.$$

For a given i, since all the μ_0 are pairwise coherent in σ_m, φ'_i is a clique of $\pi^{-1}_{\sigma\to\iota}(x_i, b_i)^\perp$. Let φ_i be a locally equivariant clique above (x_i, b_i). Then, by definition, for any ζ such that $\zeta\ \mathcal{I}_\sigma\ x_i$, there is a $\mu_0 \in |{}_m^!\sigma_m|$ such that $(\mu_0, b_i) \in \varphi_i$ and $|\mu_0| \subseteq_{\text{fin}} \zeta$. In particular, since $\xi_i\ \mathcal{I}_\sigma\ x_i$, there is a (μ_{0i}, b_i) in φ_i such that $|\mu_{0i}| \subseteq_{\text{fin}} \xi_i$. Furthermore, since φ_i is a clique of $\pi^{-1}_{\sigma\to\iota}(x_i, b_i)$, we have $\pi_\sigma(|\mu_{0i}|) = x_i$, which means that (μ_{0i}, b_i) is in φ'_i, so $\varphi_i \cap \varphi'_i \neq \emptyset$.

We have shown that, for each $i \in \{1, \ldots, n\}$, φ'_i is a locally equivariant anti-clique above (x_i, b_i).

Let us now take

$$\varphi' = \bigcup_{i=1}^{n} \varphi'_i$$

which is, as can easily be checked, a clique of $(\sigma \to \iota)_m{}^{\perp}$. Then φ has the following properties:

$\varphi' \subseteq D_{\sigma\to\iota}$, since, for any $i \in \{1, \ldots, n\}$, we know that $\varphi'_i \subseteq \pi^{-1}_{\sigma\to\iota}(x_i, b_i)^{\perp} \subseteq D_{\sigma\to\iota}$;

$\pi_{\sigma\to\iota}(\varphi') = f'$, since, for any $i \in \{1, \ldots, n\}$, φ'_i being a *non-empty* clique of $\pi^{-1}_{\sigma\to\iota}(x_i, b_i)^{\perp}$, it holds that $\pi_{\sigma\to\iota}(\varphi'_i) = \{(x_i, b_i)\}$;

for every $(x_i, b_i) \in f'$, $\varphi' \cap \pi^{-1}_{\sigma\to\iota}(x_i, b_i)$ is a locally equivariant anti-clique above (x_i, b_i), since $\varphi' \cap \pi^{-1}_{\sigma\to\iota}(x_i, b_i) = \varphi'_i$.

So φ' is an equivariant anti-clique above f', in other terms $\varphi' \; \mathcal{I}^{\perp}_{\sigma\to\iota} \; f'$. ∎

3 Equivalence of $\mathcal{R}_\sigma$ and $\mathcal{I}_\sigma$

The fact that $\mathcal{I}_\sigma$ is onto for the finite cliques of σ_s is crucial for proving by induction the equivalence of $\mathcal{R}_\sigma$ and $\mathcal{I}_\sigma$ at any simple type. We are then in conditions of proving the following proposition.

Proposition 2 *Let σ be a simple type. For any ξ and x cliques of σ_m and σ_s respectively,*

$$\xi \; \mathcal{R}_\sigma \; x \quad \textit{iff} \quad \xi \; \mathcal{I}_\sigma \; x.$$

Proof: We shall prove that, for any type σ,

$$\varphi \; \mathcal{I}_{\sigma\to\iota} \; f \quad \text{iff} \quad \forall \xi, x \;\; (\, \xi \; \mathcal{I}_\sigma \; x \Rightarrow \varphi(\xi) = f(x)\,).$$

Then, by a trivial induction on the type, it holds that, for any type σ, $\mathcal{I}_\sigma$ and $\mathcal{R}_\sigma$ are the same relation.

We shall start by proving that the lefthand side implies the righthand side. Let φ and f be cliques of $(\sigma \to \iota)_m$ and $(\sigma \to \iota)_s$ respectively, such that $\varphi \; \mathcal{I}_{\sigma\to\iota} \; f$.

Let ξ and x be cliques of σ_m and σ_s respectively, such that $\xi \; \mathcal{I}_\sigma \; x$. We shall prove that $\varphi(\xi) = f(x)$ by proving:

i) $\varphi(\xi) \subseteq f(x)$

Let $b \in \varphi(\xi)$. Then there is a $\mu_0 \in |{!_m}\sigma_m|$ such that $(\mu_0, b) \in \varphi$ and $|\mu_0| \subseteq \xi$. We know that $\varphi \; \mathcal{I}_{\sigma\to\iota} \; f$ and thus $\varphi \subseteq D_{\sigma\to\iota} \cup N_{\sigma\to\iota}$. Since $\xi \; \mathcal{I}_\sigma \; x$ and $|\mu_0| \subseteq \xi$ we have that $(\mu_0, b) \notin N_{\sigma\to\iota}$. But we also know that $(\mu_0, b) \in D_{\sigma\to\iota} \cup N_{\sigma\to\iota}$, and, therefore, (μ_0, b) is in $D_{\sigma\to\iota}$. As a consequence of $\varphi \; \mathcal{I}_{\sigma\to\iota} \; f$, we have that $\pi_{\sigma\to\iota}(\varphi \cap D_{\sigma\to\iota}) = f$, which entails that $(\pi_\sigma(|\mu_0|), b) \in f$. Since $\xi \; \mathcal{I}_\sigma \; x$, it holds that $\pi_\sigma(\xi \cap D_\sigma) = x$. Hence, as we have that $|\mu_0| \subseteq \xi$, we also have $\pi_\sigma(|\mu_0|) \subseteq x$. And, finally, we get $b \in f(x)$.

ii) $f(x) \subseteq \varphi(\xi)$

Let $b \in f(x)$. Then there is a $(x_0, b) \in f$ such that $x_0 \subseteq x$. By Definition 3, since $\varphi \; \mathcal{I}_{\sigma\to\iota} \; f$, then $\varphi \cap \pi^{-1}_{\sigma\to\iota}(x_0, b) \in \mathrm{I}_{\sigma\to\iota}(x_0, b)$. Given that $\xi \; \mathcal{I}_\sigma \; x$ and $x_0 \subseteq x$, we are in the conditions of Lemma 1 and, therefore, $\xi_0 = \xi \cap \pi^{-1}_\sigma(x_0)$ is such that $\xi_0 \; \mathcal{I}_\sigma \; x_0$. Then, by definition of $\mathrm{I}_{\sigma\to\iota}$, there is a $\mu_0 \in |{!_m}\sigma_m|$ such that $(\mu_0, b) \in \varphi$ and $|\mu_0| \subseteq \xi_0$. Which leads to $b \in \varphi(\xi)$.

We shall now prove that the righthand side implies the lefthand side. Let φ and f be cliques of $(\sigma \to \iota)_m$ and $(\sigma \to \iota)_s$ respectively, such that

$$\forall \xi, x \;\; (\, \xi \; \mathcal{I}_\sigma \; x \Rightarrow \varphi(\xi) = f(x)\,).$$

Let (μ_0, b) be in $\varphi \setminus N_{\sigma\to\iota}$. Given that $(\mu_0, b) \notin N_{\sigma\to\iota}$, there are ξ and x cliques of σ_m and σ_s respectively, such that $\xi \; \mathcal{I}_\sigma \; x$ and $|\mu_0| \subseteq \xi$. By hypothesis we have that $\varphi(\xi) = f(x)$, and, since $b \in \varphi(\xi)$, we have that $b \in f(x)$. By Definition 3 we know that $\xi_0 = \xi \cap D_\sigma$ is such that $\xi_0 \; \mathcal{I}_\sigma \; x$. So $b \in \varphi(\xi_0)$ and, hence, there is a $\nu_0 \in |{!_m}\sigma_m|$ such that $|\nu_0| \subseteq_{\mathrm{fin}} \xi_0$ and $(\nu_0, b) \in \varphi$. Both μ_0 and ν_0 are subsets of ξ, which means that $\mu_0 \coh_{!\sigma} \nu_0$. But, since $(\mu_0, b), (\nu_0, b) \in \varphi$, we have $\mu_0 \asymp_{!\sigma} \nu_0$ and, thus, $\mu_0 = \nu_0$. Therefore, given that $|\mu_0| \subseteq \xi_0 \subseteq D_\sigma$ and $\xi_0 \; \mathcal{I}_\sigma \; x$, it holds that $(\mu_0, b) \in D_{\sigma\to\iota}$.

We have shown that, for any $(\mu_0, b) \in \varphi$, if $(\mu_0, b) \notin N_{\sigma\to\iota}$ then $(\mu_0, b) \in D_{\sigma\to\iota}$. Which means that

$$\varphi \subseteq D_{\sigma\to\iota} \cup N_{\sigma\to\iota}.$$

Let us now prove that $\pi_{\sigma\to\iota}(\varphi \cap D_{\sigma\to\iota}) = f$ by proving:

i) $\pi_{\sigma\to\iota}(\varphi \cap D_{\sigma\to\iota}) \subseteq f$

Let $(\mu_0, b) \in \varphi \cap D_{\sigma\to\iota}$. From the definition of $D_{\sigma\to\iota}$ we get that $|\mu_0| \subseteq D_\sigma$ and that there are ξ and x cliques of σ_m and σ_s respectively, such that $\xi\ \mathcal{I}_\sigma\ x$ and $|\mu_0| \subseteq \xi$. Let $x_0 = \pi_\sigma(|\mu_0|)$, which is a subset of x. By Lemma 1 we know that

$$\xi_0 = \xi \cap \pi_\sigma^{-1}(x_0)$$

is such that $\xi_0\ \mathcal{I}_\sigma\ x_0$. Since $|\mu_0| \subseteq \pi_\sigma^{-1}(x_0)$ and $|\mu_0| \subseteq \xi$ then $b \in \varphi(\xi_0)$. By hypothesis $\varphi(\xi_0) = f(x_0)$ and, thus, $b \in f(x_0)$. There is then a $x_1 \subseteq_{\text{fin}} x_0$ such that $(x_1, b) \in f$. We use Lemma 1 once more to get that

$$\xi_1 = \xi \cap \pi_\sigma^{-1}(x_1)$$

obeys $\xi_1\ \mathcal{I}_\sigma\ x_1$. By hypothesis, we know that $b \in \varphi(\xi_1)$. Then there is a $\mu_1 \in |\underset{m}{!}\sigma_m|$ such that $(\mu_1, b) \in \varphi$ and $|\mu_1| \subseteq \xi_1$. Since $|\mu_0|$ and $|\mu_1|$ are subsets of ξ, then $\mu_0 \frown_\sigma \mu_1$. But (μ_0, b) and (μ_1, b) are in φ, so $\mu_0 = \mu_1$ and, then, $\pi_\sigma(|\mu_1|) = x_0$. Hence, $x_0 \subseteq \pi_\sigma(\xi_1)$. We know that $\xi_1\ \mathcal{I}_\sigma\ x_1$ which entails $\pi_\sigma(\xi_1) = x_1$ and, then, $x_0 \subseteq x_1$. But $x_1 \subseteq x_0$ and, thus, $x_0 = x_1$. Finally, $\pi_{\sigma\to\iota}(\mu_0, b) = (x_1, b)$ is in f.

ii) $f \subseteq \pi_{\sigma\to\iota}(\varphi \cap D_{\sigma\to\iota})$

Let $(x_0, b) \in f$. By Proposition 1, since x_0 is finite, there is a clique ξ of σ_m such that $\xi\ \mathcal{I}_\sigma\ x_0$. Let $\xi_0 = \xi \cap D_\sigma$, for which it holds that $\xi_0\ \mathcal{I}_\sigma\ x_0$, by Definition 3. Since $b \in f(x_0)$, by hypothesis we obtain that $b \in \varphi(\xi_0)$. And then there is a $\mu_0 \in |\underset{m}{!}\sigma_m|$ such that $(\mu_0, b) \in \varphi$ and $|\mu_0| \subseteq \xi_0$. Therefore, given that $\xi_0 \subseteq D_\sigma$, it holds that $|\mu_0| \subseteq D_\sigma$. Let $x_1 = \pi_\sigma(|\mu_0|)$. Then, since $\pi_\sigma(\xi_0) = x_0$, we have that $x_1 \subseteq x_0$. We are in the conditions of Lemma 1, and then

$$\xi_1 = \xi_0 \cap \pi_\sigma^{-1}(x_1)$$

is such that $\xi_1\ \mathcal{I}_\sigma\ x_1$. Since $b \in \varphi(\xi_1)$, then, by hypothesis, $b \in f(x_1)$. Hence, there is a $(x_2, b) \in f$ such that $x_2 \subseteq x_1$. But $x_1 \subseteq x_0$ and then $x_2 \frown_{!\sigma} x_0$. Since both (x_2, b) and (x_0, b) are in f, we have that $x_2 \asymp_{!\sigma} x_0$ and, thus, $x_2 = x_0$. But $x_1 \subseteq x_0$, which entails that $x_1 = x_0$ and, therefore, $x_0 = \pi_\sigma(|\mu_0|)$. Then $(x_0, b) = \pi_{\sigma\to\iota}(\mu_0, b)$ and, since (μ_0, b) is in both φ and $\pi_{\sigma\to\iota}^{-1}(x_0, b) \subseteq D_{\sigma\to\iota}$, we have that (x_0, b) is in $\pi_{\sigma\to\iota}(\varphi \cap D_{\sigma\to\iota})$.

Let ξ be a clique of σ_m and $(x_0, b) \in f$. If $\xi\ \mathcal{I}_\sigma\ x_0$ then, as we have shown just above, there is a $\mu_0 \in |\underset{m}{!}\sigma_m|$ such that $(\mu_0, b) \in \varphi \cap \pi_{\sigma\to\iota}^{-1}(x_0, b)$ and $|\mu_0| \subseteq \xi$. Which means that

$$\varphi \cap \pi_{\sigma\to\iota}^{-1}(x_0, b) \in \mathrm{I}_{\sigma\to\iota}(x_0, b).$$

So $\varphi\ \mathcal{I}_{\sigma\to\iota}\ f$. ∎

At this point, we trivially get that $\mathcal{R}_\sigma$ is onto for the finite cliques of σ_s, at any simple type.

Theorem 1 *Let σ be a simple type.*

i) Let x be a finite *clique of σ_s. Then there is a clique ξ of σ_m such that $\xi\ \mathcal{R}_\sigma\ x$.*

ii) $\mathcal{R}_\sigma$ is functional.

Proof: Part i) is a direct consequence of Proposition 1 and Proposition 2.

Part ii) is easy shown by induction on the types by use of Definition 2 and i). ∎

4 The extensional collapse

We shall start by defining, by means of $\mathcal{R}_\sigma$, a relation on the cliques of σ_m, the *homogeneous relation.* We shall then prove that this relation is the *extensional collapse relation,* as usually defined in the literature.

Definition 4 *Let σ be a simple type. The* homogeneous relation *$\sim_\sigma \subseteq \sigma_m \times \sigma_m$ is given by: $\xi \sim_\sigma \zeta$ if there is a clique x of σ_s such that $\xi\ \mathcal{R}_\sigma\ x$ and $\zeta\ \mathcal{R}_\sigma\ x$.*

The homogeneous relation is a partial equivalence relation at every simple type, as may easily be checked. Observe that it is partial indeed, since $P_{(((\iota\to\iota)\to\iota)\to\iota)\to\iota}$ is non-empty, as we have seen in Section 2.

Definition 5 *Let σ be a simple type. The* extensional collapse relation *$\approx_\sigma \subseteq \sigma_m \times \sigma_m$ is inductively given by:*

- *$x \approx_\iota y$ if $x = y$;*
- *$\varphi \approx_{\sigma\to\iota} \psi$ if, for every ξ and ζ cliques of σ_m, $\xi \approx_\sigma \zeta \Rightarrow \varphi(\xi) \approx_\iota \psi(\zeta)$.*

Observe that the extensional collapse relation is a logical relation. It is also, trivially, a partial equivalence relation at every simple type.

Proposition 3 *Let σ be a simple type. For every ξ and ζ cliques of σ_m,*

$$\xi \sim_\sigma \zeta \quad \textit{iff} \quad \xi \approx_\sigma \zeta.$$

Proof: We prove this result by an induction on the type. It trivially holds for type ι, so let us admit the inductive hypothesis.

Let φ and ψ be cliques of $(\sigma \to \iota)_m$. We shall start by proving that $\varphi \sim_{\sigma\to\iota} \psi$ entails $\varphi \approx_{\sigma\to\iota} \psi$. So let φ and ψ be such that $\varphi \sim_{\sigma\to\iota} \psi$. Then, by Definition 4, there is a clique f of $(\sigma \to \iota)_s$ such that $\varphi \mathcal{R}_{\sigma\to\iota} f$ and $\psi \mathcal{R}_{\sigma\to\iota} f$.

Let ξ and ζ be cliques of σ_m such that $\xi \approx_\sigma \zeta$. Then, by inductive hypothesis, $\xi \sim_\sigma \zeta$, which means that there is a clique x of σ_s such that $\xi \mathcal{R}_\sigma x$ and $\zeta \mathcal{R}_\sigma x$.

By Definition 2 we have that:

since $\xi \mathcal{R}_\sigma x$ and $\varphi \mathcal{R}_{\sigma\to\iota} f$, $\varphi(\xi) = f(x)$;

since $\zeta \mathcal{R}_\sigma x$ and $\psi \mathcal{R}_{\sigma\to\iota} f$, $\psi(\zeta) = f(x)$.

Which entails that $\varphi(\xi) = \psi(\zeta)$.

We now prove the other implication. Let φ and ψ be such that $\varphi \approx_{\sigma\to\iota} \psi$.

Let x be a *finite* clique of σ_s. By Proposition 2, there is a clique ξ of σ_m, such that $\xi \mathcal{R}_\sigma x$. Let f be the function mapping each x above to $\varphi(\xi)$. The function is well defined:

> Given ξ_1 and ξ_2 cliques of σ_m such that $\xi_1 \mathcal{R}_\sigma x$ and $\xi_2 \mathcal{R}_\sigma x$, it holds that, by Definition 4, $\xi_1 \sim_\sigma \xi_2$. Then, by application of the inductive hypothesis, $\xi_1 \approx_\sigma \xi_2$. Now, as $\approx_{\sigma\to\iota}$ is a partial equivalence relation, it holds that $\varphi \approx_{\sigma\to\iota} \varphi$ and, then, $\varphi(\xi_1) = \varphi(\xi_2)$.

Observe that f is only defined on the *finite* cliques of σ_s.

f is monotone. Let x_1 and x_2 be finite cliques of σ_s such that $x_1 \subseteq x_2$. By Proposition 2, there is a clique ξ_2 of σ_m such that $\xi_2 \mathcal{R}_\sigma x_2$. Since $\mathcal{R}_\sigma$ and $\mathcal{I}_\sigma$ are the same relation, we are in the conditions of Lemma 1 and $\xi_1 = \xi_2 \cap \pi_\sigma^{-1}(x_1)$ is such that $\xi_1 \mathcal{R}_\sigma x_1$. By monotonocity of φ we then have that $\varphi(\xi_1) \subseteq \varphi(\xi_2)$, which is the same as $f(x_1) \subseteq f(x_2)$, by construction of f.

f is stable. Let x_1 and x_2 be finite cliques of σ_s such that $x_1 \cup x_2$ is still a clique. By Proposition 1, there is a clique ζ of σ_m such that $\zeta \mathcal{I}_\sigma x_1 \cup x_2$. Let

$$\zeta_1 = \zeta \cap \pi_\sigma^{-1}(x_1) \text{ and } \zeta_2 = \zeta \cap \pi_\sigma^{-1}(x_2).$$

Since $\pi_\sigma^{-1}(x_1 \cap x_2) = \pi_\sigma^{-1}(x_1) \cap \pi_\sigma^{-1}(x_2)$, it holds that

$$\zeta_1 \cap \zeta_2 = \xi \cap \pi_\sigma^{-1}(x_1 \cap x_2),$$

so, by Lemma 1 and since $\mathcal{I}_\sigma$ and $\mathcal{R}_\sigma$ are the same relation,

$$\zeta_1 \cap \zeta_2 \mathcal{R}_\sigma x_1 \cap x_2.$$

Hence $f(x_1 \cap x_2) = \varphi(\zeta_1 \cap \zeta_2)$. But φ is stable and so we get:

$$f(x_1 \cap x_2) = \varphi(\zeta_1) \cap \varphi(\zeta_2).$$

We know that ζ_1 and ζ_2 obey, by Lemma 1, $\zeta_1 \mathcal{I}_\sigma x_1$ and $\zeta_2 \mathcal{I}_\sigma x_2$. Then, since $\mathcal{I}_\sigma$ and $\mathcal{R}_\sigma$ are the same relation, we have $\varphi(\xi_1) = f(x_1)$ and $\varphi(\xi_2) = f(x_2)$. Hence,

$$f(x_1 \cap x_2) = f(x_1) \cap f(x_2).$$

Since f is monotonous we may extend it by continuity. This extension, $\tilde{f}$, is given by[5]:

$$\tilde{f}(x) = \bigcup_{x_0 \subseteq_{\text{fin}} x} f(x_0).$$

It is easy to derive, from the same properties of f, that $\tilde{f}$ is monotone and stable. Furthermore $\tilde{f}$ is, by construction, continuous, which means that $\tilde{f}$ is a clique of $(\sigma \to \iota)_s$.

Let ξ and x be cliques of σ_m and σ_s, respectively, such that $\xi \mathcal{R}_\sigma x$. Since $\mathcal{R}_\sigma$ and $\mathcal{I}_\sigma$ are the same relation on the finite cliques of σ_s, for any $x_0 \subseteq_{\text{fin}} x$ we are in the conditions of Lemma 1. Let then

$$\xi(x_0) = \xi \cap (\pi_\sigma^{-1}(x_0) \cup N_\sigma),$$

which obeys $\xi(x_0) \mathcal{R}_\sigma x_0$, and thus, by definition, $f(x_0) = \varphi(\xi(x_0))$. As $\{\xi(x_0) \,/\, x_0 \subseteq_{\text{fin}} x\}$ is a directed family, we get, by the continuity of φ,

$$\tilde{f}(x) = \bigcup_{x_0 \subseteq_{\text{fin}} x} \varphi(\xi(x_0)) = \varphi(\bigcup_{x_0 \subseteq_{\text{fin}} x} \xi(x_0)) = \varphi(\xi).$$

Since, by Definition 4, we have that $\xi \sim_\sigma \xi$, the inductive hypothesis yields that $\xi \approx_\sigma \xi$. This entails that $\varphi(\xi) = \psi(\xi)$ and, therefore, $\psi(\xi) = \tilde{f}(x)$.

We have shown that, by construction of $\tilde{f}$, we have $\varphi \mathcal{R}_{\sigma\to\iota} \tilde{f}$ and $\psi \mathcal{R}_{\sigma\to\iota} \tilde{f}$. So, $\varphi \sim_{\sigma\to\iota} \psi$. ∎

[5] This standard construction extends any *monotone* function on finite sets to a continuous function. Monotonocity ensures that the extension is well-defined.

5 The full hierarchy of simple types

The *product hierarchy of simple types* is defined inductively by $\sigma := \iota \,|\, \sigma \to \iota \,|\, \sigma \times \sigma$. Let σ and τ be two types in this hierarchy.

We interpret the type $\sigma \times \tau$ in the set-theoretic semantics by the coherence space $(\sigma \times \tau)_s$, whose web is $\{1\} \times |\sigma_s| \cup \{2\} \times |\tau_s|$, with the following coherence relation:

$$\begin{array}{rl} (1, a_1) \frown_{\sigma\times\tau} (1, a_2) & \text{if } a_1 \frown_\sigma a_2 \\ (2, b_1) \frown_{\sigma\times\tau} (2, b_2) & \text{if } b_1 \frown_\sigma b_2 \\ (1, a) \frown_{\sigma\times\tau} (2, b) & \text{for all } a \in |\sigma| \text{ and } b \in |\tau|. \end{array}$$

The interpretation in the multiset-theoretic semantics is similar, replacing s with m.

If E and F are two sets and if $A \subseteq \{1\} \times E \cup \{2\} \times F$ we denote by A^i the set $\{a \,/\, (i, a) \in A\}$, for $i = 1, 2$.

We extend the heterogeneous relation to the type $\sigma\times\tau$ by defining, for any ξ and x cliques of $(\sigma \times \tau)_m$ and $(\sigma \times \tau)_s$ respectively:

$$\xi \; \mathcal{R}_{\sigma\times\tau} \; x \quad \text{if} \quad \xi^1 \; \mathcal{R}_\sigma \; x^1 \text{ and } \xi^2 \; \mathcal{R}_\tau \; x^2.$$

The heterogeneous structure $\mathcal{H}_{\sigma\times\tau}$ is defined as follows:

- $D_{\sigma\times\tau} = \{1\} \times D_\sigma \cup \{2\} \times D_\tau$;
- $\pi_{\sigma\times\tau}(1, \alpha) = (1, \pi_\sigma(\alpha))$ and $\pi_{\sigma\times\tau}(2, \beta) = (2, \pi_\tau(\beta))$;
- $N_{\sigma\times\tau} = \{1\} \times N_\sigma \cup \{2\} \times N_\tau$;
- given $(1, a)$ and $(2, b)$ in $|(\sigma \times \tau)_s|$,

$$\begin{array}{rcl} \mathrm{I}_{\sigma\times\tau}(1, a) & = & \{\, \{1\} \times \xi \,/\, \xi \in \mathrm{I}_\sigma(a) \,\} \\ \mathrm{I}_{\sigma\times\tau}(2, b) & = & \{\, \{2\} \times \xi \,/\, \xi \in \mathrm{I}_\tau(b) \,\}. \end{array}$$

Observe that $\mathcal{H}_{\sigma\times\tau}$ has both properties i) and ii) of Proposition 1, by use of the proposition for $\mathcal{H}_\sigma$ and $\mathcal{H}_\tau$. Then, to prove that Proposition 2 holds for the product hierarchy of simple types, we just have to combine its proof with the fact that, for any types σ and τ in this hierarchy,

$$\xi \; \mathcal{I}_{\sigma\times\tau} \; x \quad \text{iff} \quad \xi^1 \; \mathcal{I}_\sigma \; x^1 \text{ and } \xi^2 \; \mathcal{I}_\tau \; x^2,$$

as may easily be checked by use of Definition 3. Observe that, therefore, Theorem 1 holds for this hierarchy.

The extensional collapse relation is extended to the product hierarchy of simple types in the following way. Given any types σ and τ in this hierarchy, for any ξ and ζ cliques of $(\sigma \times \tau)_m$,

$$\xi \approx_{\sigma\times\tau} \zeta \quad \text{if} \quad \xi^1 \approx_\sigma \zeta^1 \text{ and } \xi^2 \approx_\tau \zeta^2.$$

One can easily show that *Proposition 3 holds for the product hierarchy of simple types.*

Let us now consider the full hierarchy of simple types. Any type of this hierarchy is of the form

$$\sigma_1 \to (\sigma_2 \to (\ldots \to (\sigma_n \to \iota) \ldots))$$

where, for each $i \in \{1, \ldots, n\}$, σ_i is a type in this same hierarchy. To each simple type σ in the full hierarchy of simple types we inductively associate a type $\overline{\sigma}$ in the product hierarchy of simple types in the following way:

$$\begin{array}{rcl} \overline{\iota} & = & \iota \\ \overline{\sigma_1 \to (\ldots \to (\sigma_n \to \iota) \ldots)} & = & \overline{\sigma}_1 \times \ldots \times \overline{\sigma}_n \to \iota \end{array}$$

We also associate to each type σ in the full hierarchy of simple types, two λ-terms, $A_\sigma : \sigma \to \overline{\sigma}$ and $B_\sigma : \overline{\sigma} \to \sigma$, by mutual induction:

$$\begin{array}{rcl} A_\iota & = & \mathrm{Id}_\iota \\ B_\iota & = & \mathrm{Id}_\iota \end{array}$$

$$\begin{array}{rcl} A_\Sigma & = & \lambda x^\Sigma . \lambda \langle y_1, \ldots, y_n \rangle^{\overline{\sigma}_1 \times \ldots \times \overline{\sigma}_n} . \\ & & \quad (x)(B_{\sigma_1})y_1 \ldots (B_{\sigma_n})y_n \\ B_\Sigma & = & \lambda y^{\overline{\Sigma}} . \lambda x_1^{\sigma_1} \ldots \lambda x_n^{\sigma_n} . \\ & & \quad (y) \langle (A_{\sigma_1})x_1, \ldots, (A_{\sigma_n})x_n \rangle \end{array}$$

where Σ denotes $\sigma_1 \to (\ldots \to (\sigma_n \to \iota) \ldots)$.

This terms have a property from which we will derive the extension of Proposition 3 to the full hierarchy of simple types. We denote $\beta\eta$-equivalence by equality.

Lemma 2 *Let σ be a type in the full hierarchy of simple types. Then:*

$$B_\sigma \circ A_\sigma = \mathrm{Id}_\sigma.$$

Observe that it also holds that $A_\sigma \circ B_\sigma = \mathrm{Id}_\sigma$, which establishes an isomorphism between the full hierarchy of simple types and the product hierarchy of simple types.

In general, neither A_σ nor B_σ are typed in those two type hierarchies. Nevertheless, they are typed in the type hierarchy given by $\sigma := \iota \mid \sigma \to \sigma \mid \sigma \times \sigma$, to which we extend both set-theoretic and multiset-theoretic intepretations. We also extend the heterogeneous relation and the extensional collapse relation to this hierarchy.

The *fundamental lemma of logical relations* yields that, for any type σ in this hierarchy, if T is a closed λ-term of type σ then, since $\mathcal{R}_\sigma$ and $\approx_\sigma$ are logical relations:

$$[\![T]\!]_m \,\mathcal{R}_\sigma\, [\![T]\!]_s \quad \text{and} \quad [\![T]\!]_m \approx_\sigma [\![T]\!]_m$$

where $[\![T]\!]_m$ and $[\![T]\!]_s$ are, respectively, the multiset-theoretic and the set-theoretic interpretations of T, defined in the usual way.

We are now in conditions of proving the following theorem.

Theorem 2 *Let σ be a type in the full hierarchy of simple types. For every ξ and ζ cliques of σ_m,*

$$\xi \sim_\sigma \zeta \quad \textit{iff} \quad \xi \approx_\sigma \zeta.$$

Proof: We start by proving that the lefthand side implies the righthand side. By Definition 4, there is a x clique of σ_s such that $\xi \,\mathcal{R}_\sigma\, x$ and $\zeta \,\mathcal{R}_\sigma\, x$.

By use of the fundamental lemma of logical relations, we get that $[\![A]\!]_m \,\mathcal{R}_{\sigma \to \overline{\sigma}}\, [\![A]\!]_s$ and then, by Definition 2,

$$[\![A]\!]_m(\xi) \,\mathcal{R}_{\overline{\sigma}}\, [\![A]\!]_s(x) \quad \text{and} \quad [\![A]\!]_m(\zeta) \,\mathcal{R}_{\overline{\sigma}}\, [\![A]\!]_s(x),$$

which entails

$$[\![A]\!]_m(\xi) \sim_{\overline{\sigma}} [\![A]\!]_m(\zeta).$$

But Proposition 3 holds for the product hierarchy of simple types and, therefore,

$$[\![A]\!]_m(\xi) \approx_{\overline{\sigma}} [\![A]\!]_m(\zeta).$$

The fundamental lemma of logical relations yields that $[\![B]\!]_m \approx_{\overline{\sigma} \to \sigma} [\![B]\!]_m$ and, hence,

$$[\![B]\!]_m([\![A]\!]_m(\xi)) \approx_\sigma [\![B]\!]_m([\![A]\!]_m(\zeta)).$$

We finally use Lemma 2 to get that

$$\xi \approx_\sigma \zeta.$$

Let us now prove that the righthand side implies the lefthand side. Using the fundamental lemma of logical relations, we get that $[\![A]\!]_m \approx_{\sigma \to \overline{\sigma}} [\![A]\!]_s$ and then, by Definition 5,

$$[\![A]\!]_m(\xi) \approx_{\overline{\sigma}} [\![A]\!]_m(\zeta).$$

But Proposition 3 holds for the product hierarchy of simple types and, therefore,

$$[\![A]\!]_m(\xi) \sim_{\overline{\sigma}} [\![A]\!]_m(\zeta).$$

Hence, by Definition 4, there is a clique x of $\overline{\sigma}_s$ such that $[\![A]\!]_m(\xi) \,\mathcal{R}_\sigma\, x$ and $[\![A]\!]_m(\zeta) \,\mathcal{R}_\sigma\, x$. By the fundamental lemma of logical relations we know that $[\![B]\!]_m \,\mathcal{R}_{\overline{\sigma} \to \sigma}\, [\![B]\!]_s$ and, therefore, by Definition 2,

$$[\![B]\!]_m([\![A]\!]_m(\xi)) \,\mathcal{R}_\sigma\, [\![B]\!]_s(x)$$

$$[\![B]\!]_m([\![A]\!]_m(\zeta)) \,\mathcal{R}_\sigma\, [\![B]\!]_s(x)$$

which entails that, by Definition 4 and Lemma 2,

$$\xi \sim_\sigma \zeta.$$

■

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