An application of the extensional collapse of the relational model of linear logic

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***Abstract*—We proved recently that the extensional collapse of the relational model of linear logic coincides with its Scott model, whose objects are preorders and morphisms are downwards closed relations. This result is obtained by the construction of a new model whose objects can be understood as preorders equipped with a realizability predicate. We present this model, which features a new duality, and explain how to use it for reducing normalization results in idempotent intersection types (usually proved by reducibility) to purely combinatorial methods. We illustrate this approach in the case of the call-by-value lambda-calculus, for which we introduce a new resource calculus.**

INTRODUCTION

Denotational semantics allows to embed syntactical computational formalisms, which feature many arbitrary choices, into much more canonical settings where basic constructs are defined in a quite abstract and universal way. Of course, a lot of information about syntax are lost in the interpretation process, and full abstraction rarely holds. Very often, this abstract and wider setting provides new intuitions about syntax and suggests interesting extensions.

The present paper deals with two very simple models of the lambda-calculus: the relational model and the Scott model. The relational model has been introduced implicitly by Girard in [Gir88] as a model of the lambda-calculus and recognized only later as a model of linear logic (LL) by several authors independently. Its objects are sets (without additional structure) and a morphism from X to Y is not a function but a relation, that is, a subset of $X \times Y$. This model is often despised and considered as degenerate because it identifies many logical constructions of LL: most dramatically, the linear negation of X is X. This model is nevertheless extremely interesting because it preserves many relevant information about programs: it is *quantitative* in the sense that the interpretation of functions allows to recover how many times an argument is used to compute a given result. For that reason, computation time can be recovered from the interpretation of terms, as shown in [DC08], [dCPdF11]. Also, it has been shown recently by De Carvalho and Tortora de Falco that the interpretation function is essentially injective on normal proof-nets.

The constructs of this model underly the interpretation of types and terms in stable models (such as coherence spaces or hypercoherence spaces). Arbitrary fixpoints of types are quite easy to compute and therefore many interesting relational models of the pure lambda-calculus and of its variants and extensions (call-by-value lambda-calculus, lambda-mu calculus etc) are available. Also, the relational model provides a natural interpretation of the differential and resource lambda-calculi and LL [ER03], [ER04], [ER05], [Tra08]. See in particular [PR10] where the relational model is explicitly considered as a model of the resource (differential) lambda-calculus and adequacy properties are proved. Remember that adequacy results express that, if a term has a non-empty interpretation (or an interpretation $\neq \bot$) normalizes in some sense (typically: head reduction terminates).

Scott semantics is of course older. It has been recognized as a model of LL a few years after Girard's discovery of LL, by Michel Huth, see [Hut93], [HJK00] and independently by Glynn Winskel [Win98], [Win04]. In this model, types are interpreted by prime algebraic complete lattices, or equivalently by preorders: indeed, any such lattice can be presented as the set of downwards closed subsets of a preorder and it is much more convenient to deal with preorders than with lattices for interpreting types and terms. The Kleisli category associated with this model of LL is (equivalent to) the category of prime-algebraic complete lattices and Scott-continuous functions. This model is less "degenerate" than the model of sets and relation, for instance the dual $S^\perp$ of a preorder S has the same underlying set, but the opposit preorder relation. This model however forgets much more information about programs than the relational model: it is a purely *qualitative* model in the sense that the interpretation of a function tells which arguments are used to compute a given result, but not how many times they are used.

This difference between the relational model and the Scott model of LL materializes itself in the fact that the Kleisli category of the second model is well-pointed, whereas the Kleisli category of the first model is not. Due to the simplicity and naturality of these models, it is tempting to think that the extensional collapse of the relational model of the lambda-calculus could be its Scott model, and indeed, this is exactly what we proved in [Ehr11]. Our approach is based on the following observation. An object in $\mathbf{Rel}$ (the relational model) is a simple set and an object in $\mathbf{Pol}$ (the preorder model) is a structure $S = (|S|, \leq_S)$ where $|S|$ is a set (the web) and $\leq_S$ is a preorder relation on $|S|$. We can define the Scott semantics of all LL connectives in such a way that the web of the $\mathbf{Pol}$ object interpreting a formula coincide with the $\mathbf{Rel}$ object (set) interpreting the same formula. We build therefore a model whose objects are pairs $E = (\langle E\rangle, \mathsf{D}(E))$ where $\langle E\rangle$ is a preorder and $\mathsf{D}(E)$ is a subset of $\mathcal{P}(|\langle E\rangle|)$ which satisfies a closure property defined by an orthogonality relation. In this model, a formula A is interpreted by such an object E (such that $\langle E\rangle$ coincides with the interpretation of A in $\mathbf{Pol}$ and $|\langle E\rangle|$ coincides with the interpretation of A in $\mathbf{Rel}$), and a closed proof π of that formula is interpreted by an element u of $\mathsf{D}(E)$. Moreover, this set u coincides with the interpretation of π in $\mathbf{Rel}$, and, thanks to the closure properties of the sets $\mathsf{D}(F)$, it can be shown that the interpretation of π in $\mathbf{Pol}$ coincides with $\downarrow u$ (where the downwards closure is taken wrt. the preorder relation of $\langle E\rangle$), which in turn means essentially that "$\mathbf{Pol}$ is the extensional collapse of $\mathbf{Rel}$" (in [Ehr11], we try to give a precise meaning to this statement).

Content: Just as $\mathbf{Rel}$ and $\mathbf{Pol}$, this new category $\mathbf{Pop}$ is a model of LL where all types have least fixpoints (for a suitable notion of inclusion between these structures). In the present paper, we use this property to prove an adequacy result for a Scott model of the call-by-value lambda-calculus, which is defined as the least fixpoint $\mathcal{U}_\mathsf{S}$ (for a suitable order relation on preorders) of the operation $S \mapsto\, !S \multimap\, !S$. We can solve the same domain equation in $\mathbf{Pop}$ and we get an object $\mathcal{U}_\mathsf{P}$ for which it is not difficult to check that $\langle\mathcal{U}_\mathsf{P}\rangle = \mathcal{U}_\mathsf{S}$ and that $\mathcal{U}_\mathsf{R} = |\langle\mathcal{U}_\mathsf{P}\rangle|$ satisfies $\mathcal{U}_\mathsf{R} = \,!\mathcal{U}_\mathsf{R} \multimap\, !\mathcal{U}_\mathsf{R}$ in $\mathbf{Rel}$. Given a term M of the call-by-value lambda-calculus, that we assume to be closed for simplicity, we can therefore compute its relational interpretation $[M]_\mathsf{R}$ which is a subset of $\mathcal{U}_\mathsf{R}$ *which belongs to* $\mathsf{D}(\mathcal{U}_\mathsf{P})$ and its Scott interpretation $[M]_\mathsf{S}$, which is a downwards closed subset of $\mathcal{U}_\mathsf{R} = |\langle\mathcal{U}_\mathsf{P}\rangle|$ (for the preorder relation of $\langle\mathcal{U}_\mathsf{P}\rangle = \mathcal{U}_\mathsf{S}$). By induction on M, and using crucially the properties of the model $\mathbf{Pop}$, one can show then that $[M]_\mathsf{S} = \downarrow[M]_\mathsf{R}$: this is an application of the "extensional collapse property" of the model $\mathbf{Pop}$. Now, adequacy of $\mathcal{U}_\mathsf{R}$ for the call-by-value lambda-calculus (that is: if $[M]_\mathsf{S} \neq \emptyset$ then M reduces to a value) can be proved purely combinatorially, introducing a call-by-value resource lambda-calculus and the fact that $\mathbf{Rel}$ satisfies a version of the Taylor formula. If $[M]_\mathsf{S} \neq \emptyset$ then $[M]_\mathsf{R} \neq \emptyset$ since $[M]_\mathsf{S} = \downarrow[M]_\mathsf{R}$ and so M reduces to a value. Whereas the standard proofs of this kind of results for Scott semantics are based on reducibility, the present approach provides a purely semantical reduction of this result to a combinatorial argument (we also give a reducibility proof in the Appendix to illustrate the difference between the two approaches). In some sense, all the reducibility argument has been encapsulated in the model $\mathbf{Pop}$. The interesting feature of this approach is that it can be used for many different calculi (standard lambda-calculus, PCF, lambda-mu calculus…) without modifying the model $\mathbf{Pop}$ whereas, when reducibility is used, a new reducibility proof has to be designed for each calculus. In our approach, a new resource calculus has to be designed each time.

The work presented here can also be understood as relating usual idempotent intersection typing systems (points in the $\mathbf{Pol}$ model can be seen as idempotent intersection types) with non-idempotent ones (using points of the $\mathbf{Rel}$ model). From this viewpoint, it might be related with [BL11], and there might be a connection between the objects $E = (\langle E\rangle, \mathsf{D}(E))$ of our extensional collapse and the orthogonality models presented in that paper, but the connection is not clear atall yet.

NOTATIONS

Given $k \in \mathbb{N}$, we use $\overline{k}$ for the set $\{1,\dots,k\}$. Given a sequence $I_1,\dots,I_n$ of subsets of $\overline{k}$ which are pairwise disjoint and such that $I_1 \cup \dots \cup I_n = \overline{k}$ (condition that we summarize by $I_1 + \dots + I_n = \overline{k}$, using $+$ to denote disjoint union), we define a permutation $f = \langle I_1,\dots,I_n\rangle \in \mathfrak{S}_k$ as the unique function $f : \overline{k} \to \overline{k}$ such, for each i, the restriction of f to I_i is strictly monotone and maps I_i to the set $\{\#I_1 + \dots + \#I_{i-1} + 1, \dots, \#I_1 + \dots + \#I_{i-1} + \#I_i\}$.

We use $[a_1,\dots,a_n]$ for the multiset made of $a_1,\dots,a_n$, taking multiplicities into account. We use $[]$ for the empty multiset and standard algebraic notations such as $m + m'$ of $\sum_i m_i$ for sums of multisets. We use $|m|$ for the support of the multiset m, which is the set of elements which appear at least once in m.

I. CATEGORICAL SEMANTICS OF LL IN A NUTSHELL

Our main reference for categorical models of LL is [Mel09].

Let $\mathcal{C}$ be a Seely category. We recall briefly that such a structure consists of a category $\mathcal{C}$, whose morphisms should be thought of as linear maps, equipped with a symmetric monoidal structure for which it is closed and $*$-autonomous wrt. a dualizing object $\perp$. The monoidal product, called tensor product, is denoted as $\otimes$, the linear function space object from X to Y is denoted as $X \multimap Y$. We use $\mathsf{ev} \in \mathcal{C}((X \multimap Y) \otimes X, Y)$ for the linear evaluation morphism and $\lambda(f) \in \mathcal{C}(Z, X \multimap Y)$ for the "linear curryfication" of a morphism $f \in \mathcal{C}(Z \otimes X, Y)$. The dual object $X \multimap \perp$ is denoted as $X^\perp$. Given an object X of $\mathcal{C}$ and a permutation $f \in \mathfrak{S}_n$, we use σ_f to denote the induced automorphism of $X^{\otimes n}$ in $\mathcal{C}$; the operation $f \to \sigma_f$ is a group homomorphism from the symmetric group $\mathfrak{S}_n$ to the group of automorphisms of $X^{\otimes n}$ in $\mathcal{C}$.

We also assume that $\mathcal{C}$ is cartesian, with a cartesian product denoted as $\&$ and a terminal object $\top$. By $*$-autonomy, this implies that $\mathcal{C}$ is also cocartesian; we use $\oplus$ for the coproduct and 0 for the initial object. In any cartesian and cocartesian category, there is a canonical morphism $\mathsf{a} \in \mathcal{C}(0,\top)$ and a canonical natural transformation $\mathsf{a}_{X,Y} \in \mathcal{C}(X \oplus Y, X \,\&\, Y)$. On says that the category is *additive* if these morphisms are isomorphisms. In that case, each homset $\mathcal{C}(X,Y)$ is equipped with a structure of commutative monoid, and all operations defined so far (composition, tensor product, linear currification) are linear wrt. this structure.

If $\mathcal{C}$ has cartesian products of all countable families $(X_i)_{i\in I}$ of objects, we say that it is countably cartesian, and in that case, $\mathcal{C}$ is also countably cocartesian. If the canonical morphism $\mathsf{a}_{(X)_{i\in I}} \in \mathcal{C}(\bigoplus_{i\in I} X_i, \&_{i\in I} X_i)$ is an isomorphism, we say that $\mathcal{C}$ is countably additive. In that case, homsets have countable sums and composition as well as all monoidal operations commute with these sums.

Last, we assume that $\mathcal{C}$ is equipped with an endofunctor $!\_$ which has a structure of comonad (unit $\mathsf{d}_X \in \mathcal{C}(!X, X)$ called *dereliction*, multiplication $\mathsf{p}_X \in \mathcal{C}(!X, !!X)$ called *digging*). Moreover, this functor must be equipped with a monoidal structure which turns it into a symmetric monoidal functor from the symmetric monoidal category $(\mathcal{C}, \&)$ to the symmetric monoidal category $(\mathcal{C}, \otimes)$: the corresponding isomorphisms $\mathsf{m} : !\top \to 1$ and $\mathsf{m}_{X,Y} : !(X \& Y) \to !X \otimes !Y$ are often called *Seely isomorphisms*. The following diagram is moreover required to be commutative.

$$\begin{array}{ccc} !X \otimes !Y & \xrightarrow{\mathsf{m}_{X,Y}} & !(X \& Y) \\ & & \downarrow \mathsf{p}_{X\&Y} \\ \downarrow \mathsf{p}_X \otimes \mathsf{p}_Y & & !!(X \& Y) \\ & & \downarrow !\langle !\pi_1, !\pi_2 \rangle \\ !!X \otimes !!Y & \xrightarrow{\mathsf{m}_{!X,!Y}} & !(!X \& !Y) \end{array}$$

A. Structural natural transformations

Using these structures, we can define a *weakening* natural transformation $\mathsf{w}_X \in \mathcal{C}(!X, 1)$ and a *contraction* natural transformation $\mathsf{c}_X \in \mathcal{C}(!X, !X \otimes !X)$ as follows. Since $\top$ is terminal, there is a canonical morphism $\mathsf{t}_X \in \mathcal{C}(X, \top)$ and we set $\mathsf{w}_X = \mathsf{m}^{-1}\, !\mathsf{t}_X$. Similarly, we have a diagonal natural transformation $\Delta_X \in \mathcal{C}(X, X \& X)$ and we set $\mathsf{c}_X = \mathsf{m}^{-1}_{X,X}\, !\Delta_X$.

One can also prove that the Kleisli category $\mathcal{C}_!$ of the comonad !_ is cartesian closed, with & as cartesian product and $!X \multimap Y$ as function space object: this is a categorical version of Girard's translation of intuitionistic logic into linear logic.

We use $\mathsf{c}^n_X : !X^{\otimes n} \to !X^{\otimes n} \otimes !X^{\otimes n}$ for the generalized contraction morphism which is defined as the following composition

$$(!X)^{\otimes n} \xrightarrow{(\mathsf{c}_X)^{\otimes n}} (!X \otimes !X)^{\otimes n} \xrightarrow{\sigma_f} (!X)^{\otimes n} \otimes (!X)^{\otimes n}$$

where $f = \langle \{1, 3, \ldots, 2n-1\}, \{2, 4, \ldots, 2n\} \rangle \in \mathfrak{S}_{2n}$.

Similarly, we define a generalized weakening morphism w^n_X as the composition

$$(!X)^{\otimes n} \xrightarrow{(\mathsf{w}_X)^{\otimes n}} (1)^{\otimes n} \xrightarrow{\lambda} 1$$

where λ is the unique canonical isomorphism induced by the monoidal structure.

Given $f \in \mathcal{C}((!X)^{\otimes n}, X)$, it is standard to define $f^! \in \mathcal{C}((!X)^{\otimes n}, !X)$, using the comonad and the monoidal structure of the functor !_. This operation is usually called *promotion* in linear logic.

Lemma 1 *For $f \in \mathcal{C}((!X)^{\otimes n}, X)$, we have*

$$\mathsf{w}_X\, f^! = \mathsf{w}^n_X$$
$$\mathsf{c}_X\, f^! = (f^! \otimes f^!)\, \mathsf{c}^n_X$$
$$\mathsf{d}_X\, f^! = f$$

For $f \in \mathcal{C}((!X)^{\otimes n} \otimes !X, X)$ and $g \in \mathcal{C}((!X)^{\otimes p}, X)$ we have

$$f^!\, ((!X)^{\otimes n} \otimes g^!) = \left(f\, ((!X)^{\otimes n} \otimes g^!)\right)^! .$$

Notice that both equated morphisms belong to $\mathcal{C}((!X)^{\otimes(n+p)}, !X)$.

Given two LL models $\mathcal{C}$ and $\mathcal{D}$, an LL functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $F : \mathcal{C} \to \mathcal{D}$ which preserves all the structure defined above. For instance, we must have $F(f \otimes g) = F(f) \otimes F(g)$, $F(\mathsf{p}_X) = \mathsf{p}_{F(X)}$ etc.

B. Weak differential LL models

The notion of categorical model recalled above allows to interpret standard classical linear logic. If one wishes to interpret differential constructs as well (in the spirit of the differential lambda-calculus or of differential linear logic), more structure and hypotheses are required. Basically, we need:

- that the cartesian and cocartesian category $\mathcal{C}$ be additive
- and that the model be equipped with a *codereliction* natural transformation $\overline{\mathsf{d}}_X \in \mathcal{C}(X, !X)$ such that $\mathsf{d}_X\, \overline{\mathsf{d}}_X = \mathrm{Id}_X$.

More conditions are required if one wants to interpret the full differential lambda-calculus of [ER03] or full differential LL as presented in e.g. [Pag09]: these conditions are a categorical axiomatization of the usual chain rule of calculus, but this rule is not required here, see [Fio07] for a complete axiomatization. When these additional conditions hold, we say that the chain rule holds in $\mathcal{C}$.

If $\mathcal{C}$ is a weak differential LL model, we can define a coweakening morphism $\overline{\mathsf{w}}_X \in \mathcal{C}(1, !X)$ and a cocontraction morphism $\overline{\mathsf{c}}_X \in \mathcal{C}(!X \otimes !X, !X)$ as we did for w_X and c_X. Similarly we also define $\overline{\mathsf{c}}^n_X \in \mathcal{C}((!X)^{\otimes n}, !X)$. Due to the naturality of $\overline{\mathsf{d}}_X$ we have $\mathsf{w}_X\, \overline{\mathsf{d}}_X = 0$ and $\mathsf{c}_X\, \overline{\mathsf{d}}_X = \overline{\mathsf{d}}_X \otimes \overline{\mathsf{w}}_X + \overline{\mathsf{w}}_X \otimes \overline{\mathsf{d}}_X$. We also define $\mathsf{d}^n_X = \mathsf{d}^{\otimes n}_X\, \mathsf{c}^n_X \in \mathcal{C}(!X, X^{\otimes n})$ and $\overline{\mathsf{d}}^n_X = \overline{\mathsf{c}}^n_X\, \overline{\mathsf{d}}^{\otimes n}_X \in \mathcal{C}(X^{\otimes n}, !X)$.

C. The Taylor formula

Let $\mathcal{C}$ be a weak differential LL model which is countably additive. Remember that each homset $\mathcal{C}(X, Y)$ is endowed with a canonical structure of commutative monoid in which countable families are summable. We assume moreover that these monoids are idempotent. This means that, if $f \in \mathcal{C}(X, Y)$, then $f + f = f$.

We say that the Taylor formula holds in $\mathcal{C}$ if, for any morphism $f \in \mathcal{C}(X, Y)$, we have

$$!f = \sum_{n=0}^{\infty} \overline{\mathsf{d}}^n_Y\, f^{\otimes n}\, \mathsf{d}^n_X$$

Remark 2 If the idempotency condition does not hold, one has to require the homsets to have a rig structure over the non-negative real numbers, and the Taylor condition must be written in the more familiar way $!f = \sum_{n=0}^{\infty} \frac{1}{n!} \overline{\mathsf{d}}^n_Y\, f^{\otimes n}\, \mathsf{d}^n_X$.

Remark 3 If the chain rule holds in $\mathcal{C}$, the Taylor condition reduces to the particular case of identity morphisms: one has just to require that $!\mathrm{Id}_X = \sum_{n=0}^{\infty} \overline{\mathsf{d}}^n_X \mathsf{d}^n_X$.

II. The extensional collapse

We present the extensional collapse construction developped in [Ehr11].

A. The relational model of LL

The model: The base category is $\mathbf{Rel}$, the category of sets and relations. Identities are diagonal relations and composition is the standard composition of relations. In this category, the isomorphisms are the bijections. The symmetric monoidal structure is given by $1 = \{*\}$ (arbitrary singleton set) and $X \otimes Y = X \times Y$, we do not give the monoidal isomorphisms which are obvious. This symmetric monoidal category (SMC) is closed, with $X \multimap Y = X \times Y$ and $\mathsf{ev} = \{(((a,b),a),b) \mid a \in X \text{ and } b \in Y\}$. It is $*$-autonomous with dualizing object $\bot = 1$ so that $X^\bot = X$ up to an obvious isomorphism.

$\mathbf{Rel}$ is countably cartesian with $\&_{i\in I} X_i = \bigcup_{i\in I}\{i\} \times X_i$ (disjoint union) and projections $\pi_i = \{((i,a),a) \mid a \in X_i\}$. It is also countably additive with $\bigoplus_{i\in I} X_i = \&_{i\in I} X_i$. The sum of a countable family of elements of $\mathbf{Rel}(X,Y)$ is its union, so that hom sets are idempotent monoids.

The exponential functor is given by $!X = \mathcal{M}_{\mathrm{fin}}(X)$ (finite multisets of elements of X) and, if $R \in \mathbf{Rel}(X,Y)$, one sets $!R = \{([a_1,\dots,a_n],[b_1,\dots,b_n]) \mid n \in \mathbb{N} \text{ and } (a_1,b_1),\dots,(a_n,b_n) \in R\}$. The Seely isomorphism $\mathsf{m} \in \mathbf{Rel}(1,!\top)$ is $\{(*,[])\}$ and the Seely natural isomorphism $\mathsf{m}_{X,Y} \in \mathbf{Rel}(!X \otimes !Y, !(X \& Y))$ is the bijection which maps $([a_1,\dots,a_n],[b_1,\dots,b_p])$ to $[(1,a_1),\dots,(1,a_n),(2,b_1),\dots,(2,b_p)])\}$. Dereliction is $\mathsf{d}_X \in \mathbf{Rel}(!X,X)$ defined by $\mathsf{d}_X = \{([a],a) \mid a \in X\}$ and digging is $\mathsf{p}_X \in \mathbf{Rel}(!X,!!X)$ defined by $\mathsf{p}_X = \{(m_1+\dots+m_k,[m_1,\dots,m_k]) \mid m_1,\dots,m_k \in !X\}$.

As easily checked, weakening is given by $\mathsf{w}_X = \{([],*)\} \in \mathbf{Rel}(!X,1)$ and binary contraction is $\mathsf{c}_X = \{(m_1 + m_2,(m_1,m_2))) \mid m_1,m_2 \in !X\}$.

This structure can also be extended to a weak differential LL model, codereliction being defined as $\overline{\mathsf{d}}_X = \{(a,[a]) \mid a \in X\} \in \mathbf{Rel}(X,!X)$. In this model, the Taylor formula holds as easily checked.

Fixpoints of types: Let $\mathbf{Rel}^\subseteq$ be the class of sets, ordere by inclusion. It is closed under arbitrary unions. A functional class $(\mathbf{Rel}^\subseteq)^n \to \mathbf{Rel}^\subseteq$ is continuous if it is monotone wrt. inclusion and preserves all directe lubs. Any continuous functional class $\Phi : \mathbf{Rel}^\subseteq \to \mathbf{Rel}^\subseteq$ admits a least fixpoint defined as usual as $\bigcup_{n\in\mathbb{N}} \Phi^n(\emptyset)$. All the LL constructions defined above are continuous functional classes.

B. The Scott model of LL

The model: A preordered set is a pair $S = (|S|, \leq_S)$ where $|S|$ is a countable set and $\leq_S$ is a transitive and reflexive binary relation on $|S|$. We denote as $\mathcal{I}(S)$ the set of all subsets of $|S|$ which are downwards closed wrt. the $\leq_S$ relation. We set $S^{\mathsf{op}} = (|S|, \geq_S)$. We use $S \times T$ for the product preorder.

Scott semantics can also be presented as a model of LL. The base category is $\mathbf{Pol}$, the category whose objects are preordered sets and where $\mathbf{Pol}(S,T) = \mathcal{I}(S^{\mathsf{op}} \times T)$. The identity morphism at S is $\mathrm{Id}_S = \{(a,a') \in |S| \times |S| \mid a' \leq_S a\}$. Composition is just the usual composition of relations.

Lemma 4 *There is an order isomorphism from $\mathbf{Pol}(S,T)$ to the set functions $\mathcal{I}(S) \to \mathcal{I}(T)$ which preserve arbitrary unions, ordered under the pointwise order. This isomorphisms maps the relation R to the function $\xi \mapsto R\,\xi = \{b \in |T| \mid \exists a \in \xi\ (a,b) \in R\}$.*

This is quite easy to prove, and this mapping from relation to functions is functorial. We equip $\mathbf{Pol}$ with a symmetric monoidal structure, taking $1 = (\{*\}, =)$ and $S \otimes T = S \times T$ (product preorder).

One checks easily that if two preorders S and S' are isomorphic as preorders through a bijection $\varphi : |S| \to |S'|$, then they are isomorphic in $\mathbf{Pol}$ by the relation $\{(a,a') \mid a' \leq_{S'} \varphi(a)\}$. The converse is not true, for instance if I is any non-empty countable set, then $(I, I\times I)$ is isomorphic to 1 in $\mathbf{Pol}$. In the first case we say that φ is a strong isomorphism from S to S'. The isomorphisms of the symmetric monoidal structure of $\mathbf{Pol}$ are the obvious strong ones. This SMC is closed, with $S \multimap T = S^{\mathsf{op}} \times T$ and linear evaluation $\mathsf{ev} \in \mathbf{Pol}((S \multimap T) \otimes S, T)$ given by $\mathsf{ev} = \{(((a',b),a),b') \mid a' \leq_S a \text{ and } b' \leq_T b\}$. $\mathbf{Pol}$ is is $*$-autonomous with dualizing object $\bot = 1$, so that, up to an obvious strong isomorphism, $S^\bot = S^{\mathsf{op}}$. Observe that as in $\mathbf{Rel}$, the cotensor product $⅋$ coincides with the tensor product $\otimes$; both categories $\mathbf{Rel}$ and $\mathbf{Pol}$ are compact closed.

$\mathbf{Pol}$ is countably cartesian: the cartesian product of a countable family $(S_i)_{i\in I}$ of preorders is $S = \&_{i\in I} S_i$ defined by $|S| = \bigcup_{i\in I}\{i\}\times|S_i|$ preordered as follows: $(i,a) \leq_S (j,b)$ if $i = j$ and $a \leq_{S_i} b$. The projections are $\pi_i = \{((i,a),a') \mid a' \leq_{S_i} a\}$. In particular, the terminal object is $(\emptyset,\emptyset)$. The category $\mathbf{Pol}$ is therefore also cocartesian, and it is countably additive with sums of morphisms defined as unions.

We define the exponential functor by $!S = (\mathcal{M}_{\mathrm{fin}}(|S|), \leq_{!S})$ where the preorder is defined as follows: $p \leq_{!S} q$ if $\forall a \in |p|\ \exists b \in |q|\ a \leq_S b$. Given $R \in \mathbf{Pol}(S,T)$, we set

$$!R = \{(p,q) \in |!S| \times |!T| \mid \forall b \in |q|\ \exists a \in |p|\ (a,b) \in R\}$$

and it is quite easy to check that $!R \in \mathbf{Pol}(!S,!T)$, and that this operation is functorial.

Remark 5 There is another possible definition, for which we use another notation: we can set $!_{\mathsf{s}}S = (\mathcal{P}_{\mathrm{fin}}(|S|), \leq_{!_{\mathsf{s}}S})$, with preorder defined just as above: $\mu \leq_{!S} \nu$ if $\forall a \in \mu\ \exists b \in \nu\ a \leq_S b$. But observe that $!S$ and $!_{\mathsf{s}}S$ are isomorphic through the relation $\mathsf{e}_S \in \mathbf{Pol}(!S, !_{\mathsf{s}}S)$ defined by $\mathsf{e}_S = \{(\mu,\mu') \mid \forall a' \in \mu'\ \exists a \in |\mu|\ a' \leq_S a\}$ (whose inverse has a similar definition). The important point is that this natural isomorphism is compatible with all the structures of both exponentials, so that (in a sense which is intuitively clear but should be made more precise) the models defined by these exponentials are equivalent. We prefer to use the multiset-based construction to present the model because it is closer to the exponential of the relational model: this simplifies greatly the presentation of the extensional collapse as we shall see, but keep in mind that we could give the same definitions with the other version (simply, replace everywhere "$\dots \in |\mu|$" by "$\dots \in \mu$"!).

The Seely isomorphism $\mathsf{m} \in \mathbf{Pol}(1,!\top)$ is $\{(*,[])\}$ and the Seely natural isomorphism $\mathsf{m}_{S_1,S_2} \in \mathbf{Pol}(!S_1 \otimes$

$!S_2, !(S \,\&\, T))$ is $\{((p_1,p_2),q) \mid (i,a) \in |q| \Rightarrow \exists a' \in |p_i|\ a \leq_{S_i} a'\}$. Dereliction $\mathsf{d}_S \in \mathbf{Pol}(!S,S)$ is $\mathsf{d}_S = \{(p,a) \mid \exists a' \in |p|\ a \leq_S a'\}$ and digging $\mathsf{p}_S \in \mathbf{Pol}(!S,!!S)$ is $\mathsf{p}_S = \{(p,[p_1,\dots,p_n] \mid i \in \mathbb{N} \text{ and } \forall i\ p_i \leq_{!S} p\}$.

As easily checked, weakening is given by $\mathsf{w}_S = \{(p,*) \mid p \in |!S|\} \in \mathbf{Rel}(!S,1)$ and binary contraction is $\mathsf{c}_S = \{(p,(p_1,p_2))) \mid p,p_1,p_2 \in |!S|\ p_1+p_2 \leq_{!S} p\}$.

Unlike the relational model, this structure cannot be extended into a weak differential LL model.

Proposition 6 *There is no natural transformation* $\overline{\mathsf{d}}_S \in \mathbf{Pol}(S,!S)$ *such that* $\mathsf{d}_S\,\overline{\mathsf{d}}_S = \mathrm{Id}_S$.

Proof. We prove first that necessarily $\overline{\mathsf{d}}_S = \{(a,p) \in |S| \times |!S| \mid p \leq_{!S} [a]\}$. First, let $(a,p) \in \overline{\mathsf{d}}_S$. Let $a' \in |p|$. By definition of d_S, we have $(p,a') \in \mathsf{d}_S$, and hence $(a,a') \in \mathsf{d}_S\,\overline{\mathsf{d}}_S = \mathrm{Id}_S$. Therefore $a' \leq_S a$ and hence $p \leq_{!S} [a]$. Conversely, let $a \in |S|$. We have $(a,a) \in \mathrm{Id}_S$ and therefore there exists p such that $(a,p) \in \overline{\mathsf{d}}_S$ and $(p,a) \in \mathsf{d}_S$. By the second property, we can find $a' \in |p|$ such that $a \leq_S a'$. We have $[a] \leq_{!S} p$ and $(a,p) \in \overline{\mathsf{d}}_S \in \mathbf{Pol}(S,!S)$. Therefore $(a,[a]) \in \overline{\mathsf{d}}_S$. It follows that, for any p such that $p \leq_{!S} [a]$, one has $(a,p) \in \overline{\mathsf{d}}_S$.

Let $S = (\{0\},=)$ and $T = (\{1,2\},=)$. Let $R = \{(0,1),(0,2)\}$, we have $R \in \mathbf{Pol}(S,T)$. Observe that $([0],[1,2]) \in !R$ (warning: this is of course not true in $\mathbf{Rel}$) so that $(0,[1,2]) \in !R\,\overline{\mathsf{d}}_S$. But there is no $b \in |T|$ such that $(b,[1,2]) \in \overline{\mathsf{d}}_T$ and hence we do not have $(0,[1,2]) \in \overline{\mathsf{d}}_T\,R$, and this shows that $\overline{\mathsf{d}}_S$ is not a natural transformation. □

Observe however that the inclusion $\overline{\mathsf{d}}_T\,R \subseteq !R\,\overline{\mathsf{d}}_S$ holds, so that $\overline{\mathsf{d}}_S$ enjoys a lax naturality property.

Fixpoints of types: Let S and T be preorders, we write $S \subseteq T$ if $|S| \subseteq |T|$ and, for any $a,a' \in |S|$, one has $a \leq_S a'$ iff $a \leq_T a'$. This defines an order relation on the class of preorders and we use $\mathbf{Pol}^{\subseteq}$ for this partially ordered class. It is clear that any countable directed family in $\mathbf{Pol}^{\subseteq}$ has a lub and that all the LL constructions presented above are continuous. It is also clear that any continuous functional $\Phi : \mathbf{Pol}^{\subseteq} \to \mathbf{Pol}^{\subseteq}$ has a least fixpoint.

C. The collapsing model of LL

The last model that we consider combines the two models above. It is based on a new duality that we introduce now.

The model: Let S be a preorder and let $u,u' \subseteq |S|$. We write $u \perp u'$ if $u \cap u' = \emptyset \Rightarrow (\downarrow_S u) \cap u' = \emptyset$; this means intuitively that u' cannot separate u from its downwards closure.

Observe that $(\downarrow_S u) \cap u' = \emptyset$ holds iff $(\downarrow_S u) \cap (\downarrow_{S^{\mathsf{op}}} u') = \emptyset$ so that $u \perp u'$ holds relatively to S iff $u' \perp u$ holds relatively to S^{op}. Given $D \subseteq \mathcal{P}(|S|)$, we define $D^{\perp(S)} \subseteq \mathcal{P}(|S|)$ by $D^{\perp(S)} = \{u' \subseteq |S| \mid \forall u \in D\ u \perp u'\}$. It is clear that $D \subseteq D^{\perp(S)\perp(S^{\mathsf{op}})}$ and that $D_1 \subseteq D_2 \Rightarrow {D_2}^{\perp(S)} \subseteq {D_1}^{\perp(S)}$, so that $D^{\perp(S)} = D^{\perp(S)\perp(S^{\mathsf{op}})\perp(S)}$. Observe that $\mathcal{I}(S^{\mathsf{op}}) \subseteq D^{\perp(S)} \subseteq \mathcal{P}(|S|)$ so that, when D is "closed" in the sense that $D = D^{\perp(S)\perp(S^{\mathsf{op}})}$, one has $\mathcal{I}(S) \subseteq D \subseteq \mathcal{P}(|S|)$.

The objects of the model are called *preorders with projections* and are pairs $E = (\langle E\rangle, \mathsf{D}(E))$ where $\langle E\rangle$ is a preorder and $\mathsf{D}(E) \subseteq \mathcal{P}(|\langle E\rangle|)$ satisfies $(\mathsf{D}(E))^{\perp(\langle E\rangle)\perp(\langle E\rangle^{\mathsf{op}})} \subseteq \mathsf{D}(E)$, that is $(\mathsf{D}(E))^{\perp(\langle E\rangle)\perp(\langle E\rangle^{\mathsf{op}})} = \mathsf{D}(E)$. If E is a preorder with projections, we set of course $E^\perp = (\langle E\rangle^{\mathsf{op}}, (\mathsf{D}(E))^{\perp(\langle E\rangle)})$.

Let E and F be prorders with projections. One defines $E \otimes F$ by $\langle E \otimes F\rangle = \langle E\rangle \times \langle F\rangle$ and $\mathsf{D}\,(E \otimes F) = \{u \times v \mid u \in \mathsf{D}(E) \text{ and } v \in \mathsf{D}(F)\}^{\perp(\langle E\rangle\times\langle F\rangle)\perp(\langle E^\perp\rangle\times\langle F^\perp\rangle)}$. Let $E \multimap F = (E \otimes F^\perp)^\perp$.

Lemma 7 *Let* $R \subseteq |\langle E \multimap F\rangle|$. *One has* $R \in \mathsf{D}(E \multimap F)$ *iff any of the following equivalent conditions holds.*

- *For any* $u \in \mathsf{D}(E)$ *and any* $v' \in \mathsf{D}(F^\perp)$, *one has* $R \cap (u \times v') = \emptyset \Rightarrow R \cap (\downarrow_{\langle E\rangle} u \times \uparrow_{\langle F\rangle} v') = \emptyset$.
- *For any* $u \in \mathsf{D}(E)$, *one has* $R\,u \in \mathsf{D}(F)$ *and* $R\!\downarrow\! u \subseteq \downarrow\!(R\,u)$.
- *For any* $u \in \mathsf{D}(E)$, *one has* $R\,u \in \mathsf{D}(F)$ *and* $\downarrow\!(R\,u) = (\downarrow_{\langle E\rangle\multimap\langle F\rangle} R)\,(\downarrow\! u)$.

Proof. See [Ehr11]. □

The category $\mathbf{Pop}$ of preorders with projections has the structures defined above as objects, and $\mathbf{Pop}(E,F) = \mathsf{D}(E \multimap F)$. By Lemma 7 $\mathrm{Id}_E = \{(a,a) \mid a \in |\langle E\rangle|\} \in \mathbf{Pop}(E,E)$, and if $Q \in \mathbf{Pop}(E,F)$ and $P \in \mathbf{Pop}(F,G)$, then $P\,Q \in \mathbf{Pop}(E,G)$ and so identities and composition of $\mathbf{Pop}$ are defined in the usual relational way.

This category is $*$-autonomous: we have already seen the definition of the tensor product on objects. On morphisms, it is defined just as in $\mathbf{Rel}$. The internal hom object $E \multimap F$ has also been defined above, and the linear evaluation relation is defined as in $\mathbf{Rel}$ again. Of course, one has to check carefully that all these relations are $\mathbf{Pop}$ morphisms, this done in [Ehr11]. Notice that, as shown in that paper, this category is not compact closed.

The category $\mathbf{Pop}$ is countably cartesian, $E = \&_{i\in I} E_i$ is defined by $\langle E\rangle = \&_{i\in I}\langle E_i\rangle$ and $w \subseteq \mathsf{D}(E)$ iff $\pi_i\,w \in \mathsf{D}(E_i)$ for each $i \in I$ (where π_i is the ith projection in the relational model). The projections morphism in $\mathbf{Pop}$ are those of the relational model. The category $\mathbf{Pop}$ is therefore also countably cocartesian, and one checks easily that it is countably additive.

We define now the exponential $!E$ of a preorder with projection E. One sets $\langle !E\rangle = !\langle E\rangle$ and therefore, we have $|\langle !E\rangle| = !|\langle E\rangle| = \mathcal{M}_{\mathrm{fin}}(|\langle E\rangle|)$ by our definition of $!E$ based on multisets and not on sets, see Remark 5. We set $\mathsf{D}(!E) = \{u^! \mid u \in \mathsf{D}(E)\}^{\perp(\langle !E\rangle)\perp(\langle !E\rangle^{\mathsf{op}})}$, where $u^! = \mathcal{M}_{\mathrm{fin}}(u)$. The main tool for dealing with this construction is the following property.

Proposition 8 *Let E and F be preorders with projections and let* $R \in \mathbf{Rel}(|\langle E\rangle|, |\langle F\rangle|)$. *One has* $R \in \mathbf{Pop}(!E,F)$ *iff, for any* $u \in \mathsf{D}(E)$

- $R\,u^! \in \mathsf{D}(F)$
- $R\,(\downarrow_{\langle E\rangle} u)^! \subseteq \downarrow_{\langle F\rangle}(R\,u^!)$.

Proof. See [Ehr11]. □

The Seely isomorphisms, and the dereliction and digging natural transformations are defined exactly as in $\mathbf{Rel}$. One uses Proposition 8 to prove that they are indeed morphisms in $\mathbf{Pop}$.

Fixpoints of types: Let E and F be preorders with projections. We write $E \subseteq F$ if $\langle E\rangle \subseteq \langle F\rangle$, $\mathsf{D}(E) \subseteq \mathsf{D}(F)$ and, for any $v \in \mathsf{D}(F)$, one has $v \cap |\langle E\rangle| \in \mathsf{D}(E)$ and $\downarrow_{\langle F\rangle} v \cap |\langle E\rangle| \subseteq \downarrow_{|\langle E\rangle|}(v \cap |\langle E\rangle|)$. This is an order relation on the class of preorders with projections, and we write $\mathbf{Pop}^{\subseteq}$ for the corresponding partially ordered class. It is shown in [Ehr11] that this this partially ordered class is complete (all directed lubs exist) and we define as usual the notion of continuous functional $(\mathbf{Pop}^{\subseteq})^n \to \mathbf{Pop}^{\subseteq}$, one checks that all constructions of linear logic are continuous functionals, and that any continuous functional $\Phi : \mathbf{Pop}^{\subseteq} \to \mathbf{Pop}^{\subseteq}$ admits a least fixpoint $\bigcup_{n=0}^{\infty} \Phi^n(\emptyset)$.

Forgetful LL functors: There is an obvious functor $\rho : \mathbf{Pop} \to \mathbf{Rel}$ defined on objects by $\rho(E) = |\langle E\rangle|$ and which is the identity on morphisms. With a preorder with projection E, we can also associate a preorder $\sigma(E) = \langle E\rangle$. This operation is extended to morphisms as follows: let $R \in \mathbf{Pop}(E, F)$, we set $\sigma(R) = \downarrow_{\langle E\rangle \multimap \langle F\rangle} R$.

Lemma 9 *Both ρ and σ are LL functors.*

The proof can be found in [Ehr11], the statement concerning ρ being straightforward. Concerning σ, LL functoriality is made possible by the presence of the sets $\mathsf{D}(E)$. For instance functoriality results directly from Lemma 7 and Lemma 4, but it is clear that, given preorders S, S' and S'' and relations $R \in \mathbf{Rel}(|S|, |S'|)$ and $R' \in \mathbf{Rel}(|S'|, |S''|)$, the inclusion $(\downarrow_{S' \multimap S''} R')\,(\downarrow_{S \multimap S'} R) \subseteq \downarrow_{S \multimap S''}(R'\, R)$ does not hold in general.

Lemma 10 *When restricted to inclusions, ρ induces a continuous function $\mathbf{Pop}^{\subseteq} \to \mathbf{Rel}^{\subseteq}$ and σ induces a continuous functional $\mathbf{Pop}^{\subseteq} \to \mathbf{Rel}^{\subseteq}$.*

III. THE CALL-BY-VALUE LAMBDA-CALCULUS

Our syntax for the call-by-value lambda-calculus is a slight modification of the ordinary lambda-calculus syntax. Indeed, with these notations, denotational semantics can be defined in a very natural way.

- If V is a value, then $\langle V\rangle$ is a term
- if M and N are terms, then $M\,N$ is a term
- if x is a variable, then x is a value
- if M is a term and x is a variable, then $\lambda x\, M$ is a value.

As usual, we work up to α-equivalence, which means that we make no difference between $\lambda x\, M$ and $\lambda y\,(M\,[y/x])$ as soon as y does not occur free in M. We use Λ_{t} for the set of terms, Λ_{v} for the set of values and Λ_{e} for the disjoint union of these two sets, whose elements will be called expressions and denoted with letters $P, Q, \dots$.

A. General reduction relation

We define now a reduction relation β_{V} on these expressions, by the following rules. More precisely, this reduction relation can be seen as the disjoint union of two reduction relations, $\beta_{\mathsf{V}} \subseteq (\Lambda_{\mathsf{t}} \times \Lambda_{\mathsf{t}}) \cup (\Lambda_{\mathsf{v}} \times \Lambda_{\mathsf{v}})$.

$$\frac{}{\langle \lambda x\, M\rangle\,\langle V\rangle\ \beta_{\mathsf{V}}\ M\,[V/x]} \qquad \frac{M\ \beta_{\mathsf{V}}\ M'}{M\,N\ \beta_{\mathsf{V}}\ M'\,N}$$

$$\frac{N\ \beta_{\mathsf{V}}\ N'}{M\,N\ \beta_{\mathsf{V}}\ M\,N'} \qquad \frac{M\ \beta_{\mathsf{V}}\ M'}{\lambda x\, M\ \beta_{\mathsf{V}}\ \lambda x\, M'}$$

We use β_{V}^* for the transitive closure of β_{V}.

B. Confluence

We prove now a confluence property for this calculus. For this purpose, we adapt the standard Tait-Martin-Löf technique of parallel reduction. We define the parallel reduction relation ρ_{V} by the following rules

$$\frac{}{P\ \rho_{\mathsf{V}}\ P} \qquad \frac{M\ \rho_{\mathsf{V}}\ M' \qquad V\ \rho_{\mathsf{V}}\ V'}{\langle \lambda x\, M\rangle\,\langle V\rangle\ \rho_{\mathsf{V}}\ M'\,[V'/x]}$$

$$\frac{M\ \rho_{\mathsf{V}}\ M' \qquad N\ \rho_{\mathsf{V}}\ N'}{M\,N\ \rho_{\mathsf{V}}\ M'\,N'} \qquad \frac{M\ \rho_{\mathsf{V}}\ M'}{\lambda x\, M\ \rho_{\mathsf{V}}\ \lambda x\, M'}$$

Observe again that $\rho_{\mathsf{V}} \subseteq (\Lambda_{\mathsf{t}} \times \Lambda_{\mathsf{t}}) \cup (\Lambda_{\mathsf{v}} \times \Lambda_{\mathsf{v}})$.

Lemma 11 $\rho_{\mathsf{V}} \subseteq \beta_{\mathsf{V}}^*$

Proof. By induction on the ρ_{V}-deduction tree leading to $P\ \rho_{\mathsf{V}}\ P'$, we prove that $P\ \beta_{\mathsf{V}}^*\ P'$. The only case which deserves comment is when $P = \langle \lambda x\, M\rangle\,\langle V\rangle$, $P' = M'\,[V'/x]$ and the deduction tree ends with the rule

$$\frac{M\ \rho_{\mathsf{V}}\ M' \qquad V\ \rho_{\mathsf{V}}\ V'}{P = \langle \lambda x\, M\rangle\,\langle V\rangle\ \rho_{\mathsf{V}}\ P' = M'\,[V'/x]}$$

By inductive hypothesis we have $M\ \beta_{\mathsf{V}}^*\ M'$ and $V\ \beta_{\mathsf{V}}^*\ V'$. Therefore $\langle \lambda x\, M\rangle\,\langle V\rangle\ \beta_{\mathsf{V}}^*\ \langle \lambda x\, M'\rangle\,\langle V'\rangle\ \beta_{\mathsf{V}}\ M'\,[V'/x]$. □

It is also clear that $\beta_{\mathsf{V}} \subseteq \rho_{\mathsf{V}}$ and hence $\rho_{\mathsf{V}}^* = \beta_{\mathsf{V}}^*$.

Lemma 12 *If $P\ \rho_{\mathsf{V}}\ P'$ and $V\ \rho_{\mathsf{V}}\ V'$, then $P\,[V/x]\ \rho_{\mathsf{V}}\ P'\,[V'/x]$.*

Proof. By induction on the ρ_{V}-deduction tree leading to $P\ \rho_{\mathsf{V}}\ P'$. The only interesting case is when $P = (\lambda y\, M)\langle W\rangle$ and $P' = M'\,[W'/y]$ with $M\ \rho_{\mathsf{V}}\ M'$ and $W\ \rho_{\mathsf{V}}\ W'$ (we can assume that $y \neq x$). Then we have $P\,[V/x] = (\lambda y\, M\,[V/x])\langle W\,[V/x]\rangle$. Moreover, by inductive hypothesis we have $M\,[V/x]\ \rho_{\mathsf{V}}\ M'\,[V'/x]$ and $W\,[V/x]\ \rho_{\mathsf{V}}\ W'\,[V'/x]$, and hence, applying the definition of ρ_{V}, we get $P\,[V/x]\ \rho_{\mathsf{V}}\ M'\,[V'/x]\,[W'\,[V'/x]/y] = P'\,[V'/x]$ since we can assume that y does not occur free in W'. □

Theorem 13 *If $P\ \rho_{\mathsf{V}}\ P_i$ for $i = 1, 2$ then there exists P' such that $P_i\ \rho_{\mathsf{V}}\ P'$ for $i = 1, 2$.*

Proof. If $P_i = E$ for $i = 1$ or $i = 2$ then one concludes immediately.

Assume that $P = M\,N$ and $P_i = M_i\,N_i$ with $M\ \rho_{\mathsf{V}}\ M_i$ and $N\ \rho_{\mathsf{V}}\ N_i$ for $i = 1, 2$. In that case the inductive hypothesis applies immediately. There is a similar case with $P = \lambda x\, M$.

Assume next that $P = (\lambda x\, M)\langle V\rangle$, that $M\ \rho_{\mathsf{V}}\ M_i$ and $V\ \rho_{\mathsf{V}}\ V_i$, $P_1 = (\lambda x\, M_1)\langle V_1\rangle$ and $P_2 = M_2\,[V_2/x]$. By inductive hypothesis we can find M' and V' such that $M_i\ \rho_{\mathsf{V}}\ M'$ and $V_i\ \rho_{\mathsf{V}}\ V'$ for $i = 1, 2$. By definition of ρ_{V} we have $P_1\ \rho_{\mathsf{V}}\ M'\,[V'/x]$. By Lemma 12 we have $P_2\ \rho_{\mathsf{V}}\ M'\,[V'/x]$.

Up to symmetries, the last case is when $P = (\lambda x\, M)\langle V\rangle$, $M\ \rho_{\mathsf{V}}\ M_i$ and $V\ \rho_{\mathsf{V}}\ V_i$, and $P_i = M_i\,[V_i/x]$ for $i = 1, 2$. We conclude similarly, applying twice Lemma 12. □

C. Weak reduction

We define now a reduction relation $\hat{\beta}_{\mathsf{V}}$ which is included in β_{V} and which consists in reducing only redexes not occurring inside a value (that is, under a λ). It plays the same role as head reduction in the standard λ-calculus. It is defined by the following rules.

$$\frac{}{(\lambda x\, M)\langle V\rangle \;\hat{\beta}_{\mathsf{V}}\; M\,[V/x]} \qquad \frac{M \;\hat{\beta}_{\mathsf{V}}\; M'}{M\,N \;\hat{\beta}_{\mathsf{V}}\; M'\,N}$$

$$\frac{N \;\hat{\beta}_{\mathsf{V}}\; N'}{M\,N \;\hat{\beta}_{\mathsf{V}}\; M\,N'}$$

IV. Linear-logic based models

We present here a general notion of model for this calculus, which corresponds to a translation of intuitionistic logic into linear logic alluded to by Girard in [Gir87] and called by him "boring". This interpretation is compatible with the translation of the call-by-value lambda-calculus into linear logic given in [MOTW99]. Let $\mathcal{C}$ be an LL model.

A. Interpretation of the calculus

We define a $\mathcal{C}$-model of call-by-value as a triple $(U, \mathsf{app}, \mathsf{lam})$ where U is an object of $\mathcal{C}$, and $\mathsf{app} \in \mathcal{C}(U, !U \multimap !U)$ and $\mathsf{lam} \in \mathcal{C}(!U \multimap !U, U)$ are such that $\mathsf{app}\,\mathsf{lam} = \mathrm{Id}_{!U\multimap !U}$.

We give the interpretation of expressions in such a structure. Given an expression P and a sequence of variables $\vec{x} = (x_1, \dots, x_n)$ adapted to P (this means that the sequence is repetition-free and contains all the free variables of P), we define $[P]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, X)$ where $X = U$ if P is a value and $X = !U$ if P is a term. The definition is by induction on P, and we consider first the cases where P is a term.

Assume first that $P = \langle V\rangle$. By inductive hypothesis we have $[V]^{\vec{x}} : (!U)^{\otimes(n)} \to U$, and we set $[P]^{\vec{x}} = ([V]^{\vec{x}})^! : (!U)^{\otimes(n)} \to !U$.

Assume next that $P = M\,N$. By inductive hypothesis, we have $[M]^{\vec{x}}, [N]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, !U)$. Therefore $\mathsf{app}\,\mathsf{d}_U\,[M]^{\vec{x}} \in \mathcal{C}((!U)^{\otimes n}, !U \multimap !U)$. So we set

$$[P]^{\vec{x}} = \mathsf{ev}\,((\mathsf{app}\,\mathsf{d}_U\,[M]^{\vec{x}}) \otimes [N]^{\vec{x}})\,\mathsf{c}_U^n \in \mathcal{C}((!U)^{\otimes n}, !U)$$

Now we interpret values. Assume first that P is a variable, so that $P = x_i$ for an uniquely determined $i \in \{1, \dots, n\}$. Then we have $\mathsf{w}_U^{\otimes(i-1)} \otimes \mathsf{d}_U \otimes \mathsf{w}_U^{\otimes(n-i)} : (!U)^{\otimes n} \to 1^{\otimes(i-1)} \otimes U \otimes (1)^{\otimes(n-i)} \simeq U$ (we keep this isomorphism implicit). We set $[M]^{\vec{x}} = \mathsf{w}_U^{\otimes(i-1)} \otimes \mathsf{d}_U \otimes \mathsf{w}_U^{\otimes(n-i)}$.

Assume last that $P = \lambda x\, M$. We can assume that x does not occur in $\vec{x}$. By inductive hypothesis, we have $[M]^{\vec{x},x} \in \mathcal{C}((!U)^{\otimes n} \otimes !U, !U)$ and hence $\lambda([M]^{\vec{x},x}) \in \mathcal{C}((!U)^{\otimes n}, !U \multimap !U)$ and therefore we set

$$[P]^{\vec{x}} = (\mathsf{lam}\,\lambda([P]^{\vec{x},x})) \in \mathcal{C}((!U)^{\otimes n}, U)\,.$$

Lemma 14 (Substitution Lemma) *Let P be an expression, x a variable and V a value. Let $\vec{x}$ which does not contain x, is adapted to V and such that $\vec{x}, x$ is adapted to E. We have $[P\,[V/x]]^{\vec{x}} = [P]^{\vec{x},x}\,((!U)^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n$ where n is the length of $\vec{x}$.*

Proof. By induction on P. We deal first with the cases where P is a term. Assume that $P = M\,N$. We have

$$\begin{aligned}
&[P]^{\vec{x},x}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \\
&= \mathsf{ev}\,((\mathsf{app}\,\mathsf{d}_U\,[M]^{\vec{x},x}) \otimes [N]^{\vec{x},x})\,\mathsf{c}_U^{n+1}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \\
&= \mathsf{ev}\,((\mathsf{app}\,\mathsf{d}_U\,[M]^{\vec{x},x}) \otimes [N]^{\vec{x},x}) \\
&\qquad ((\mathrm{Id}_{!U}^{\otimes n} \otimes [V]^{\vec{x}}) \otimes (\mathrm{Id}_{!U}^{\otimes n} \otimes [V]^{\vec{x}}))\,(\mathsf{c}_U^n \otimes \mathsf{c}_U^n)\,\mathsf{c}_U^n \\
&\qquad \text{by Lemma 1 and by the definition of } \mathsf{c}_U^n \\
&= \mathsf{ev}\,((\mathsf{app}\,\mathsf{d}_U\,[M\,[V/x]]^{\vec{x}}) \otimes [N\,[V/x]]^{\vec{x}})\,\mathsf{c}_U^n \\
&\qquad \text{by inductive hypothesis.}
\end{aligned}$$

Assume now that $P = \langle W\rangle$. Then $[P]^{\vec{x}} = ([W]^{\vec{x},x})^!$ so that

$$\begin{aligned}
&[P]^{\vec{x},x}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n = ([W]^{\vec{x},x})^!\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \\
&\quad = ([W]^{\vec{x},x}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!))^!\,\mathsf{c}_U^n \quad \text{by Lemma 1} \\
&\quad = ([W]^{\vec{x},x}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n)^! \quad \text{by definition of } \mathsf{c}_U^n \\
&\quad = ([W\,[V/x]]^{\vec{x}})^! \quad \text{by inductive hypothesis.} \\
&\quad = [P\,[V/x]]^{\vec{x}}
\end{aligned}$$

We consider now the cases where P is a value. Assume first that $P = x$. Then $[P]^{\vec{x},x} = \mathsf{w}_U^{\otimes n} \otimes \mathsf{d}_U$ and hence $[P]^{\vec{x},x}\,((!U)^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n = [V]^{\vec{x}}$ by Lemma 1.

Assume next that $P = x_i$ for an uniquely determined $i \in \{1, \dots, n\}$. Then

$$\begin{aligned}
&[P]^{\vec{x},x}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \\
&= ((\mathsf{w}_U^{\otimes(i-1)} \otimes \mathsf{d}_U \otimes \mathsf{w}_U^{\otimes(n-i)}) \otimes \mathsf{w}_U)\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \\
&= ((\mathsf{w}_U^{\otimes(i-1)} \otimes \mathsf{d}_U \otimes \mathsf{w}_U^{\otimes(n-i)}) \otimes \mathsf{w}_U^{\otimes n})\,\mathsf{c}_U^n \quad \text{by Lemma 1} \\
&= \mathsf{w}_U^{\otimes(i-1)} \otimes \mathsf{d}_U \otimes \mathsf{w}_U^{\otimes(n-i)} \\
&\qquad \text{by neutrality of weakening wrt. contraction} \\
&= [x_i]^{\vec{x}} \quad \text{as required.}
\end{aligned}$$

Assume last that $E = \lambda y\, M$ with y not occurring in $\vec{x}, x$.

$$\begin{aligned}
&[P]^{\vec{x},x}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \\
&= \mathsf{lam}\,\lambda([M]^{\vec{x},x,y})\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \\
&= \mathsf{lam}\,\lambda([M]^{\vec{x},x,y}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes ([V]^{\vec{x}})^! \otimes \mathrm{Id}_{!U}))\,\mathsf{c}_U^n \\
&\qquad \text{by monoidal closedness} \\
&= \mathsf{lam}\,\lambda([M]^{\vec{x},y,x}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes \mathrm{Id}_{!U} \otimes ([V]^{\vec{x}})^!))\,\mathsf{c}_U^n \\
&= \mathsf{lam}\,\lambda([M]^{\vec{x},y,x}\,(\mathrm{Id}_{!U}^{\otimes n} \otimes \mathrm{Id}_{!U} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^{n+1}) \\
&\qquad \text{by monoidal closedness} \\
&= \mathsf{lam}\,\lambda([M\,[V/x]]^{\vec{x},y}) \quad \text{by inductive hypothesis}
\end{aligned}$$

∎

Theorem 15 *Let $\vec{x}$ be adapted to the expressions P and P' and assume that $P\;\beta_{\mathsf{V}}\;P'$. Then $[P]^{\vec{x}} = [P']^{\vec{x}}$.*

Proof. By induction on the β_{V}-deduction tree leading to $P\;\beta_{\mathsf{V}}\;P'$. As usual, n denotes the length of $\vec{x}$.

Assume that $P = (\langle\lambda x\, M\rangle)\langle V\rangle$ and that $P' = M\,[V/x]$.

$$\begin{aligned}[P]^{\vec{x}} &= \mathsf{ev}\,((\mathsf{app}\,\mathsf{d}_U\,([\lambda x\, M]^{\vec{x}})^!) \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \\ &= \mathsf{ev}\,((\mathsf{app}\,\mathsf{lam}\,\lambda([M]^{\vec{x},x})) \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \\ &= \mathsf{ev}\,((\lambda([M]^{\vec{x},x})) \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \quad \text{since } \mathsf{app}\,\mathsf{lam} = \mathrm{Id} \\ &= [M]^{\vec{x},x}\,((!U)^{\otimes n} \otimes ([V]^{\vec{x}})^!)\,\mathsf{c}_U^n \quad \text{by mon. closedness} \\ &= [M']^{\vec{x}} \quad \text{by Lemma 14.}\end{aligned}$$

In the other cases, one applies directly the inductive hypothesis. □

V. A RELATIONAL MODEL AND THE ASSOCIATED TYPE SYSTEM

Let $\Phi_{\mathsf{R}} : \mathbf{Rel}^{\subseteq} \to \mathbf{Rel}^{\subseteq}$ be the continuous functional defined by $\Phi_{\mathsf{R}}(X) = !X \multimap !X$. Let $\mathcal{U}_{\mathsf{R}}$ be its least fixpoint, then we have $\mathcal{U}_{\mathsf{R}} = !\mathcal{U}_{\mathsf{R}} \multimap !\mathcal{U}_{\mathsf{R}}$ so that $\mathcal{U}_{\mathsf{R}}$ is a $\mathbf{Rel}$-model of call-by-value with $\mathsf{app} = \mathsf{lam} = \mathrm{Id}$.

A. Non-idempotent intersection types

We introduce a typing system for deriving judgments of shape $\Gamma \vdash M : m$ where M is a term, $m \in !\mathcal{U}_{\mathsf{R}}$ and Γ is a context, that is, a finite partial function from variables to $!\mathcal{U}_{\mathsf{R}}$ where the x_is are pairwise distinct variables and $m_1, \ldots, m_n \in !U$, and judgments of shape $\Gamma \vdash V : a$ where V is a value and $a \in \mathcal{U}_{\mathsf{R}}$. The sum of contexts $\Gamma + \Delta$ is defined pointwise (using the sum of multisets), when Γ and Δ have the same domain. The typing rules for terms are

$$\frac{\Gamma \vdash M : [(p,q)] \qquad \Delta \vdash N : p}{\Gamma + \Delta \vdash M\,N : q}$$

$$\frac{\Gamma_1 \vdash V : a_1 \qquad \cdots \qquad \Gamma_k \vdash V : a_k}{\Gamma_1 + \cdots + \Gamma_k \vdash \langle V\rangle : [a_1, \ldots, a_k]}$$

and the typing rules for values are

$$\frac{}{x_1 : [], \ldots, x_n : [], x : [a] \vdash x : a} \qquad \frac{\Gamma, x : p \vdash M : q}{\Gamma \vdash \lambda x\, M : (p,q)}$$

Proposition 16 *Let P be an expression and let $\vec{x} = (x_1, \ldots, x_n)$ be a list of variables adapted to P. Let $\vec{p} \in (!\mathcal{U}_{\mathsf{R}})^n$ and let $\alpha \in X$ (where $X = \mathcal{U}_{\mathsf{R}}$ if P is a value and $X = !\mathcal{U}_{\mathsf{R}}$ if P is a term). Then one has $(\vec{p}, \alpha) \in [P]_{\mathsf{R}}^{\vec{x}}$ iff the typing judgment $x_1 : p_1, \ldots, x_n : p_n \vdash P : \alpha$ is derivable.*

The proof is a simple verification, by induction on the structure of P.

B. A resource calculus

We introduce a resource calculus whose terms can be used to denote typing derivations in the typing system described above.

1) Notation: given a finite family $(a_i)_{i \in I}$ and a predicate P on I, we use $[a_i \mid P(i)]$ for the multiset whose elements are the a_i such that $P(i)$ holds, taking multiplicities into account.

2) Syntax: we describe first the syntax of our resource calculus.

- If s and t are terms, then $s\,t$ is a term.
- If $v_1, \ldots, v_n$ are values, then $\langle v_1, \ldots, v_n\rangle$ is a term.
- If x is a variable, then x is a value.
- If x is a variable and s is a term, then $\lambda x\, s$ is a value.

We call terms and values *simple expressions*. An expression is a set of simple expressions[1], that we write as sums to insist on the algebraic flavor of the definitions. The above syntactic constructs are extended to non simple expressions, by linearity. For instance, if $v = \sum_{i=1}^n v_i$ and $w = \sum_{j=1}^m w_j$ are sets of simple values, the expression $\langle v, w\rangle$ denotes the set $\sum_{i=1,j=1}^{n,m} \langle v_i, w_j\rangle$. And if $s = \sum_{i=1}^n s_i$ is a set of simple terms, then $\lambda x\, s$ denotes $\sum_{i=1}^n \lambda x\, s_i$, which is a set of simple values.

Given a simple expression e and simple values $v_1, \ldots, v_n$, we define the linear substitution $\partial_x(e; v_1, \ldots, v_n)$ by

$$\partial_x(e; v_1, \ldots, v_n) = \begin{cases} \sum_{f \in \mathfrak{S}_n} & e\,[v_1/x_{f(1)}, \ldots, v_n/x_{f(n)}] \\ & \text{if } n = \deg_x e \\ 0 & \text{otherwise} \end{cases}$$

where $\deg_x e$ is the number of free occurrences of x in e and $x_1, \ldots, x_n$ are these occurrences (in the case $n = \deg_x e$).

3) Reduction rules: we can give now the reduction rules of the calculus. We define a reduction relation denoted as δ from simple expressions to expressions by the following rules.

$$\frac{}{\langle\lambda x\, s\rangle\,\langle v_1, \ldots, v_n\rangle \;\delta\; \partial_x(s; v_1, \ldots, v_n)}$$

$$\frac{}{\langle v_1, \ldots, v_n\rangle\, t \;\delta\; 0} \text{ if } n \neq 1 \qquad \frac{s \;\delta\; s'}{\lambda x\, s \;\delta\; \lambda x\, s'}$$

$$\frac{s \;\delta\; s'}{s\,t \;\delta\; s'\,t} \qquad \frac{t \;\delta\; t'}{s\,t \;\delta\; s\,t'}$$

$$\frac{v \;\delta\; v'}{\langle v, v_1, \ldots, v_n\rangle \;\delta\; \langle v', v_1, \ldots, v_n\rangle}$$

This reduction is extended as usual to non simple expressions: we define a reduction relation $\tilde{\delta}$ on non simple terms by the following rule.

$$\frac{e_1 \;\delta\; e_1', \ldots, e_n \;\delta\; e_n'}{\sum_{i=1}^n e_i + f \;\tilde{\delta}\; \sum_{i=1}^n e_i' + f}$$

Lemma 17 *Let e be a simple expression and let $v_1, \ldots, v_n$ be simple values. If $e \;\delta\; e'$, then $\partial_x(e; v_1, \ldots, v_n) \;\tilde{\delta}\; \partial_x(e'; v_1, \ldots, v_n)$, and if $v_1 \;\delta\; v_1'$, then $\partial_x(e; v_1, v_2, \ldots, v_n) \;\tilde{\delta}\; \partial_x(e'; v_1', v_2, \ldots, v_n)$.*

The proof is straightforward.

Theorem 18 *The reduction relation $\tilde{\delta}$ satisfies the diamond property.*

Proof. It suffices to prove that, if e is a simple expression and if $e \;\delta\; e_i$ for $i = 1, 2$, there exists e' such that $e_i \;\tilde{\delta}\; e'$ for $i = 1, 2$. The proof is by induction on e.

[1] Linear combinations should be used more generally, but this particular case will be sufficient for the present paper.

The only non trivial case is when $e = \langle\lambda x\, s\rangle\,\langle v_1,\dots,v_n\rangle$, $e_1 = \partial_x(s; v_1,\dots,v_n)$ and e_2 is of shape $e_2 = \langle\lambda x\, s'\rangle\,\langle v_1,\dots,v_n\rangle$ with $s\ \delta\ s'$ or $e_2 = \langle\lambda x\, s\rangle\,\langle v_1',v_2,\dots,v_n\rangle$ with $v_1\ \delta\ v_1'$. In both cases, we simply apply Lemma 17. □

Lemma 19 *There is no infinite sequence $(e_i)_{i\in\mathbb{N}^+}$ of simple expressions such that, for each i, $e_i\ \tilde\delta\ e'$ with $e_{i+1}\in e'$.*

Proof. One defines a size function on expressions. For instance we can set $|x| = 1$, $|\lambda x\, s| = |s|$, $|s\, t| = |s| + |t|$ and $|\langle v_1,\dots,v_k\rangle| = \sum_{j=1}^{k}|v_j|$. In other words $|e|$ is the number of occurrences of variables in e. □

C. Categorical denotational semantics

Let U be a $\mathcal{C}$-model of call-by-value, where we assume moreover that $\mathcal{C}$ is a weak differential LL model which is countably additive and where homsets have idempotent sums. We show how to interpret the call-by-value resource calculus in such a structure.

We introduce first a convenient notation. Let $g_1,\dots,g_k \in \mathcal{C}((!U)^{\otimes n}, U)$. We set $\langle g_1,\dots,g_k\rangle = \overline{\mathsf{d}}_U^k\,(g_1\otimes\cdots\otimes g_k)\,\mathsf{c}_U^{n,k}$, where $\mathsf{c}_U^{n,k}\in\mathcal{C}((!U)^{\otimes n}, ((!U)^{\otimes n})^{\otimes k})$ is an obvious generalization of c_U^n.

Given a simple expression e and an adapted sequence of variables $\vec{x}$, we define $[e]^{\vec{x}}\in\mathcal{C}((!U)^{\otimes n}, X)$ where $X = U$ if e is a value and $X = !U$ if e is a term. The definition is by induction on e. For the syntactical constructs which are similar to those of the call-by-value lambda-calculus (namely: variables, application and abstraction), the interpretation is the same as in Section IV-A. To complete the definition we have just to define the semantics of $\langle v_1,\dots,v_k\rangle$. By inductive hypothesis we have defined $g_j = [v_j]^{\vec{x}}\in\mathcal{C}((!U)^{\otimes n}, U)$ and we set $[\langle v_1,\dots,v_k\rangle]^{\vec{x}} = \langle g_1,\dots,g_k\rangle$. If e is an expression, that is a set of simple expressions $e = \sum_{i\in I} e_i$ and a list of variables $\vec{x}$ adapted to all x_is, we set $[e]^{\vec{x}} = \sum_{i\in I}[e_i]^{\vec{x}}$ which is well defined because we have assumed that the sum of morphisms is idempotent in $\mathcal{C}$.

Lemma 20 *If $e\ \delta\ e'$ and $\vec{x}$ is adapted to e and e', then $[e]^{\vec{x}} = [e']^{\vec{x}}$.*

Proof. It suffices to prove the result in the case where e is simple, by induction on e. The proof uses the following property of linear substitution wrt. the interpretation (substitution lemma). Let e be a simple expression and $v_1,\dots,v_k$ be simple values. Let $\vec{x}, x$ a sequence of variable adapted to e and to all v_js. Let n be the length of $\vec{x}$. Then we have

$$[\partial_x(e; v_1,\dots,v_k)]^{\vec{x}} = [e]^{\vec{x},x}\,(!U^{\otimes n}\otimes\left\langle[v_1]^{\vec{x}},\dots,[v_1]^{\vec{x}}\right\rangle)\,\mathsf{c}_U^n$$

and this is proved by a simple induction on e. □

For any expression P of the call-by-value lambda-calculus, we define a set $\mathcal{T}(P)$ of simple expressions by induction.

$$\mathcal{T}(x) = \{x\}\qquad \mathcal{T}(\lambda x\, M) = \{\lambda x\, s \mid s\in\mathcal{T}(M)\}$$
$$\mathcal{T}(M\, N) = \{s\, t \mid s\in\mathcal{T}(M) \text{ and } t\in\mathcal{T}(N)\}$$
$$\mathcal{T}(\langle V\rangle) = \{\langle v_1,\dots,v_k\rangle \mid k\in\mathbb{N} \text{ and } \forall i\ v_i\in\mathcal{T}(V)\}$$

Lemma 21 *Let P be an expression and let V be a value. Let $e\in\mathcal{T}(P)$ and $v_1,\dots,v_k\in\mathcal{T}(V)$. Then $\partial_x(e; v_1,\dots,v_k)\subseteq\mathcal{T}(P\,[V/x])$.*

Proof. Easy induction on P. □

Lemma 22 *If the Taylor formula holds in $\mathcal{C}$ then for any expression P and any $\vec{x}$ adapted to P we have $[P]^{\vec{x}} = \sum_{e\in\mathcal{T}(P)}[e]^{\vec{x}}$.*

Proof. Easy induction on P. □

D. Adequacy in $\mathbf{Rel}$

Lemma 23 *Let P, P' be expressions and let $e\in\mathcal{T}(P)$. If $P\ \hat\beta_{\mathsf{V}}\ P'$ then there exists $e'\subseteq\mathcal{T}(P')$ such that $e\ \delta\ e'$.*

Proof. Simple inspection, using Lemma 21. □

Theorem 24 *Let P be an expression. Let $\vec{x}$ be adapted to P and let n be the length of $\vec{x}$. Let $m_1,\dots,m_n\in !\mathcal{U}_{\mathsf{R}}$ and let $\alpha\in X$ where $X = \mathcal{U}_{\mathsf{R}}$ if P is a value and $X = !\mathcal{U}_{\mathsf{R}}$ if P is a term. If $x_1 : m_1,\dots,x_n : m_n\vdash P : \alpha$ then P is $\hat\beta_{\mathsf{V}}$ strongly normalizing.*

Proof. By Proposition 16, our hypothesis means that $(\vec{m},\alpha)\in[P]^{\vec{x}}$. By Lemma 22 there exists $e\in\mathcal{T}(P)$ such that $(\vec{m},\alpha)\in[e]^{\vec{x}}$. If $P\ \hat\beta_{\mathsf{V}}\ P'$ there then there exists $f\subseteq\mathcal{T}(P')$ such that $e\ \delta\ f$ by Lemma 23. By Lemma 20 we have $(\vec{m},\alpha)\in[f]^{\vec{x}}$ and hence there exists $e'\in f$ such that $(\vec{m},\alpha)\in[e']^{\vec{x}}$. Therefore, for any reduction $P = P_1\ \hat\beta_{\mathsf{V}}\ P_2\dots\hat\beta_{\mathsf{V}}\ P_l$ we can find $e_1\in\mathcal{T}(P_1),\dots,e_l\in\mathcal{T}(P_l)$ with $|e_1| > |e_2| > \cdots > |e_l|$. □

VI. A Scott model and the associated type system

Let $\Phi_{\mathsf{S}} : \mathbf{Pol}^{\subseteq}\to\mathbf{Pol}^{\subseteq}$ be the continuous functional defined by $\Phi_{\mathsf{S}}(S) = !S\multimap !S$. Let $\mathcal{U}_{\mathsf{S}}$ be the least fixpoint of Φ_{S}, then $\mathcal{U}_S$ (equipped with two identity morphisms) is a $\mathbf{Pol}^{\subseteq}$-model of the call-by-value lambda-calculus. We use $\leq$ for the preorder $\leq_{\mathcal{U}_{\mathsf{S}}}$.

A. Idempotent intersection types

We introduce a typing system for deriving judgments of shape $\Gamma\vdash_{\mathsf{S}} M : m$ where M is a term, $m\in|!\mathcal{U}_{\mathsf{S}}|$ and Γ is a context, that is, a finite partial function from variables to $|!\mathcal{U}_{\mathsf{S}}|$ where the x_is are pairwise distinct variables and $m_1,\dots,m_n\in|!\mathcal{U}_{\mathsf{S}}|$, and judgments of shape $\Gamma\vdash_{\mathsf{S}} V : a$ where V is a value and $a\in|\mathcal{U}_{\mathsf{S}}|$. The typing rules for terms are

$$\frac{\Gamma\vdash_{\mathsf{S}} M : [(p,q)]\qquad \Gamma\vdash_{\mathsf{S}} N : p}{\Gamma\vdash_{\mathsf{S}} M\, N : q}$$
$$\frac{\Gamma\vdash_{\mathsf{S}} V : a_1\qquad\cdots\qquad\Gamma\vdash_{\mathsf{S}} V : a_k}{\Gamma\vdash_{\mathsf{S}}\langle V\rangle : [a_1,\dots,a_k]}$$

and the typing rules for values are

$$\frac{[a]\leq m}{\Gamma, x : m\vdash_{\mathsf{S}} x : a}\qquad\frac{\Gamma, x : p\vdash_{\mathsf{S}} M : q}{\Gamma\vdash_{\mathsf{S}}\lambda x\, M : (p,q)}$$

Proposition 25 *Let P be an expression and let $\vec{x} = (x_1, \ldots, x_n)$ be a list of variables adapted to P. Let $\vec{m} \in (|!\mathcal{U}_\mathsf{S}|)^n$ and let $\alpha \in |S|$ (where $S = \mathcal{U}_\mathsf{S}$ if P is a value and $S = !\mathcal{U}_\mathsf{S}$ if P is a term). Then one has $(\vec{m}, \alpha) \in [P]^{\vec{x}}_\mathsf{S}$ iff the typing judgment $x_1 : m_1, \ldots, x_n : m_n \vdash P : \alpha$ is derivable.*

The proof is a simple verification, by induction on the structure of P.

B. Adequacy in the idempotent case

One can prove an analog of Theorem 24 for this idempotent typing system, but the same technique does not apply because, as we have seen with Proposition 6, $\mathbf{Pol}$ is no a model of the call-by-value resource calculus. The standard method to prove adequacy in this model is by reducibility, we give an example of such a proof in the Appendix section.

Theorem 26 *Let P be an expression. Let $\vec{x}$ be adapted to P and let n be the length of $\vec{x}$. Let $m_1, \ldots, m_n \in |!\mathcal{U}_\mathsf{S}|$ and let $\alpha \in |X|$ where $X = \mathcal{U}_\mathsf{S}$ if P is a value and $X = !\mathcal{U}_\mathsf{S}$ if P is a term. If $x_1 : m_1, \ldots, x_n : m_n \vdash_\mathsf{S} P : \alpha$ then P is $\hat{\beta}_\mathsf{V}$ strongly normalizing.*

C. Adequacy, using preorders with projections

Let $\Phi_\mathsf{P} : \mathbf{Pop}^\subseteq \to \mathbf{Pop}^\subseteq$ be the continuous functional defined by $\Phi_\mathsf{P}(E) = !E \multimap !E$. Let $\mathcal{U}_\mathsf{P}$ be the least fixpoint of Φ_P, then $\mathcal{U}_\mathsf{P}$ (equipped with two identity morphisms) is a $\mathbf{Pop}$-model of the call-by-value lambda-calculus.

By Lemma 10, we have $\rho(\mathcal{U}_\mathsf{P}) = \mathcal{U}_\mathsf{R}$ and $\sigma(\mathcal{U}_\mathsf{P}) = \mathcal{U}_\mathsf{S}$.

Let P be an expression and let $\vec{x}$ be a sequence of variables adapted to P, let n be the length of $\vec{x}$. Because ρ is an LL functor, we have $[P]^{\vec{x}}_\mathsf{P} = \rho([P]^{\vec{x}}_\mathsf{P}) = [P]^{\vec{x}}_\mathsf{R}$, and similarly we have $\sigma([P]^{\vec{x}}_\mathsf{P}) = [P]^{\vec{x}}_\mathsf{S}$. These properties are proved by a straightforward induction on P. As a consequence, using the definition of the functor σ, we get the following result, which relates the relational semantics of an expression to its Scott semantics.

Theorem 27 *Let P be an expression and let $\vec{x}$ be a sequence of variables of length n, adapted to P. Then $[P]^{\vec{x}}_\mathsf{S} = \downarrow[P]^{\vec{x}}_\mathsf{R}$ where the downwards closure is taken in $(!\mathcal{U}_\mathsf{S})^{\otimes n} \multimap S$ (with $S = \mathcal{U}_\mathsf{S}$ if P is a value and $S = !\mathcal{U}_\mathsf{S}$ if P is a term).*

This gives us an alternative proof of Theorem 26. We deal with the case of a term, but the proof is of course similar for values. So assume that $x_1 : m_1, \ldots, x_k : m_k \vdash_\mathsf{S} M : m$ which means that $(\vec{m}, m) \in [M]^{\vec{x}}_\mathsf{S}$ where $m_1, \ldots, m_n, m \in |!\mathcal{U}_\mathsf{S}|$. Then by Theorem 27, we can find $m'_1, \ldots, m'_n, m' \in |!\mathcal{U}_\mathsf{S}| = \mathcal{U}_\mathsf{R}$ with $m'_i \leq m_i$ and $m \leq m'$ (where $\leq$ is the preorder on $\mathcal{U}_\mathsf{S}$) and such that $(m'_1, \ldots, m'_n, m') \in [M]^{\vec{x}}_\mathsf{R}$. By Theorem 24, M is $\hat{\beta}_\mathsf{V}$ strongly normalizing.

Conclusion

We have shown how to use a purely semantical construction (the model $\mathbf{Pop}$) to reduce the proof of an adequacy theorem usually proved by reducibility to a purely combinatorial argument and we have illustrated this approach in the call-by-value lambda-calculus. In further work, we'll apply this approach to other languages and other notion of normalization (e.g. strong normalization for the general β-reduction), in order to understand better how the reducibility structure (interpretation of types etc) in encoded in the model.

References

[BL11] Alexis Bernadet and Stéphane Lengrand. Filter models: Non-idempotent intersection types, orthogonality and polymorphism. In *CSL*, Lecture Notes in Computer Science, pages 51–66. Springer-Verlag, 2011. To appear.

[DC08] Daniel De Carvalho. Execution Time of λ-Terms via Denotational Semantics and Intersection Types. Research Report RR-6638, INRIA, 2008.

[dCPdF11] Daniel de Carvalho, Michele Pagani, and Lorenzo Tortora de Falco. A semantic measure of the execution time in linear logic. *Theoretical Computer Science*, 412(20):1884–1902, 2011.

[Ehr11] Thomas Ehrhard. The Scott model of Linear Logic is the extensional collapse of its relational model. *Theoretical Computer Science*, 2011. To appear. A draft version is available on http://www.pps.jussieu.fr/~ehrhard.

[ER03] Thomas Ehrhard and Laurent Regnier. The differential lambda-calculus. *Theoretical Computer Science*, 309(1-3):1–41, 2003.

[ER04] Thomas Ehrhard and Laurent Regnier. Differential interaction nets. In *Proceedings of WoLLIC'04*, volume 103 of *Electronic Notes in Theoretical Computer Science*, pages 35–74. Elsevier Science, 2004.

[ER05] Thomas Ehrhard and Laurent Regnier. Uniformity and the Taylor expansion of ordinary lambda-terms. Technical report, Institut de mathématiques de Luminy, 2005. submitted to Theoretical Computer Science.

[Fio07] Marcelo P. Fiore. Differential structure in models of multiplicative biadditive intuitionistic linear logic. In Simona Ronchi Della Rocca, editor, *TLCA*, volume 4583 of *Lecture Notes in Computer Science*, pages 163–177. Springer, 2007.

[Gir87] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Gir88] Jean-Yves Girard. Normal functors, power series and the λ-calculus. *Annals of Pure and Applied Logic*, 37:129–177, 1988.

[HJK00] Michael Huth, Achim Jung, and Klaus Keimel. Linear types and approximation. *Mathematical Structures in Computer Science*, 10(6):719–745, 2000.

[Hut93] Michael Huth. Linear Domains and Linear Maps. In Stephen D. Brookes, Michael G. Main, Austin Melton, Michael W. Mislove, and David A. Schmidt, editors, *MFPS*, volume 802 of *Lecture Notes in Computer Science*, pages 438–453. Springer-Verlag, 1993.

[Mel09] Paul-André Melliès. Categorical semantics of linear logic. *Panoramas et Synthèses*, 27, 2009.

[MOTW99] John Maraist, Martin Odersky, David N. Turner, and Philip Wadler. Call-by-name, call-by-value, call-by-need and the linear lambda calculus. *Theoretical Computer Science*, 228(1-2):175–210, 1999.

[Pag09] Michele Pagani. The cut-elimination theorem for differential nets with promotion. In Pierre-Louis Curien, editor, *TLCA*, volume 5608 of *Lecture Notes in Computer Science*, pages 219–233. Springer, 2009.

[PR10] Michele Pagani and Simona Ronchi Della Rocca. Solvability in resource lambda-calculus. In C.-H. Luke Ong, editor, *FOSSACS*, volume 6014 of *Lecture Notes in Computer Science*, pages 358–373. Springer-Verlag, 2010.

[Tra08] Paolo Tranquilli. Intuitionistic Differential Nets and Lambda-Calculus. *Theoretical Computer Science*, 2008. To appear.

[Win98] Glynn Winskel. A linear metalanguage for concurrency. In Armando Martin Haeberer, editor, *AMAST*, volume 1548 of *Lecture Notes in Computer Science*, pages 42–58. Springer-Verlag, 1998.

[Win04] Glynn Winskel. Linearity and non linearity in distributed computation. In Thomas Ehrhard, Jean-Yves Girard, Paul Ruet, and Philip Scott, editors, *Linear Logic in Computer Science*, volume 316 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2004.

Appendix

Lemma 28 *If $M_1\, M_2\ \hat{\beta}_{\mathsf{V}}^*\ \langle V\rangle$ then there is a term N and a value W such that $M_1\ \hat{\beta}_{\mathsf{V}}^*\ \langle\lambda x\, N\rangle$, $M_2\ \hat{\beta}_{\mathsf{V}}^*\ \langle W\rangle$ and $N\,[W/x]\ \hat{\beta}_{\mathsf{V}}^*\ V$.*

Proof. Straightforward induction on the length of the reduction $M_1\, M_2\ \hat{\beta}_{\mathsf{V}}^*\ \langle V\rangle$. Notice that the base case holds trivially. □

Lemma 29 *If $M\ \hat{\beta}_{\mathsf{V}}^*\ \langle V\rangle$ and $M\ \hat{\beta}_{\mathsf{V}}\ M'$, then $M'\ \hat{\beta}_{\mathsf{V}}^*\ \langle V\rangle$.*

Proof. The proof is by induction on M. Since $M\ \hat{\beta}_{\mathsf{V}}\ M'$, M must be of shape $M = M_1\, M_2$ (in the other cases, the property holds trivially). We have three cases to consider.

If $M_1 = \langle\lambda x\, N\rangle$ and $M_2 = \langle W\rangle$, we must have $M' = N\,[W/x]$ and the reduction $M\ \hat{\beta}_{\mathsf{V}}^*\ \langle V\rangle$ is $M\ \hat{\beta}_{\mathsf{V}}\ M'\ \hat{\beta}_{\mathsf{V}}^*\ \langle V\rangle$; the announced property holds.

In the other cases, we know by Lemma 28 that $M_1\ \hat{\beta}_{\mathsf{V}}^*\ \langle\lambda x\, N\rangle$, $M_2\ \hat{\beta}_{\mathsf{V}}^*\ \langle W\rangle$ and $N\,[W/x]\ \hat{\beta}_{\mathsf{V}}^*\ \langle V\rangle$.

If $M' = M_1'\, M_2$ with $M_1\ \hat{\beta}_{\mathsf{V}}\ M_1'$, we know by inductive hypothesis that $M_1'\ \hat{\beta}_{\mathsf{V}}^*\ \langle\lambda x\, N\rangle$. Therefore $M'\ \hat{\beta}_{\mathsf{V}}^*\ \langle\lambda x\, N\rangle\,\langle W\rangle\ \hat{\beta}_{\mathsf{V}}^*\ Q$.

If $M' = M_1\, M_2'$ with $M_2\ \hat{\beta}_{\mathsf{V}}\ M_2'$, we know by inductive hypothesis that $M_2'\ \hat{\beta}_{\mathsf{V}}^*\ \langle W\rangle$. Therefore $M'\ \hat{\beta}_{\mathsf{V}}^*\ \langle\lambda x\, N\rangle\,\langle W\rangle\ \hat{\beta}_{\mathsf{V}}^*\ V$ and we are done. □

Let $\mathcal{N}$ be the set of all terms M such that $M\ \hat{\beta}_{\mathsf{V}}^*\ \langle V\rangle$ for a value V. Let L be the following term:

$$L = L_0\, L_0 \text{ where } L_0 = \langle\lambda x\, \langle\lambda y\, \langle x\rangle\, \langle x\rangle\rangle\rangle\ .$$

Observe that $L \in \mathcal{N}$ and that, for any value V, we have $L\,\langle V\rangle\ \hat{\beta}_{\mathsf{V}}^*\ L$. We could use any term with the same properties instead of L.

We say that a set $\mathcal{X}$ of terms is *saturated* if it satisfies the following properties

1) $\mathcal{X} \subseteq \mathcal{N}$.
2) For any terms M and N such that $M\ \hat{\beta}_{\mathsf{V}}\ N$, one has $M \in \mathcal{X}$ iff $N \in \mathcal{X}$.
3) $L \in \mathcal{X}$.

Lemma 30 *The set $\mathcal{N}$ is saturated.*

Proof. The only non straightforward property is that, if $M\ \hat{\beta}_{\mathsf{V}}\ M'$ and $M \in \mathcal{N}$, then $M' \in \mathcal{N}$. This results from Lemma 29. □

Lemma 31 *If $(\mathcal{X}_i)_{i\in I}$ is a family of saturated sets, with $I \neq \emptyset$, then $\bigcap_{i\in I} \mathcal{X}_i$ is saturated.*

The proof is straightforward.

Given two sets of lambda-terms $\mathcal{X}$ and $\mathcal{Y}$, we set as usual

$$\mathcal{X} \multimap \mathcal{Y} = \{M \mid \forall N \in \mathcal{X}\ M\, N \in \mathcal{Y}\}\ .$$

Lemma 32 *If $\mathcal{X}$ and $\mathcal{Y}$ are saturated, then $\mathcal{X} \multimap \mathcal{Y}$ is saturated.*

Proof. Let $M \in \mathcal{X} \multimap \mathcal{Y}$. Since $L \in \mathcal{X}$, we have $M\, L \in \mathcal{Y}$ and hence $M\, L \in \mathcal{N}$. If $M \notin \mathcal{N}$, then any term N such that $M\, L\ \hat{\beta}_{\mathsf{V}}^*\ N$ is of shape $N = M'\, L'$ with $M\ \hat{\beta}_{\mathsf{V}}^*\ M'$ (and of course $M' \notin \mathcal{N}$) and $L\ \hat{\beta}_{\mathsf{V}}^*\ L'$: this contradicts the fact that $M\, L \in \mathcal{N}$. So we have $M \in \mathcal{N}$ and hence $\mathcal{X} \multimap \mathcal{Y} \subseteq \mathcal{N}$.

Let M and M' be terms such that $M\ \hat{\beta}_{\mathsf{V}}\ M'$. Let $N \in \mathcal{X}$, we have $M\, N\ \hat{\beta}_{\mathsf{V}}\ M'\, N$ and hence, since $\mathcal{Y}$ is saturated, we have $M\, N \in \mathcal{Y}$ iff $M'\, N \in \mathcal{Y}$. Therefore $M \in \mathcal{X} \multimap \mathcal{Y}$ iff $M' \in \mathcal{X} \multimap \mathcal{Y}$.

Last, we check that $L \in \mathcal{X} \multimap \mathcal{Y}$. Let $M \in \mathcal{X}$. Since $\mathcal{X} \subseteq \mathcal{N}$, we have $M\ \hat{\beta}_{\mathsf{V}}^*\ \langle V\rangle$ where V is a value. Therefore $L\, M\ \hat{\beta}_{\mathsf{V}}^*\ L\,\langle V\rangle\ \beta_{\mathsf{V}}\ L \in \mathcal{Y}$. Since $\mathcal{Y}$ is saturated, we have therefore $L\, M \in \mathcal{Y}$. □

Given $a \in |\mathcal{U}_{\mathsf{S}}|$ and $m \in |!\mathcal{U}_{\mathsf{S}}|$, we define sets $a^\bullet$ and $m^\bullet$ of terms. The definition is by induction on the size. We set

$$(p,q)^\bullet = p^\bullet \multimap q^\bullet$$
$$[a_1,\dots,a_n]^\bullet = \mathcal{N} \cap a_1^\bullet \cap \cdots \cap a_n^\bullet\ .$$

It results from lemmas 31 and 32, that these sets $a^\bullet$ and $m^\bullet$ are saturated.

Lemma 33 *Let $a, a' \in |\mathcal{U}_{\mathsf{S}}|$. If $a \leq a'$, then $a'^\bullet \subseteq a^\bullet$. Let $m, m' \in |!\mathcal{U}_{\mathsf{S}}|$. If $m \leq m'$ (in $!\mathcal{U}_{\mathsf{S}}$) then $m'^\bullet \subseteq m^\bullet$.*

Proof. By induction in the size of a and of m. Assume first that $a \leq a'$. By definition of $\mathcal{U}_{\mathsf{S}}$, we must have $a = (p,q)$ and $a' = (p',q')$ with $q \leq q'$ and $p' \leq p$ (in $!\mathcal{U}_{\mathsf{S}}$). By inductive hypothesis we have $p^\bullet \subseteq p'^\bullet$ and $q'^\bullet \subseteq q^\bullet$. It follows easily that $a'^\bullet = p'^\bullet \multimap q'^\bullet \subseteq p^\bullet \multimap q^\bullet = a^\bullet$, as required. Assume now that $m \leq m'$ (in $!\mathcal{U}_{\mathsf{S}}$). Let $M \in m'^\bullet$, and let us prove that $M \in m^\bullet = \bigcap_{a\in|m|} a^\bullet$. So let $a \in |m|$. We know that there exists $a' \in |m'|$ such that $a \leq a'$ because $m \leq m'$. By inductive hypothesis we have $a'^\bullet \subseteq a^\bullet$. By definition of $m'^\bullet$ we have $m'^\bullet \subseteq a'^\bullet$ and we conclude that $M \in a^\bullet$ because $M \in m'^\bullet$. □

We can now prove the main property.

Proposition 34 *If $x_1 : r_1, \dots, x_n : r_n \vdash P : \alpha$ and if $V_1, \dots, V_n$ are values such that $\langle V_i\rangle \in r_i^\bullet$ for $i = 1, \dots, n$, then we have $P\,[V_1/x_1, \dots, V_n/x_n] \in \alpha^\bullet$ if P is a term and $\langle P\,[V_1/x_1, \dots, V_n/x_n]\rangle \in \alpha^\bullet$ if P is a value.*

Proof. By induction on the deduction of $x_1 : r_1, \dots, x_n : r_n \vdash P : \alpha$. Given any expression Q, we use Q' to denote the expression $Q\,[V_1/x_1, \dots, V_n/x_n]$.

Assume first that $P = x_n$ and that the deduction consists of the axiom

$$\frac{}{x_1 : r_1, \dots, x_{n-1} : r_{n-1}, x_n : r_n \vdash_{\mathsf{S}} x_n : a}$$

where we have $[a] \leq r_n$. By Lemma 33 we have therefore $r_n^\bullet \subseteq a^\bullet$. But $\langle V\rangle \in r_n^\bullet$ by assumption and hence $P' = \langle V\rangle \in a^\bullet$ as required,

Assume next that the deduction ends with

$$\frac{\Gamma \vdash_{\mathsf{S}} M : [(p,q)] \qquad \Gamma \vdash_{\mathsf{S}} N : p}{\Gamma \vdash_{\mathsf{S}} M\, N : q}$$

with $P = M\, N$ (so P is a term) and $\Gamma = (x_1 : r_1, \dots, x_n : r_n) = \Gamma$ and $\alpha = q$. For each $i = 1, \dots, n$, we have $\langle V_i\rangle \in r_i^\bullet$ and hence by inductive hypothesis $M' \in (p,q)^\bullet$ and $N' \in p^\bullet$. Since $(p,q)^\bullet = p^\bullet \multimap q^\bullet$, we get $P' = M'\, N' \in q^\bullet$ as required, since P is a term and $\alpha = q$.

Assume next that the proof ends with

$$\frac{\Gamma \vdash_{\mathsf{S}} V : a_1 \qquad \cdots \qquad \Gamma \vdash_{\mathsf{S}} V : a_k}{\Gamma \vdash_{\mathsf{S}} \langle V \rangle : [a_1, \ldots, a_k]}$$

with $P = \langle V \rangle$ and $\Gamma = (x_1 : r_1, \ldots, x_n : r_n) = \Gamma$. For each $j \in \{1, \ldots, k\}$ we have by inductive hypothesis $\langle V' \rangle \in a_j^\bullet$. Hence $\langle V' \rangle \in \bigcap_{j=1}^{k} a_j^\bullet$. Last, we obviously have $\langle V' \rangle \in \mathcal{N}$ and hence $P' = \langle V' \rangle \in \alpha^\bullet$.

Assume last that the proof ends with

$$\frac{\Gamma, x : p \vdash_{\mathsf{S}} M : q}{\Gamma \vdash_{\mathsf{S}} \lambda x\, M : (p, q)}$$

with $P = \lambda x\, M$ (so that P is a value) and $\alpha = (p, q)$. We must prove that $\langle P' \rangle = \langle \lambda x\, M' \rangle \in p^\bullet \multimap q^\bullet$. Let $N \in p^\bullet$, we must prove that $\langle \lambda x\, M' \rangle\, N \in q^\bullet$. Since $N \in p^\bullet \subseteq \mathcal{N}$, we know that $N\ \hat{\beta}_{\mathsf{V}}^{*}\ \langle W \rangle$ for some value W. Moreover, since $p^\bullet$ is saturated and since $N \in p^\bullet$, we have $\langle W \rangle \in p^\bullet$. By inductive hypothesis we have therefore $M'\,[W/x] \in q^\bullet$. Now $\langle \lambda x\, M' \rangle\, N\ \ \hat{\beta}_{\mathsf{V}}^{*}\ \ \langle \lambda x\, M' \rangle \langle W \rangle\ \ \hat{\beta}_{\mathsf{V}}\ \ M'\,[W/x] \in q^\bullet$ and hence $\langle \lambda x\, M' \rangle\, N \in q^\bullet$ because $q^\bullet$ is saturated. $\square$

Theorem 26 follows, using Proposition 34 and the fact that any saturated set is included in $\mathcal{N}$.