Stable and measurable functions on positive cones

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General goal: Extend the Probabilist Coherence Space (PCS) denotational semantics to "continuous data-types" such as the real line.
$\rightsquigarrow$ models of functional "programming languages" computing probability distributions, real PCF.

PCS duality

Let I be a finite or countable set.

If $u, u' \in (\mathbb{R}^+)^I$, set

$$\langle u, u' \rangle = \sum_{i \in I} u_i u'_i \in \overline{\mathbb{R}^+}$$

and if $\mathcal{F} \subseteq (\mathbb{R}^+)^I$ then

$$\mathcal{F}^\perp = \{u' \in (\mathbb{R}^+)^I \mid \forall u \in \mathcal{F}\ \langle u, u' \rangle \leq 1\}\,.$$

PCS: $X = (|X|, \mathrm{P}X)$ where $|X|$ countable and $\mathrm{P}X \subseteq (\mathbb{R}^+)^{|X|}$ such that $\mathrm{P}X = \mathrm{P}X^{\perp\perp}$.
Equivalently: $\mathrm{P}X$ is downwards-closed (for the product order), convex and directed-complete.

Additional conditions to avoid ∞ coefficients:

- $\forall a \in |X| \, \exists x \in \mathrm{P}X \;\; x_a \neq 0$
- $\forall a \in |X| \; \{x_a \mid x \in \mathrm{P}X\}$ bounded

Examples:

- 1 with $|1| = \{*\}$ (singleton) and $\mathsf{P}1 = \{(0,\alpha) \mid \alpha \in [0,1]\} = [0,1]$.
- N with $|\mathsf{N}| = \mathbb{N}$ and $u \in (\mathbb{R}^+)^{\mathbb{N}} \in \mathsf{PN}$ if $\sum_{i=0}^{\infty} u_i \leq 1$. Sub-probability distributions on $\mathbb{N}$.
- $\mathsf{N}^\perp$ with $\mathsf{N}^\perp = \mathbb{N}$ and $u' \in (\mathbb{R}^+)^{\mathbb{N}} \in \mathsf{PN}^\perp$ if $\forall i \in \mathbb{N}\ u'_i \in [0,1]$. These are not at all probability distributions!

Standard ℓ_1/ℓ_∞ duality in Banach spaces.

Linear morphisms

In the category of PCS's, a morphism from X to Y is an element of $\mathrm{P}(X \multimap Y)$ where $X \multimap Y$ is the PCS given by

- $|X \multimap Y| = |X| \times |Y|$
- given $s \in (\mathbb{R}^+)^{|X|\times|Y|}$, one has $s \in \mathrm{P}(X \multimap Y)$ if

$$\forall u \in \mathrm{P}X \quad s\,u = \left(\sum_{a\in|X|} s_{a,b} u_a\right)_{[b\in|Y|} \in \mathrm{P}Y$$

that is

$$\forall u \in \mathrm{P}X\,\forall v' \in \mathrm{P}Y^{\perp} \qquad \sum_{a\in|X|,b\in|Y|} s_{a,b} u_a v'_b \leq 1$$

Example: a morphism from N to N (element of P(N $\multimap$ N)) is a sub-stochastic matrix indexed by $\mathbb{N} \times \mathbb{N}$.

Just as ordinary coherence spaces, PCS, with these linear morphisms

- are a model of full classical LL
- with fixpoints (hence a model of PCF etc)
- with fixpoints of types (hence contain various models of the pure lambda-calculus, of FPC, of CBPV with recursive types etc)
- 2nd order LL, polymorphism? (never explored)

All these languages extended with probabilistic primitives, for instance: random integers in a given range, (fair) coin etc.

Examples of type constructions:

- $|1| = \{*\}$, $\mathrm{P}1 = [0,1]$
- $|X \mathbin{\&} Y| = |X| + |Y|$ and
  $\mathrm{P}(X \mathbin{\&} Y) = \{x \oplus y \mid x \in \mathrm{P}X \text{ and } y \in \mathrm{P}Y\} \simeq \mathrm{P}X \times \mathrm{P}Y$
- $|X \oplus Y| = |X| + |Y|$ and
  $\mathrm{P}(X \oplus Y) = \{px \oplus (1-p)y \mid x \in \mathrm{P}X,\ y \in \mathrm{P}Y \text{ and } p \in [0,1]\}$

So $|1 \oplus 1| = \{\mathbf{t}, \mathbf{f}\}$ and
$\mathrm{P}(1 \oplus 1) = \{p\mathbf{t} + q\mathbf{f} \mid p, q \in \mathbb{R}^+, p + q \leq 1\}$

$\mathsf{N} = 1 \oplus \mathsf{N}$ (“least” fixpoint) so that $|\mathsf{N}| = \mathbb{N}$ and $\mathrm{P}\mathsf{N}$ is the set of sub-probability distributions on $\mathbb{N}$.

Similarly one defines probabilistic types of lists, trees, streams etc.

Morphisms in the CCC of PCS

By standard LL/categorical considerations we know that we have a CCC: the "Kleisli category" of the "!" comonad that we have not described.

One defines $X \Rightarrow Y$ by

- $|X \Rightarrow Y| = \mathcal{M}_{\mathrm{fin}}(|X|) \times |Y|$
- If $u \in (\mathbb{R}^+)^{|X|}$ and $m \in \mathcal{M}_{\mathrm{fin}}(|X|)$, set $u^m = \prod_{a \in |X|} u_a^{m(a)}$
- Then $t \in (\mathbb{R}^+)^{|X \Rightarrow Y|}$ is in $\mathrm{P}(X \Rightarrow Y)$ if for all $u \in \mathrm{P}X$,

$$\widehat{t}(u) = \left(\sum_{m \in \mathcal{M}_{\mathrm{fin}}(|X|)} t_{m,b} u^m \right)_{b \in |Y|} \in \mathrm{P}Y$$

NB: such a t defines therefore a function $\widehat{t} : \mathrm{P}X \to \mathrm{P}Y$ and t is easily seen to be determined by this function ($\widehat{s} = \widehat{t} \Rightarrow s = t$).

These morphisms are closed under composition: this defines the CCC $\mathbf{Pcoh}_!$. In other words, given $s \in \mathsf{P}(X \Rightarrow Y)$ and $t \in \mathsf{P}(Y \Rightarrow Z)$, there is $t \circ s \in \mathsf{P}(X \Rightarrow Z)$ such that

$$\widehat{t \circ s} = \widehat{t} \circ \widehat{s}$$

and there is an $\mathrm{Id} \in \mathsf{P}(X \Rightarrow X)$, given by $\mathrm{Id}_{[a],a} = 1$ and $\mathrm{Id}_{m,a} = 0$ if $m \neq [a]$. Of course $\widehat{\mathrm{Id}}(x) = x$.

- The cartesian product of X and Y is $X \,\&\, Y$.
- The object of morphisms from X to Y is $X \Rightarrow Y$.

$\mathrm{P}X$ is naturally ordered by: $x \leq y$ if $\forall a \in |X|\ x_a \leq y_a$. Then $\mathrm{P}X$ is directed-complete (it is actually an ω-continuous cpo).

For any $s \in \mathrm{P}(X \Rightarrow Y)$, the function $\widehat{s}$ is Scott continuous, and hence we can interpret (*e.g.* PCF) fixpoint operators.

Example of morphisms in this CCC

A $s \in \mathrm{P}(1 \Rightarrow 1)$ is a family $(s_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{R}^+$ such that $\sum_{n=0}^{\infty} s_n \leq 1$, and the associated function is $\widehat{s} : [0,1] \to [0,1]$ given by $\widehat{s}(x) = \sum_{n=0}^{\infty} s_n x^n$.

Functions of this kind are

- very smooth (analytic)
- very monotonic (all derivatives are ≥ 0)

Example of such function $[0,1] \to [0,1]$: $f(x) = \frac{1}{3} + \frac{x^2}{2}$.

An example coming from a simple program

Consider $\vdash M : 1 \to 1$ given by the recursive definition

$$M\,x = \mathrm{ifz}(\mathrm{coin}_{1/2}, (), x; M\,x; M\,x)$$

where coin_p reduces to $\underline{0}$ with proba p and to $\underline{1}$ with proba $1 - p$, and ";" is the "unary conditional" ($P; Q \rightsquigarrow Q$ if $P \rightsquigarrow ()$).
Then M is represented by the least function $f : [0,1] \to [0,1]$ (in $\mathrm{P}[1 \Rightarrow 1]$) such that $f(x) = \frac{1}{2} + \frac{1}{2} x f(x)^2$ so that

$$f(x) = \frac{1 - \sqrt{1 - x}}{x} \text{ if } x \neq 0 \text{ and } f(0) = \frac{1}{2}$$

NB: $f'(1) = \infty$! Possible singularities on the border of $\mathrm{P}X$.

Completely similar characterization of $s \in \mathrm{P}((1 \,\&\, 1) \Rightarrow 1)$.

The "weak parallel or" wpor : $[0,1]^2 \to [0,1]$ defined by $\mathrm{wpor}(x,y) = x + y - xy$ is not a morphism (there is a negative coefficient).

NB 1: adding such a morphism to the model is incompatible with the fact that all morphisms are analytic and the presence of least fixpoints. We would be able to define a function $f : [0,1] \to [0,1]$ by $f(x) = \mathrm{wpor}(x, f(x)) = x + f(x) - xf(x)$ as a least fixpoint. But then $f(0) = 0$ and $f(x) = 1$ if $x > 0$ so f is not analytic.
NB 2: f is Scott continuous, but not continuous for the standard topology of $\mathbb{R}$.

A $s \in \mathrm{P}((1 \oplus 1) \Rightarrow 1)$ is a family $(s_{n,m})_{n,m \in \mathbb{N}}$ such that

$$\forall p \in [0,1] \quad \sum_{n,m \in \mathbb{N}} s_{n,m} p^n (1-p)^m \leq 1$$

For instance: $f : \mathrm{P}(1 \oplus 1) \to [0,1]$ given by $f(x) = 4 x_{\mathbf{t}} x_{\mathbf{f}}$ is such a morphism because $p \in [0,1] \Rightarrow p(1-p) \leq \frac{1}{4}$.

NB 1: this function is not definable in "PCF" (but the function $f(x) = 2 x_{\mathbf{t}} x_{\mathbf{f}}$ is).

NB 2: we have no "parallel or" function in the model (requires negative coefficients), but we have an analogue of Gustave's function $g \in \mathrm{P}((1 \oplus 1) \,\&\, (1 \oplus 1) \,\&\, (1 \oplus 1) \Rightarrow 1)$ given by $g(x,y,z) = x_{\mathbf{t}} y_{\mathbf{f}} + y_{\mathbf{t}} z_{\mathbf{f}} + z_{\mathbf{t}} x_{\mathbf{f}}$. But $\frac{1}{2} g$ is definable whereas no εwpor (for $\varepsilon > 0$) is!

Main properties of this model

For probabilistic PCF (and its extensions with recursive types etc):

- Adequacy: if $\vdash M : \iota$, then the interpretation $[M]$ of M, which is an element of PN (ι is interpreted as the PCS N) satisfies:

  For all $n \in \mathbb{N}$, $[M]_n$ is the probability of M to reduce to $\underline{n}$.
- Equational full abstraction: semantical equality $\Rightarrow$ observational equivalence (same probability to reduce to $\underline{0}$ in any context of ground type ι).

Inequational full abstraction fails for the standard order of the model: $x, y \in \mathrm{P}X$ satisfy $x \leq y$ if $\forall a \in |X|\ x_a \leq y_a$.

Major limitation of PCS: no "continuous" types

Main limitation of this model: apparently, does not allow to interpret "continuous types" like the real line $\mathbb{R}$ (very important for the semantics of Machine Learning oriented languages).

Idea to overcome it: our morphisms are functions acting on the cpos $\mathsf{P}X$. Introduce more general such cpos and find a notion of morphisms generalizing those of PCS.

However, the Cantor space is (almost) here!

Let X be the PCS which is the "least solution" (there is a natural order relations on PCS such that all connectives of LL are Scott continuous wrt. this order) of

$$X = 1 \mathbin{\&} (X \oplus X)\,.$$

Then $|X| \simeq \{0,1\}^*$.
NB: if $\theta \in \{0,1\}^{\mathbb{N}}$ then $x \in (\mathbb{R}^+)^{|X|}$ given by

$$x_s = \begin{cases} 1 & \text{if } s \text{ prefix of } \theta \\ 0 & \text{otherwise} \end{cases}$$

is an element of $\mathsf{P}X$: $\mathsf{P}X$ contains the Cantor space.

The elements x of $\mathsf{P}X$ such that

$$\forall s \in |X| \quad x_s = x_{s0} + x_{s1}$$

are exactly the sub-probability measures of the Cantor space with its standard Borelian σ-algebra.

$X = 1 \mathbin{\&} (\mathsf{N} \otimes X)$ contains all sub-probability measures on Baire space.
We don't know how to do the same for $[0,1]$, seems impossible.

Warning: $X = 1 \oplus (\mathsf{N} \otimes X) \rightsquigarrow$ lists of integers with all sub-probability distributions; a much simpler PCS.

The cone generated by a PCS

Given a PCS X we can consider its associated "cone" $\mathsf{C}(X)$: the set of all $x \in (\mathbb{R}^+)^{|X|}$ such that $\varepsilon x \in \mathsf{P}X$ for some $\varepsilon > 0$.Then

- $\mathsf{C}(X)$ is an $\mathbb{R}^+$-semi module: if $x, y \in \mathsf{C}(X)$ and $\alpha, \beta \in \mathbb{R}^+$ then $\alpha x + \beta y = (\alpha x_a + \beta y_a)_{a \in |X|} \in \mathsf{C}(X)$ (with the usual algebraic properties).
- The canonical ordre relation can be defined by: $x \leq y$ if there is $z \in \mathsf{C}(X)$ such that $y = x + z$.
- There is a "norm" $\|\_\| : \mathsf{C}(X) \to \mathbb{R}^+$ defined by $\|x\| = \sup_{x' \in \mathsf{P}X^\perp} \langle x, x' \rangle$ which satisfies $\|\alpha x\| = \alpha \|x\|$, $\|x + y\| \leq \|x\| + \|y\|$ and $x \leq y \Rightarrow \|x\| \leq \|y\|$.
- Any monotonic bounded sequence of elements of $\mathsf{C}(X)$ has a least upper bound.

$\mathsf{C}(X)$ is a kind of "order complete positive Banach space".

Positive cone

We generalize this situation. A (Selinger) *cone* is an $\mathbb{R}^+$-semi module P (there are operations $+$ and $\mathbb{R}^+$ scalar multiplication satisfying the usual laws) equiped with a function $\|\_\|_P : P \to \mathbb{R}^+$ such that:

- $x + y = x' + y \Rightarrow x = x'$ (simplifiability).
- $\|\alpha x\|_P = \alpha\|x\|_P$, $\|x + y\|_P \leq \|x\|_P + \|y\|_P$ and $\|x\|_P = 0 \Rightarrow x = 0$ ($\|\_\|_P$ is a norm).
- Defining $x \leq y$ by $\exists z\ y = x + z$, we have $\|x\|_P \leq \|y\|_P$ (monotonicity of the norm).
- Any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of P such that $\forall n \in \mathbb{N}\ x_n \leq x_{n+1}$ and $\exists\alpha \in \mathbb{R}^+\ \forall n \in \mathbb{N}\ \|x_n\|_P \leq \alpha$ (the sequence is bounded) has a lub $\sup_{n\in\mathbb{N}} x_n \in P$.

Subtraction

If $x \leq y$, there is a unique z such that $x + z = y$, by simplifiability.

This z is denoted as $y - x$.

This subtraction, when defined, satisfies all the usual algebraic laws.

Main motivating examples of cones

- For any PCS X, $\mathsf{C}(X) = \{x \in (\mathbb{R}^+)^{|X|} \mid \exists \varepsilon > 0\ \varepsilon x \in \mathsf{P}X\}$ is a cone with $\|x\|_{\mathsf{C}(X)} = \sup_{x' \in \mathsf{P}X^\perp} \langle x, x' \rangle$.
- Let $(\mathcal{X}, \Sigma_\mathcal{X})$ be a measurable space. We define a cone $\mathsf{Meas}(\mathcal{X})$ as the set of all $\mathbb{R}^+$-valued measures μ on $\Sigma_\mathcal{X}$ (in particular $\mu(\mathcal{X}) < \infty$). Algebraic operations defined in the obvious way. $\|\mu\|_{\mathsf{Meas}(\mathcal{X})} = \mu(\mathcal{X})$.

What morphisms?

Unit ball of P: $\mathcal{B}P = \{x \in P \mid \|x\|_P \leq 1\}$.

It is a poset where all monotone sequences have a lub (we do not consider uncoutable directed subsets because we will have to use the Monotone Convergence Theorem at some point).

By analogy with (the CCC of) PCS's, a morphism from P to Q should be a Scott continuous function $\mathcal{B}P \to \mathcal{B}Q$.

But if we take all Scott continuous functions, we don't get a CCC.

Problem: the curryfied version $\Lambda(\mathrm{wpor})$ of $\mathrm{wpor} : [0,1] \times [0,1] \to [0,1]$ should be a monotonic function $[0,1] \to P$ where P is a cone of Scott continuous functions $[0,1] \to [0,1]$. Remember $\mathrm{wpor}(x,y) = x + y - xy$ is Scott continuous.

$$\Lambda(\mathrm{wpor})(x)(y) = \mathrm{wpor}(x,y) = x + y - xy$$

We should have $\Lambda(\mathrm{wpor})(0) \leq \Lambda(\mathrm{wpor})(1)$ in this cone P where the operations are defined pointwise.

$$\Lambda(\mathrm{wpor})(0)(y) = y$$
$$\Lambda(\mathrm{wpor})(1)(y) = 1$$

So $f = \Lambda(\mathrm{wpor})(1) - \Lambda(\mathrm{wpor})(0) : [0,1] \to [0,1]$ is the function defined by $f(y) = 1 - y$ which is not monotonic.

Local cone

Let P be a cone and $u \in \mathcal{B}P$.

We define a new cone P_u as follows

- $P_u = \{x \in P \mid \exists\varepsilon > 0\ \varepsilon x + u \in \mathcal{B}P\}$
- algebraic operations defined as in P.
- $\|x\|_{P_u} = \inf\{1/\varepsilon \mid \varepsilon > 0 \text{ and } \varepsilon x + u \in \mathcal{B}P\}$

Fact: P_u is a cone.

Observe that $\mathcal{B}(P_u) = \{x \in P \mid x + u \in \mathcal{B}P\} \subseteq \mathcal{B}P$.

Let P and Q be cones and $f : \mathcal{B}P \to Q$.

Assume that f is monotonic. Then, given $u \in \mathcal{B}P$, we have $\forall x \in \mathcal{B}(P_u)\ f(x) \leq f(x+u)$.

So we can define a function $\Delta f(\_;u) : \mathcal{B}(P_u) \to Q$ by $\Delta f(x;u) = f(x+u) - f(x)$. We require this function to be also monotonic.

Given $v \in \mathcal{B}P$ such that $u + v \in \mathcal{B}P$, we can consider the function $\Delta f(\_; u, v) : \mathcal{B}(P_{u,v}) \to Q$ given by

$$\begin{aligned}\Delta f(x; u, v) &= \Delta(\Delta f(\_; u))(x; v) \\ &= f(x + u + v) - f(x + u) - f(x + v) + f(x)\end{aligned}$$

We require this function to be also monotonic.

And the same for all functions $\Delta f(\_; u_1, \dots, u_n)$ for $u_1, \dots, u_n \in \mathcal{B}P$ such that $\sum_{i=1}^{n} u_i \in \mathcal{B}P$.

$P_{\vec{u}} = P_{\sum_{i=1}^{n} u_i}$

Third iterated difference:
$\Delta f(x; u_1, u_2, u_3) = f(x + u_1 + u_2 + u_3) - f(x + u_1 + u_2) - f(x + u_1 + u_3) - f(x + u_2 + u_3) + f(x + u_1) + f(x + u_2) + f(x + u_3) - f(x)$.

More generally, if $\vec{u} \in \mathcal{B}P^n$ with $\sum_{i=1}^{n} u_i \in \mathcal{B}P$ and $x \in \mathcal{B}(P_{\vec{u}})$ and $\varepsilon \in \{+, -\}$, one defines

$$\Delta^{\varepsilon} f(x; \vec{u}) = \sum_{I \in \mathcal{P}_{\varepsilon}(n)} f(x + \sum_{i \in I} u_i) \in Q$$

where $\mathcal{P}_{+}(n)$ (resp. $\mathcal{P}_{-}(n)$) is the set of all $I \subseteq \{1, \dots, n\}$ such that $n - \#I$ is even (resp odd).

Absolutely monotonic and stable functions

Definition

The function $f : \mathcal{B}P \to Q$ is *absolutely monotonic* if, for all $\vec{u} \in \mathcal{B}P^n$ with $\sum_{i=1}^n u_i \in \mathcal{B}P$ and $x \in \mathcal{B}P_{\vec{u}}$, one has

$$\Delta^- f(x; \vec{u}) \leq \Delta^+ f(x; \vec{u})$$

Then we set $\Delta f(x; \vec{u}) = \Delta^+ f(x; \vec{u}) - \Delta^- f(x; \vec{u})$. This generalizes our previous examples, and the function $\Delta f(\_; \vec{u})$ is monotonic.

Definition

The function $f : \mathcal{B}P \to Q$ is *stable* if it is absolutely monotonic, bounded (that is $\exists\alpha\,\forall x \in \mathcal{B}P\ \|f(x)\|_Q \leq \alpha$) and Scott continuous, that is:
For all monotonic sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{B}P$, one has $\sup_{n\in\mathbb{N}} f(x_n) = f(\sup_{n\in\mathbb{N}} x_n)$.

Fact

If X and Y are PCS's and $s \in \mathrm{P}(X \Rightarrow Y)$, then the function $\widehat{s} : \mathrm{P}X \to \mathrm{P}Y$ is stable.

NB: Raphaëlle Crubillé proved the converse (based on work by Bernstein in the... 1930's).

The cone of stable functions

We define a cone $P \Rightarrow Q$ as follows.

- The elements of $P \Rightarrow Q$ are the stable functions $\mathcal{B}P \to Q$.
- Addition and scalar multiplication are defined pointwise $((f+g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$; these functions are stable by linearity of the operators $\Delta^{\varepsilon}\_(x;\vec{u}))$.
- $\|f\|_{P \Rightarrow Q} = \sup_{x \in \mathcal{B}P} \|f(x)\|_Q$.

Then $f \leq g$ is equivalent to the following condition: for all $n \in \mathbb{N}$, all $\vec{u} \in \mathcal{B}P^n$ such that $\sum u_i \in \mathcal{B}P$ and all $x \in \mathcal{B}(P_{\vec{u}})$, one has

$$\Delta f(x;\vec{u}) \leq \Delta g(x;\vec{u})$$

that is

$$\Delta^+ f(x;\vec{u}) + \Delta^- g(x;\vec{u}) \leq \Delta^- f(x;\vec{u}) + \Delta^+ g(x;\vec{u})$$

Composing stable functions

If $f \in \mathcal{B}(P \Rightarrow Q)$ then $f : \mathcal{B}P \to \mathcal{B}Q$.

Theorem

Let $f \in \mathcal{B}(P \Rightarrow Q)$ and $g \in \mathcal{B}(Q \Rightarrow R)$. Then $g \circ f \in \mathcal{B}(P \Rightarrow R)$.

The proof is not straightforward because morphisms are not defined by a preservation property (like Scott continuity of Berry stability).

So we have a category **Cstab** whose objects are the cones and morphisms, the stable functions.

Intermezzo: why do we call these functions "stable"?

Reminder: a coherence space is a structure $E = (|E|, \frown_E)$ where $|E|$ is a countable set and $\frown_E$ is a binary, reflexive and symmetric relation on $|E|$.

Cliques of E: $\mathrm{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in x\ a \frown_E a'\}$.

$(\mathrm{Cl}(E), \subseteq)$ is a cpo, any subset of a clique is a clique.

A function $f : \mathrm{Cl}(E) \to \mathrm{Cl}(F)$ is stable if it is monotonic, Scott continuous and "conditionally multiplicative", that is:

$$\forall x, x' \in \mathrm{Cl}(E) \quad x \cup x' \in \mathrm{Cl}(E) \Rightarrow f(x \cap x') = f(x) \cap f(x')$$

Let $f, g : \mathrm{Cl}(E) \to \mathrm{Cl}(F)$ be stable.
f is stably less than g ($f \leq g$) if

$$\forall x, x' \in \mathrm{Cl}(E) \quad x \subseteq x' \Rightarrow f(x) = f(x') \cap g(x)$$

Reformulating stability in coherence spaces

Let E be a coherence space. If $u \in \mathrm{Cl}(E)$, define a "local" coherence space E_u as follows:
$|E_u| = \{a \in |E| \setminus u \mid \forall a' \in u\ a \coh_E a'\}$. So if $x \in \mathrm{Cl}(E_u)$, $x + u$ (disjoint union) is in $\mathrm{Cl}(E)$.

Let $f : \mathrm{Cl}(E) \to \mathrm{Cl}(F)$ be a function and $u \in \mathrm{Cl}(E)$. We define $\Delta f(\_; u) : \mathrm{Cl}(E_u) \to \mathrm{Cl}(F)$ by
$\Delta f(x; u) = f(x + u) \setminus f(x) \in \mathrm{Cl}(F)$.

Theorem

A Scott continuous function $f : \mathcal{Cl}(E) \to \mathcal{Cl}(F)$ is stable iff for all $u \in \mathcal{Cl}(E)$, the function $\Delta f(\_\,; u) : \mathcal{Cl}(E_u) \to \mathcal{Cl}(F)$ is monotonic.

If f is stable, then $\Delta f(\_\,; u) : \mathcal{Cl}(E_u) \to \mathcal{Cl}(F)$ is also stable.

So there is no need to consider $\Delta f(x; u_1, \dots, u_n)$ for $n \geq 2$: the corresponding conditions are redundant (this due to the idempotency of union).

Theorem

Let $f, g : \mathcal{Cl}(E) \to \mathcal{Cl}(F)$ be stable functions.

One has $f \leq g$ (for the stable order) iff

- $\forall x \in \mathcal{Cl}(E)\ f(x) \subseteq g(x)$
- $\forall u \in \mathcal{Cl}(E) \forall x \in \mathcal{Cl}(E_u)\ \Delta f(x; u) \subseteq \Delta g(x; u)$.

Back to cones. The CCC **Cstab**

The cartesian product is defined in the obvious way: $P \times Q$ with norm defined by $\|(x, y)\|_{P \times Q} = \max(\|x\|_P, \|y\|_Q)$.

We have already defined the cone $P \Rightarrow Q$. The evaluation map $\mathrm{Ev} : (P \Rightarrow Q) \times P \to Q$ is defined by $\mathrm{Ev}(f, x) = f(x)$. It is stable.

If $f : \mathcal{B}R \times \mathcal{B}P \to \mathcal{B}Q$ is stable, it is a very nice exercise to prove that the function $\Lambda(f) : \mathcal{B}R \to \mathcal{B}(P \Rightarrow Q)$ defined as usual by $\Lambda(f)(z)(x) = f(z, x)$ takes actually its values in $\mathcal{B}(P \Rightarrow Q)$ and is stable.

What is the trouble with measurability?

Types of our target language are interpreted as cones. There is a type ρ of real numbers, and $[\rho] = \mathrm{Meas}(\mathbb{R})$ (with respect to the standard Borel σ-algebra). For simplicity, ρ is our unique ground type.

A closed term M such that $\vdash M : \rho$ will be interpreted as an element $[M]$ of $\mathcal{B}(\mathrm{Meas}(\mathbb{R}))$, that is, as a sub-probability measure.

For each $r \in \mathbb{R}$, there is a constant $\underline{r}$ of our language $\vdash \underline{r} : \rho$. We set $[\underline{r}] = \delta_r \in \mathcal{B}(\mathrm{Meas}(\mathbb{R}))$.

There is also a constant $\vdash \mathsf{sample} : \rho$. Intuitively, sample draws a real number in $[0, 1]$ with uniform probability.

The language is CBN but has a "let" construct *restricted to the ground type of real numbers* (we omit contexts for readability):

$$\frac{\vdash M : \rho \qquad x : \rho \vdash N : \sigma}{\vdash \mathsf{let}\, x = M \,\mathsf{in}\, N : \sigma}$$

This construction is crucial: it draws a real number r according to the sub-probability measure defined by M and reduces to $N\,[\underline{r}/x]$.

In our model **Cstab**: $\mu = [M]$ is an element of $\mathcal{B}(\mathsf{Meas}(\mathbb{R}))$. And $[N]_x : \mathcal{B}(\mathsf{Meas}(\mathbb{R})) \to [\sigma]$ is a stable function.

So we have a function $\gamma : [N]_x \circ \delta : \mathbb{R} \to [\sigma]$ where δ is the function $r \mapsto \delta_r$. We have $\gamma(r) = [N\,[\underline{r}/x]]$ (by Substitution Lemma).

So we would like to set
$[\mathsf{let}\, x = M \,\mathsf{in}\, N] = \int_{\mathbb{R}} \gamma(r)\mu(dr) = \int_{\mathbb{R}} [N]_x\,(\delta_r)\mu(dr)$

This integral does not make sense *a priori* for two reasons:

- We don't know how to integrate functions ranging in an arbitrary cone, but this is not a serious issue because $\sigma = \sigma_1 \rightarrow \cdots \rightarrow \sigma_k \rightarrow \rho$ so we can replace our problem with: given $\gamma : \mathbb{R} \times \mathcal{B}P \rightarrow \mathsf{Meas}(\mathbb{R})$, define $\gamma_\mu : \mathcal{B}P \rightarrow \mathsf{Meas}(\mathbb{R})$ by

$$\gamma_\mu(x)(U) = \int_{\mathbb{R}} \gamma(r, x)(U)\mu(dr)$$

  for $U \in \Sigma_{\mathbb{R}}$.
- More seriously, given $x \in \mathcal{B}P$ and $U \in \Sigma_{\mathbb{R}}$, there is no reason *a priori* for the function $r \mapsto \gamma(r, x)(U)$ (from $\mathbb{R}$ to $\mathbb{R}$) to be measurable.

Our solution

Equip all cones with a family of sets of "measurability tests" $(\mathsf{M}^n(P))_{n\in\mathbb{N}}$ where each element l of $\mathsf{M}^n(P)$ is a function $l : \mathbb{R}^n \times P \to \mathbb{R}^+$ with the following properties:

- For each $\vec{r} \in \mathbb{R}^n$, the function $x \mapsto l(\vec{r}, x)$ is linear (commutes with all linear combinations in P) and Scott continuous from P to $\mathbb{R}^+$.
- For each $x \in P$, the function $\vec{r} \mapsto l(\vec{r}, x)$ is measurable.
- For each measurable $h : \mathbb{R}^k \to \mathbb{R}^n$, $l \circ h \in \mathsf{M}^k(P)$.
- $0 \in \mathsf{M}^n(P)$ for all n.

Next we say that a function $\gamma : \mathbb{R}^n \to P$ is a *measurable path* if:

- $\gamma(\mathbb{R}^n)$ is bounded in P
- and for all $l \in \mathsf{M}^k(P)$, the function $l * \gamma : \mathbb{R}^{k+n} \to \mathbb{R}^+$ defined by $(l * \gamma)(\vec{r}, \vec{s}) = l(\vec{r}, \gamma(\vec{s}))$ is measurable.

Last, a morphism $P \to Q$ in our new category $\mathbf{Cstab}_\mathsf{m}$ is a stable function $f : \mathcal{B}P \to \mathcal{B}Q$ such that, for any measurable path $\gamma : \mathbb{R}^n \to \mathcal{B}P$, the function $f \circ \gamma$ is a measurable path.

Such an f will be called a stable measurable function.

Construction of measurability tests

If $\mathcal{X}$ is a measurable space, we equip $\mathsf{Meas}(\mathcal{X})$ with the following measurability tests: $\mathsf{M}^n(\mathsf{Meas}(\mathcal{X})) = \{e_U \mid U \in \Sigma_{\mathcal{X}}\}$ where $e_U(\vec{r}, \mu) = \mu(U)$ for $\vec{r} \in \mathbb{R}^n$ and $\mu \in \mathsf{Meas}(\mathcal{X})$.

Hence a path $\gamma : \mathbb{R}^n \to \mathsf{Meas}(\mathcal{X})$ is a bounded function $\mathbb{R}^n \to \mathsf{Meas}(\mathcal{X})$ such that the map $\vec{r} \mapsto \gamma(\vec{r})(U)$ is measurable for each $U \in \Sigma_{\mathcal{X}}$, that is, γ is a stochastic kernel from $\mathbb{R}^n$ to $\mathcal{X}$.

$P \Rightarrow Q$ is the cone of all stable and measurable functions $\mathcal{B}P \to Q$, and this cone is equiped with the following measurability tests:

- Given $\gamma : \mathbb{R}^n \to \mathcal{B}P$ a measurable path and $m \in \mathsf{M}^n(Q)$, we define $\gamma \triangleright m : \mathbb{R}^n \times (P \Rightarrow Q) \to \mathbb{R}^+$ by $(\gamma \triangleright m)(\vec{r}, f) = m(\vec{r}, f(\gamma(\vec{r})))$.
- $\mathsf{M}^n(P \Rightarrow Q)$ is the set of all these $\gamma \triangleright m$.

To prove the completeness property of $P \Rightarrow Q$, we need the Monotone Convergence Theorem, so we can consider only lubs of countable families. This is enough for fixpoints!

No surprise in the definition of $P \times Q$.

This defines a CCC $\mathbf{Cstab}_{\mathsf{m}}$ where we can interpret our target language and prove an adequacy theorem.

This solves indeed our integration problem.

$$\frac{\vdash M : \rho \qquad x : \rho \vdash N : \sigma}{\vdash \mathsf{let}\, x = M \,\mathsf{in}\, N : \sigma}$$

We take $\sigma = \rho$ to simplify a bit the notations.

- $\mu = [M] \in \mathcal{B}(\mathsf{Meas}(\mathbb{R}))$,
- $f = [N]_x : \mathcal{B}(\mathsf{Meas}(\mathbb{R})) \to \mathcal{B}(\mathsf{Meas}(\mathbb{R}))$ is stable and mesurable.

The map $\delta : \mathbb{R} \to \mathsf{Meas}(\mathbb{R})$ defined by $\delta(r) = \delta_r$ is a measurable path, because, for any $U \in \Sigma_{\mathbb{R}}$, the test $e_U \in \mathsf{M}^n(\mathsf{Meas}(\mathbb{R}))$ satisfies:

$$(e_U * \delta)(\vec{r}, r) = e_U(\vec{r}, \delta(r)) = \delta_r(U) = \begin{cases} 1 & \text{if } r \in U \\ 0 & \text{otherwise.} \end{cases}$$

so $e_U * \delta$ is measurable since $U \in \Sigma_{\mathbb{R}}$.

Because f is (stable) measurable, it follows that $f \circ \delta : \mathbb{R} \to \mathrm{Meas}(\mathbb{R})$ is a measurable path which means that for all $U \in \Sigma_{\mathbb{R}}$, the function

$$\begin{aligned} \mathbb{R} &\to \mathbb{R}^+ \\ r &\mapsto f(\delta_r)(U) \end{aligned}$$

is measurable.

So we can define $[\mathrm{let}\, x = M \,\mathrm{in}\, N] \in \mathrm{Meas}(\mathbb{R})$ as the measure ν given by

$$\nu(U) = \int_{\mathbb{R}} f(\delta_r)(U)\mu(dr)$$

Conclusion: a few questions

- Conjecture: this is an equationally fully abstract model of our "real probabilistic PCF" target language.
- We have a natural notion of measurable linear maps on cones. Does it give rise to a model of ILL? Probably. Of classical LL? Probably not, but can we find a class of measurable cones for which it is true, and which contains the cones $\mathsf{Meas}(\mathcal{X})$?

- Representation theorem for a sufficiently large class of cones (including $\mathsf{Meas}(\mathcal{X})$), typically replacing the webs $|X|$ of PCS with more structured spaces? Related to the previous question. This seems a crucial step in the development of an "intersection type systems" adapted to languages with continuous data types like $\mathbb{R}$.
- Probabilistic sequentiality, strong stability?
- Connection with other approaches (in particular: Staton quasi-Borel spaces, Keimel and Plotkin Kegelspitzen)?