Resource lambda-calculus: the differential viewpoint

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History (˜1985): Girard

Girard had differential intuitions when developping LL, as explained in the conclusion of his Linear Logic TCS paper.

He had the intuition of a link between *linear head reduction* (Krivine's machine) and differentiation/Taylor expansion.

But he had no concrete model nor syntactic reduction rules supporting this intuition.

History (~1990): Boudol

Process calculi feature a distinction between replicable and non replicable resources. For instance in CCS

$$\bar{a} \cdot P \mid a \cdot Q \mid R \leadsto P \mid Q \mid R$$
$$\bar{a} \cdot P \mid !a \cdot Q \mid R \leadsto P \mid !a \cdot Q \mid Q \mid R$$

They also feature an intrinsic non-determinism:

$$\bar{a} \cdot P \mid a \cdot Q_1 \mid a \cdot Q_2 \mid R \leadsto P \mid Q_1 \mid a \cdot Q_2 \mid R$$
$$\bar{a} \cdot P \mid a \cdot Q_1 \mid a \cdot Q_2 \mid R \leadsto P \mid a \cdot Q_1 \mid Q_2 \mid R$$

Boudol designed a lambda-calculus inspired by a translation of the lambda-calculus into the pi-calculus (an extension of CCS).

Terms are applied to multisets of arguments, each of them is

- either to be used only once
- or replicable.

$$\text{example: } \langle M\rangle[N_1, N_2, N_3, N_4^\infty, N_5^\infty]$$

This is possible thanks to an explicit substitution mechanism, which is essential to postpone substitution until the moment where the variable to be substituted occurs in head position.

Does not take non-determinism into account.

History (~2000): Kfoury

Very close to Boudol's ideas

- without replicable arguments,
- with moreover the idea that resource terms should be used as approximations of lambda-terms.

He restricts his attention to resource terms appearing as such approximations (we call them "uniform").

History (~2002): E. and Regnier

Differential LL stems from two concrete models of LL:

- Köthe sequence spaces, which can be seen as locally convex spaces
- Finiteness spaces, which can be seen as linearly topologized vector spaces.

In these models, the morphisms of the Kleisli category of the $!\_$ comonad admit differentials.

SYNTAX

Differential lambda-calculus

A differential simple term is

- a variable x
- or an abstraction $\lambda x\, s$ where x is a variable and s is a simple term
- or an ordinary application $(s)\, u$ where s is a simple terms and u is a term
- or a differential application $\mathrm{D}s \cdot t$ where s and t are simple terms.

Terms can be added (linearly combined with some coefficients).

Typing

$$\frac{\Gamma \vdash s : A \to B \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathrm{D}s \cdot t : A \to B}$$

Can be iterated

$$\frac{\Gamma \vdash s : A \to B \qquad \Gamma \vdash t_1 : A \quad \cdots \quad \Gamma \vdash t_n : A}{\Gamma \vdash \mathrm{D}^n s \cdot (t_1, \dots, t_n) : A \to B}$$

Congruence based on

$$\mathrm{D}^2 s \cdot (t_1, t_2) \sim \mathrm{D}^2 s \cdot (t_2, t_1)$$

required for confluence.

Linear substitution (differentiation)

$\frac{\partial s}{\partial x} \cdot t$ is s where exactly one occurrence of x has been replaced by t; this requires turning non-linear occurrences into linear ones (last case):

$$\frac{\partial y}{\partial x} \cdot t = \begin{cases} t & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial \lambda y\, s}{\partial x} \cdot t = \lambda y\, \frac{\partial s}{\partial x} \cdot t$$

$$\frac{\partial \mathrm{D} u \cdot s}{\partial x} \cdot t = \mathrm{D}\left(\frac{\partial u}{\partial x} \cdot t\right) \cdot s + \mathrm{D} u \cdot \left(\frac{\partial s}{\partial x} \cdot t\right)$$

$$\frac{\partial\,(s)\,u}{\partial x} \cdot t = \left(\frac{\partial s}{\partial x} \cdot t\right) u + \left(\mathrm{D} s \cdot \left(\frac{\partial u}{\partial x} \cdot t\right)\right) u$$

Reduction rules

Two reduction rules:

$$(\lambda x\, s)\, u \rightsquigarrow s\,[u/x]$$

$$\mathrm{D}(\lambda x\, s) \cdot t \rightsquigarrow \lambda x \left(\frac{\partial s}{\partial x} \cdot t \right)$$

The resource term $\langle M \rangle [N_1, N_2, N_3, N_4^{\infty}, N_5^{\infty}]$ would be written

$$\left(\mathrm{D}^3 M \cdot (N_1, N_2, N_3)\right) (N_4 + N_5)$$

(iterated differential application).

Differential linear logic

This extension of the lambda-calculus, admits a linear logic counterpart.

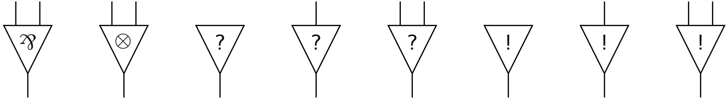
- No new connectives
- and three new exponential rules, dual to the standard rules of dereliction, weakening and contraction.

Introduces a new symmetry in LL.

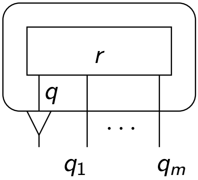
This system is easier to present using *interaction nets* (Lafont).

The basic ingredients of DiLL nets

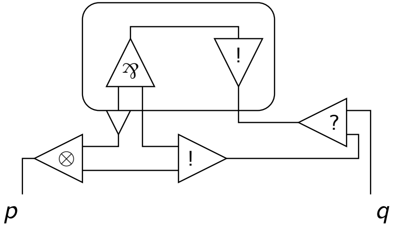
Cells:



Boxes:



An example of net:

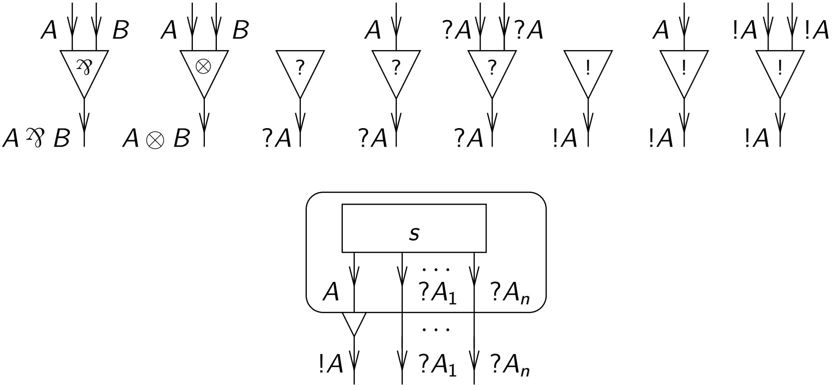


We also need to add nets which have the same interface (set of free ports) and there is a 0 net of each interface. Coefficients (in any semi-ring) can also be used.

All construction *but boxes* are linear wrt these constructions.

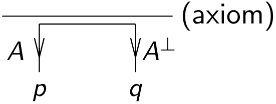
Typing

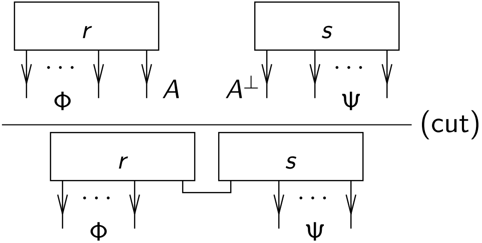
Types, which are LL formulae, can be associated with *oriented wires* (reversing the orientation of a wire turns its type A into $A^\perp$). The following typing constraints must be satisfied:



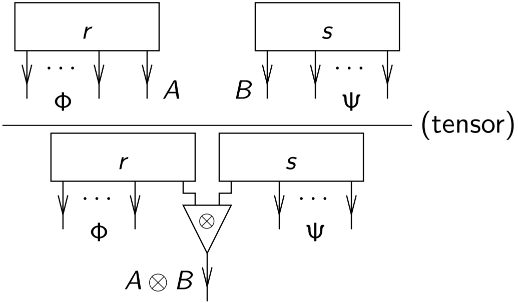
Logically correct nets

We give an inductive definition of logically correct nets. This also defines a sequent calculus for DiLL.





$$\frac{}{\vdash A, A^{\perp}} \qquad \frac{\vdash \Phi, A \qquad \vdash A^{\perp}, \Psi}{\vdash \Phi, \Psi}$$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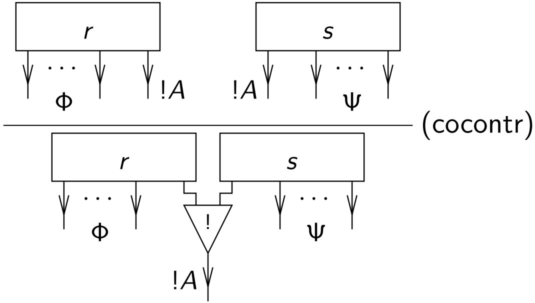
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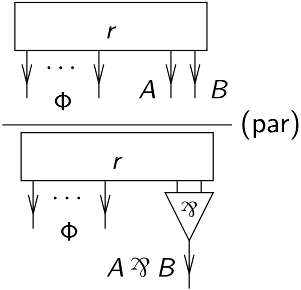
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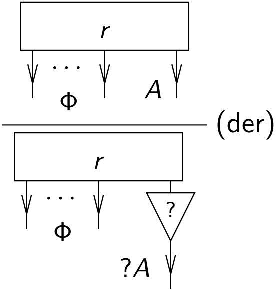
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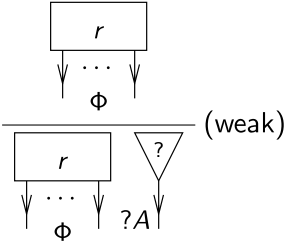
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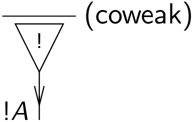
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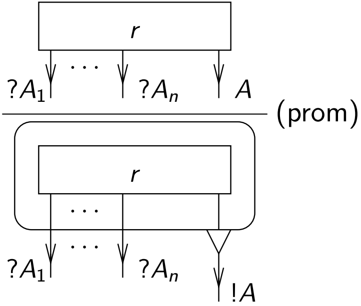
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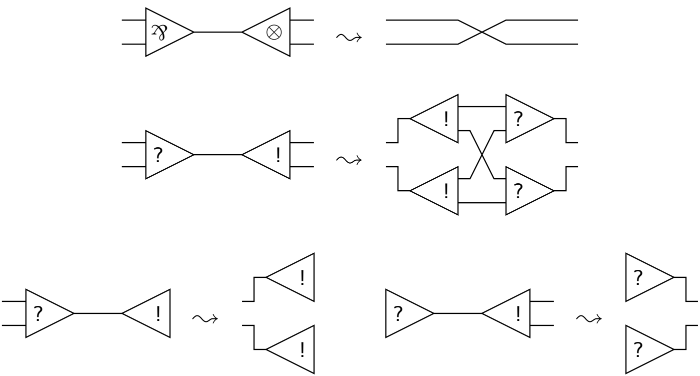
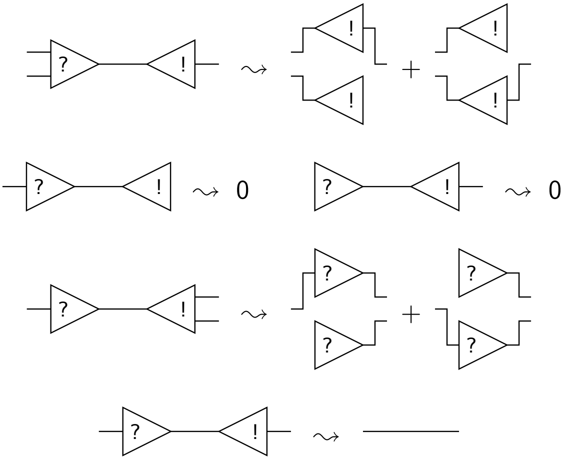
Reduction rules





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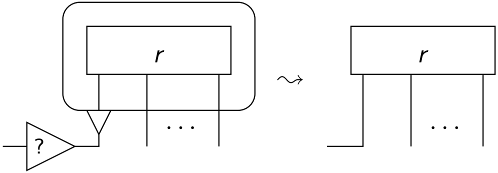
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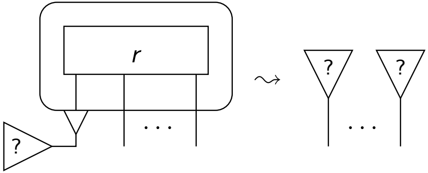
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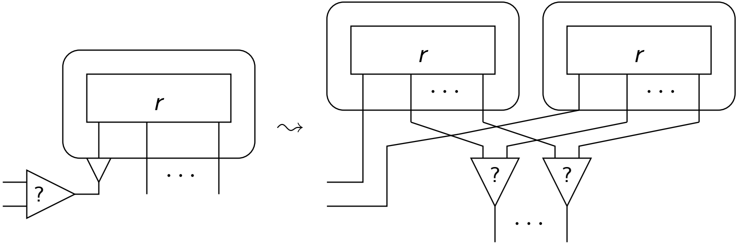
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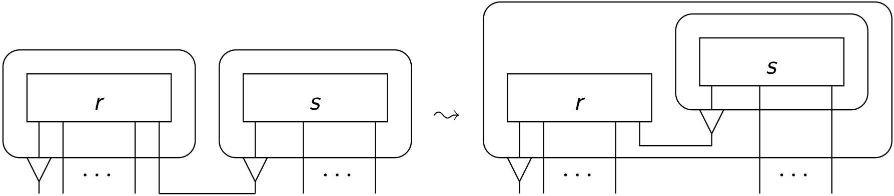
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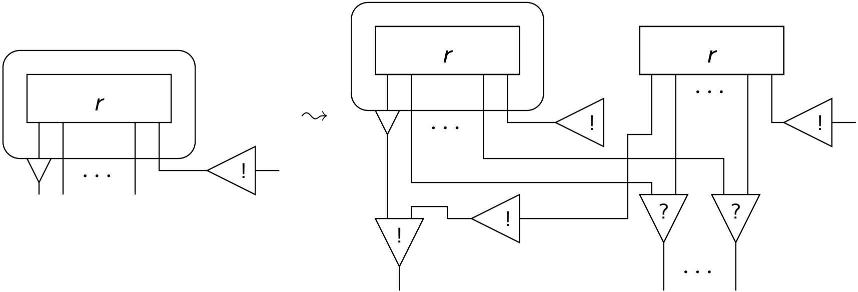

This is a formalization of the chain rule of Calculus:

$$(g \circ f)'(0) \cdot u = g'(f(0)) \cdot (f'(0) \cdot u)$$

This system has good properties: confluence (Tranquilli) and normalization (Pagani and Tranquilli, Gimenez).

SEMANTICS

Finiteness spaces

Differential LL comes from denotational models where objects can be seen as *topological vector spaces*.

The simplest of these models is based on *finiteness spaces* which can be seen as *linearly topological vector spaces* (ltvs), a notion introduced by Lefschetz in 1942.

Scalars are taken in a field **k** endowed with the *discrete topology*, so any field can be used: $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{F}_{27}$...

Definition of an ltvs

Let E be a $\mathbf{k}$-vector space.

A *linear topology* on E is a topology λ such that there is a filter $\mathcal{L}$ of linear subspaces of E with the following property:

a subset U of E is λ-open iff for any $x \in U$ there exists $V \in \mathcal{L}$ such that $x + V \subseteq U$. One says that such a filter $\mathcal{L}$ *generates* the topology $\mathcal{L}$.

A $\mathbf{k}$-ltvs is a $\mathbf{k}$-vector space equipped with a linear topology. Observe that E is Hausdorff iff $\bigcap \mathcal{L} = \{0\}$; from now on we assume always that this is the case.

Linear boundedness

Let E be an ltvs and let U be an open linear subspace of E.

Let $\pi_U : E \to E/U$ be the canonical projection. $\ker \pi_U = U$ is open, hence π_U is continuous if E/U has the discrete topology. So the quotient topology is the discrete topology.

A subspace B of E is linearly bounded if $\pi_U(B)$ is finite dimensional, for all linear open subspace or E. In other words, for any linear open subspace U, there is a finite dimensional subspace A of E such that $B \subseteq U + A$.

E is *locally linearly bounded* if it has a linear open subspace which is linearly bounded.

Completeness

Net in E: family $(x_d)_{d\in D}$ of elements of E indexed by a directed set D.

$(x_d)_{d\in D}$ converges to $x \in E$ if, for any neighborhood U of 0, there exists $d \in D$ such that $\forall e \in D\ e \geq d \Rightarrow x_e - x \in U$. Because E is Hausdorff, a net converges to at most one point.

A net $(x_d)_{d\in D}$ is Cauchy if, for any neighborhood U of 0, there exists $d \in D$ such that $\forall e, e' \in D\ e, e' \geq d \Rightarrow x_e - x_{e'} \in U$. This latter statement is equivalent to $\forall e \in D\ e \geq d \Rightarrow x_e - x_d \in U$.

One says that E is complete if any Cauchy net in E converges.

Hypocontinuous multilinear maps

Let $E_1, \dots, E_n$ and F be $\mathbf{k}$-ltvs's.

An n-multilinear function $f : E_1 \times \cdots \times E_n \to F$ is *hypocontinuous* if

- for any $i \in \{1, \dots, n\}$,
- any linear open subspace $V \subseteq F$
- and any linearly bounded subspaces $B_1 \subseteq E_1$,...,$B_{i-1} \subseteq E_{i-1}$, $B_{i+1} \subseteq E_{i+1}$,...,$B_n \subseteq E_n$

there exists an open linear subspace $U \subseteq E_i$ such that $f(B_1 \times \cdots \times B_{i-1} \times U \times B_{i+1} \times \cdots \times B_n) \subseteq V$.

For $n = 1$, this simply means that $f^{-1}(V)$ is open, for any linear open subspace of F (f is continuous).

Finiteness spaces

Let I be a set. Given $\mathcal{F} \subseteq \mathcal{P}(I)$, we define $\mathcal{F}^\perp \subseteq \mathcal{P}(I)$ by

$$\mathcal{F}^\perp = \{u' \subseteq I \mid \forall u \in \mathcal{F}\ u \cap u' \text{ is finite}\}\,.$$

We have $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{G}^\perp \subseteq \mathcal{F}^\perp$, $\mathcal{F} \subseteq \mathcal{F}^{\perp\perp}$ and therefore $\mathcal{F}^{\perp\perp\perp} = \mathcal{F}^\perp$.

A *finiteness space* is a pair $X = (|X|, \mathsf{F}(X))$ where $|X|$ (the *web*) is a set and $\mathsf{F}(X) \subseteq \mathcal{P}(|X|)$ (the *finitary sets*) satisfies $\mathsf{F}(X) = \mathsf{F}(X)^{\perp\perp}$. The following properties follow easily from the definition

- if $u \subseteq |X|$ is finite then $u \in \mathsf{F}(X)$
- if $u, v \in \mathsf{F}(X)$ then $u \cup v \in \mathsf{F}(X)$
- if $u \subseteq v \in \mathsf{F}(X)$, then $u \in \mathsf{F}(X)$.

The **k**-vector space $\mathbf{k}\langle X \rangle$ is the set of all families $x \in \mathbf{k}^{|X|}$ such that the set $\mathrm{Supp}(x) = \{a \in |X| \mid x_a \neq 0\}$ belongs to $\mathsf{F}(X)$.

Given $u' \in \mathsf{F}(X)^\perp$, we define a linear subspace of $\mathbf{k}\langle X\rangle$ by

$$\mathsf{V}(u') = \{x \in \mathbf{k}\langle X\rangle \mid \mathsf{Supp}(x) \cap u' = \emptyset\}\,.$$

Remark: $\forall u', v' \in \mathsf{F}(X)^\perp \quad u' \subseteq v' \Leftrightarrow \mathsf{V}(v') \subseteq \mathsf{V}(u')$.

Hence $\{\mathsf{V}(u') \mid u' \in \mathsf{F}(X)^\perp\}$ is a filter of linear subspaces of $\mathbf{k}\langle X\rangle$.

$\bigcap_{u' \in \mathsf{F}(X)^\perp} \mathsf{V}(u') = \{0\}$ (because $\forall a \in |X|\ \{a\} \in \mathsf{F}(X)^\perp$), so this filter defines an Hausdorff linear topology on $\mathbf{k}\langle X\rangle$: the *canonical topology* of $\mathbf{k}\langle X\rangle$.

Theorem

For any finiteness space X, the ltvs $\mathbf{k}\langle X\rangle$ is Cauchy-complete.

$(x(d))_{d\in D}$ a Cauchy net. Let $a \in |X|$. Take $u' = \{a\}$ in the definition of a Cauchy net: there exist $x_a \in \mathbf{k}$ and $d_a \in D$ such that $\forall e \geq d_a\ x(e)_a = x_a$.

One proves that $x = (x_a)_{a\in|X|} \in \mathbf{k}\langle X\rangle$

Frome Cauchy condition:
$\forall u' \in \mathsf{F}(X)^{\perp}\, \exists d \in D\, \forall e \geq d\, \forall a \in u' \quad x(e)_a = x_a$

$x \in \mathbf{k}\langle X\rangle$: let $u' \in \mathsf{F}(X)^{\perp}$, let $d \in D$ be such that $\forall e \geq d\, \forall a \in u'\ x(e)_a = x_a$. Then $\mathrm{Supp}(x) \cap u' = \mathrm{Supp}(x(d)) \cap u'$ is finite, so $\mathrm{Supp}(x) \in \mathsf{F}(X)^{\perp\perp} = \mathsf{F}(X)$, that is $x \in \mathbf{k}\langle X\rangle$.

Theorem

A finiteness space X is metrizable iff there exists a sequence $(u'_n)_{n\in\mathbb{N}}$ of elements of $\mathsf{F}(X)^\perp$ which is monotone $(n \leq m \Rightarrow u'_n \subseteq u'_m)$ and such that $\forall u' \in \mathsf{F}(X)^\perp\, \exists n \in \mathbb{N}\ u' \subseteq u'_n$.

If this is the case, define

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{if } x \neq y \text{ and } n \text{ is the least integer} \\ & \text{such that } u'_n \cap \mathrm{Supp}(x-y) \neq \emptyset\,. \end{cases}$$

it is an ultrametric distance: $d(x,z) \leq \max(d(x,y), d(y,z))$.

Theorem

The ltvs $\mathbf{k}\langle !?1\rangle$ *is not metrizable* ($!?1 = ((1 \Rightarrow 1) \Rightarrow 1)^{\perp}$)

Set $X = !?1$: $|X| = \mathcal{M}_{\mathrm{fin}}(\mathbb{N})$ and a subset u for $|X|$ belongs to $\mathrm{F}(X)$ iff $\exists n \in \mathbb{N}\ u \subseteq \mathcal{M}_{\mathrm{fin}}(\{0, \dots, n\})$.

Take a monotone sequence $(u'_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{F}(X)^{\perp}$.

Let $n \in \mathbb{N}$, we have $\{p[n] \mid p \in \mathbb{N}\} \in \mathcal{M}_{\mathrm{fin}}(X)$ and hence $u'_n \cap \{p[n] \mid p \in \mathbb{N}\}$ is finite. Therefore we can find a function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}\ f(n)[n] \notin u'_n$. Let $u' = \{f(n)[n] \mid n \in \mathbb{N}\}$. Then $u' \in \mathrm{F}(X)^{\perp}$ since, for any $n \in \mathbb{N}$, $u' \cap \mathcal{M}_{\mathrm{fin}}([0, n]) = \{f(i)[i] \mid i \in [0, n]\}$ is finite. But for all $n \in \mathbb{N}$ we have $f(n)[n] \in u' \setminus u'_n$ and so $u' \not\subseteq u'_n$.

Linearly bounded subspaces

Given $u \subseteq |X|$, let $\mathsf{D}(u) = \{x \in \mathbf{k}\langle X\rangle \mid \mathsf{Supp}(x) \subseteq u\}$. This is a linear subspace of $\mathbf{k}\langle X\rangle$.

Theorem

A linear subspace B of $\mathbf{k}\langle X\rangle$ is linearly bounded iff there exists $u \in \mathsf{F}(X)$ such that $B \subseteq \mathsf{D}(u)$.

As a consequence, $\mathbf{k}\langle X\rangle$ is locally linearly bounded iff

$$\exists u \in \mathsf{F}(X) \exists u' \in \mathsf{F}(X)^{\perp}\ u \cup u' = |X|\,.$$

Remark: if this property holds, $\mathbf{k}\langle X\rangle$ is metrizable. Take $u'_n = u' \cup \{a_1, \dots, a_n\}$ where (a_n) is any enumeration of $|X|$.

Tensor product

$|X \otimes Y| = |X| \times |Y|$ and

$$\mathsf{F}(X \otimes Y) = \{u \times v \mid u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y)\}^{\perp\perp}$$

and one can prove that $w \in \mathsf{F}(X \otimes Y)$ iff $\pi_1(w) \in \mathsf{F}(X)$ and $\pi_2(w) \in \mathsf{F}(Y)$.

Let $\tau : \mathbf{k}\langle X\rangle \times \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle X \otimes Y\rangle$ be defined by $\tau(x, y) = x \otimes y = (x_a y_b)_{(a,b)\in|X\otimes Y|}$. The map τ is bilinear hypocontinuous and classifies all bilinear hypocontinuous maps on $\mathbf{k}\langle X\rangle \times \mathbf{k}\langle Y\rangle$.

The category $\mathbf{Fin}(\mathbf{k})$ of finiteness spaces and linear continuous maps is symmetric monoidal with this tensor product.

It is an SMCC, with linear function space $X \multimap Y = (X \otimes Y^{\perp})^{\perp}$ where $X^{\perp}$ defined by $|X^{\perp}| = |X|$ and $\mathsf{F}(X^{\perp}) = \mathsf{F}(X)^{\perp}$.

Given a matrix $M \in \mathbf{k}\langle X \multimap Y\rangle$ and a vector $x \in \mathbf{k}\langle X\rangle$, we set

$$M\,x = \left(\sum_{a \in |X|} M_{a,b} x_a\right)_{b \in |Y|}$$

all these sums are *finite* and the resulting family of scalars is in $\mathbf{k}\langle X\rangle$. Such matrices represent linear continuous maps fully and faithfully.

$\mathbf{k}\langle X \multimap Y\rangle$ is linearly homeomorphic to the ltvs of continuous linear function $\mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$ endowed with the topology of "uniform convergence on all linearly bounded subspaces": generated by the subspaces

$$\{f \in \mathbf{k}\langle X \multimap Y\rangle \mid f(B) \subseteq V\}$$

for $B \subseteq \mathbf{k}\langle X\rangle$ linearly bounded subspace and $V \subseteq \mathbf{k}\langle Y\rangle$ linear open subspace.

The category $\mathbf{Fin}(\mathbf{k})$ is $*$-autonomous ($\perp = 1$ with $|1| = \{*\}$).

Additives: product

Given a countable family of finiteness spaces $(X_i)_{i\in I}$, we set $X = \&_{i\in I} X_i$ by $|X| = \bigcup_{i\in I} |X_i|$ and $w \subseteq |X|$ is in $\mathsf{F}(X)$ if

$$\forall i \in I \quad w_i = \{a \in |X_i| \mid (i,a) \in w\} \in \mathsf{F}(X_i)$$

Then $\mathbf{k}\langle \&_{i\in I} X_i\rangle$ is isomorphic to $\prod_{i\in I} \mathbf{k}\langle X_i\rangle$, with the product topology.

With obvious projections, it is the cartesian product of the family $(X_i)_{i\in I}$.

Additives: coproduct

$$\bigoplus_{i\in I} X_i = \left(\bigwith_{i\in I} {X_i}^{\perp}\right)^{\perp}$$

so that $\mathbf{k}\langle\bigoplus_{i\in I} X_i\rangle \subseteq \prod_{i\in I}\mathbf{k}\langle X_I\rangle$ is the space of all families $(x_i)_{i\in I}$ such that $x_i = 0$ for all but a finite numbers of i's, with the topology generated by *all* products $\prod_{i\in I} V_i$ (V_i linear open in $\mathbf{k}\langle X_i\rangle$): much finer than the product topology when I is infinite.

Example: take $I = \mathbb{N}$ and $X_i = 1$, so that $\mathbf{k}\langle X_i\rangle = \mathbf{k}$. Then $\mathbf{k}\langle\bigwith_{i\in I} X_i\rangle = \mathbf{k}^{\mathbb{N}}$ with the product topology and $\mathbf{k}\langle\bigoplus_{i\in I} X_i\rangle = \mathbf{k}^{(\mathbb{N})}$ (all $x\in\mathbf{k}^{\mathbb{N}}$ such that $x_i = 0$ for almost all i) with the discrete topology.

Let E and F be ltvs's.

A map $f : E \to F$ is *homogeneous polynomial of degree d* if there exists a hypocontinuous d-linear map $h : E^d \to F$ such that

$$\forall x \in E \quad f(x) = h(x, \ldots, x)$$

And f is polynomial if it is a finite sum of homogeneous polynomial maps.

Remark: A polynomial map $\mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$ is not necessarily continuous. It is continuous if $\mathbf{k}\langle X\rangle$ is locally linearly bounded (but we have seen that this property does not hold in general: it is already false for $\mathbf{k}\langle !?1\rangle$).

Define $\mathcal{A}(E, F)$ as the Cauchy-completion of the **k**-vector space of all polynomial maps $E \to F$ with the topology of uniform convergence on all linearly bounded subspaces: a basic open is

$$\{f \in \mathcal{A}(E, F) \mid f(B) \subseteq V\}$$

where $B \subseteq E$ is linearly bounded and $V \subseteq F$ is a linear open subspace.

Exponentials

Define $!X$ by $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ (finite multisets) and

$$\mathsf{F}(!X) = \{\mathcal{M}_{\mathrm{fin}}(u) \mid u \in \mathsf{F}(X)\}^{\perp\perp}.$$

Just as for the tensor product, one can prove that $U \subseteq |!X|$ belongs to $\mathsf{F}(!X)$ iff $\bigcup_{m \in U} \mathsf{Supp}(m) \in \mathsf{F}(X)$.

Theorem

$\mathbf{k}\langle !X \multimap Y\rangle$ *is linearly homeomorphic to* $\mathcal{A}(\mathbf{k}\langle X\rangle, \mathbf{k}\langle Y\rangle)$ *(with the topology of uniform convergence on all linearly bounded subspaces).*

$!\_$ is a monoidal comonad on $\mathbf{Fin}(\mathbf{k})$ and we have defined in that way a model off LL, a (new-)Seely category.

In particular we have the following natural transormations

- dereliction $\mathsf{d}_X : !X \to X$ with $(\mathsf{d}_X)_{m,a} = \delta_{m,[a]}$
- weakening $\mathsf{w}_X : !X \to 1$ with $(\mathsf{w}_X)_{m,*} = \delta_{m,[]}$
- contraction $\mathsf{c}_X : !X \to !X \otimes !X$ with $(\mathsf{c}_X)_{m,(l,r)} = \delta_{m,l+r}$

We also have the following natural transformations

- codereliction $\overline{\mathrm{d}}_X : X \to {!X}$ with $(\mathrm{d}_X)_{a,m} = \delta_{[a],m}$
- coweakening $\overline{\mathrm{w}}_X : 1 \to {!X}$ with $(\overline{\mathrm{w}}_X)_{*,m} = \delta_{[],m}$
- cocontraction $\overline{\mathrm{c}}_X : {!X} \otimes {!X} \to {!X}$ with $(\overline{\mathrm{c}}_X)_{(l,r),m} = \delta_{l+r,m} \binom{m}{l} = \delta_{l+r,m} \binom{m}{r}$

where

$$\binom{m}{l} = \prod_{a \in |X|} \binom{m(a)}{l(a)}$$

(almost all the factors in that product are equal to 1).

A model of finite differential LL (ie: differential LL without promotion) is a $*$-autonomous category $\mathcal{C}$ which is additive (hom-sets have an addition and a 0 and everything is linear wrt to this structure) and has an operation

$X \mapsto (!X, \mathsf{c}_X, \mathsf{w}_X, \mathsf{d}_X, \overline{\mathsf{c}}_X, \overline{\mathsf{w}}_X, \overline{\mathsf{d}}_X)$ where $!X$ is an object of $\mathcal{C}$, $(!X, \mathsf{c}_X, \mathsf{w}_X, \overline{\mathsf{c}}_X, \overline{\mathsf{w}}_X)$ is a bicommutative bialgebra and a few more diagrams commute, which correspond to the reduction rules of DiLL.

Derivatives and anti-derivatives

We have morphisms $\partial_X : !X \to !X \otimes X$ and $\overline{\partial}_X : !X \otimes X \to !X$ defined by $\partial_X = (\mathsf{d}_X \otimes !X) \circ \mathsf{c}_X$ and $\overline{\partial}_X = \bar{\mathsf{c}}_X \circ (\bar{\mathsf{d}}_X \otimes !X)$.

Let $f : !X \to Y$, to be considered as an "intuitionistic" map $X \to Y$ (in finiteness spaces, it is an analytic function $\mathbf{k}\langle X \rangle \to \mathbf{k}\langle Y \rangle$).

Then the morphism $f \circ \overline{\partial}_X : !X \otimes X \to Y$ should be considered as the derivative (differential, Jacobian...) of f: thanks to monoidal closeness it can be seen as an intuitionistic morphism $X \to (X \multimap Y)$.

We say that $\mathcal{C}$ *has anti-derivatives* if

$$J_X = \mathsf{Id}_{!X} + (\overline{\partial}_X \circ \partial_X) : !X \to !X$$

is an iso and then we set $I_X = {J_X}^{-1}$.

Remark: in **Fin**(**k**), one has

$$(\mathsf{Id}_{!X} + (\overline{\partial}_X \circ \partial_X))_{I,r} = (1 + \#I)\delta_{I,r}$$

and so **Fin**(**k**) has anti-derivatives as soon as **k** is of characteristic 0.

Intuition

Let $f : !X \to Y$, to be considered as an "intuitionistic" map $X \to Y$ (in finiteness spaces, it is an analytic function $\mathbf{k}\langle X\rangle \to \mathbf{k}\langle Y\rangle$).

Then the map $f \circ I_X : !X \to Y$ represents the intuitionistic map $g : X \to Y$ defined by

$$g(x) = \int_0^1 f(tx)dt\,.$$

A categorical Poincaré's Lemma

Given $f : !X \otimes X \to Y$, when is it the case that f is the differential of a morphism $g : !X \to Y$?

A necessary condition is that $f \circ (\overline{\partial}_X \otimes X) : !X \otimes X \otimes X \to Y$ be symmetric in its two last parameters:

$$f \circ (\overline{\partial}_X \otimes X) \circ \sigma_{2,3} = f \circ (\overline{\partial}_X \otimes X)$$

Theorem

If $\mathcal{C}$ has anti-derivatives, this condition is sufficient for f to be a differential.

Take $g = f \circ (I_X \otimes \mathrm{Id}_X) \circ \partial_X : !X \to Y$, that is, intuitively, $g(x) = \int_0^1 (f(tx) \cdot x) dt$.

Conclusion

We start from denotational models to develop new syntactic objects. The problem is then to understand their operational relevance and possible applications. What do we know about DiLL?

- It has good internal properties (confluence and normalization).
- It simplifies and clarifies resource lambda-calculi.
- Taylor expansion in the differential lambda-calculus is related to Krivine's machine, as suggested by Girard (E., Regnier).
- One can represent concurrent processes (solos, π) in DiLL interaction nets (E., Laurent).

What about anti-derivatives? Can we solve some kind of differential equations? Could it be a new way to specify programs/algorithms? Work in progress...