Dictoses

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Introduction

This paper presents a notion of categories which are models of the calculus of constructions. These categories are in fact equivalent to the "locally cartesian closed categories with a internal small-complete category" that Hyland presented in [5]. However, our point of view is quite different, since we shall not make reference to the completeness of an internal category, but get it as a corollary of our axioms. In fact, the notion of internal completeness is very subtle.

So what we present here are locally cartesian closed categories with an object of propositions and an impredicative quantification principle, and furthermore a reflection principle of types over propositions. Our claim is that this notion is a natural generalisation of toposes, where proofs are first-class citizens, in contrast with toposes where proofs are not interpreted. That's why we called these categories "dictoses".

In this short paper, we just present the notion, give some very basic facts about dictoses (preservation by localisation, internal completeness of the internal category of propositions), and give the example of ω-sets (for reflection only, since the remainder is well known). We don't claim that this is a very original piece of work, but we only hope to provide another sight on objects which have been studied by category theorists for years.

1 Dictoses

The object of this section is to introduce a categorical notion intended to generalize elementary toposes, taking proofs into acount. Actually, it is well known that toposes are higher order intuitionistic theories of types (see [8,9]), with equality. But in these theories, the notion of truth is quite poor, with respect to the so-called "Heyting semantics" of intuitionistic logic. In a topos, a proposition is true if it is equal to "true" constant. So the proof used in order to establish the truth of this proposition isn't taken in any way into account in the semantics, and in a logical (and computer scientist, cf. Curry-Howard paradigm of programs-as-proofs) point of view, it seems to be a serious drawback.

1.1 Higher order locally cartesian closed categories

We recall first that a locally cartesian closed category is a category with finites limits (thus it has a terminal object) such that all slice categories are cartesian closed. If $\mathbf{C}$ is such a category, the last condition means that for any morphism $f : X \to Y$, the pull-back functor $f^* : \mathbf{C}/Y \to \mathbf{C}/X$ has a right adjoint $\Pi f : \mathbf{C}/X \to \mathbf{C}/Y$, and that the Beck-Chevalley condition is satisfied. Roughly speaking, this means that for any pull-back diagram in $\mathbf{C}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

the following is a (pseudo-)commutative diagram of functors

$$\begin{array}{ccc} \mathbf{C}/X & \xrightarrow{\Pi f} & \mathbf{C}/Y \\ {\scriptstyle g^*}\uparrow & & \uparrow{\scriptstyle h^*} \\ \mathbf{C}/X' & \xrightarrow{\Pi f'} & \mathbf{C}/Y' \end{array}$$

(for more details, see for instance [6] where a perfectly clean treatment is done of this property).

About LCCC's, we shall need the following property:

Lemma 1 *Let* $\mathbf{C}$ *be a LCCC, and let* X *be an object of it. Then* $\mathbf{C}/X$ *is locally cartesian closed as well, and the functor*

$$\begin{array}{rcl} X^* : \mathbf{C} & \rightarrow & \mathbf{C}/X \\ Y & \mapsto & (\pi_2 : Y \times X \rightarrow X) \\ (f : Y \rightarrow Z) & \mapsto & f \times \mathrm{Id}_X \end{array}$$

is left exact and preserves local exponentials.

If we consider a LCCC as an intuitionistic theory of types in the spirit of Martin-Löf's first order system of [10], this result means essentially that we can do the same kind of reasonning, assuming a constant hypothesis X.

Now we introduce the intermediate notion of higher order locally cartesian closed category, which doesn't seem to be rich enough to provide interesting results. A LCCC $\mathcal{E}$ is a HLCCC if it has a special morphism $T : \Omega' \rightarrow \Omega$ such that, for any $\varphi : X \rightarrow \Omega$ and any $f : X \rightarrow Y$ there exists a morphism $\forall f\,(\varphi) : Y \rightarrow \Omega$ such that $\widetilde{\forall f\,(\varphi)} \cong \Pi f\,(\tilde{\varphi}) : Y \rightarrow \Omega$ where $\tilde{\varphi}$ is a shorthand for $\varphi^*(T)$. This last requirement will sometimes be called "axiom of impredicative product".

A category of this kind is a model of the Calculus of Constructions (see [3] for more details), with a strong sum and an equality type in the spirit of [10].

1.2 The internal full subcategory of propositions

We recall a classical construction which can be carried out in any locally cartesian closed category $\mathbf{C}$. In such a category, let $u : U \rightarrow C_0$ be any morphism. We consider the projections $\pi_1, \pi_2 : C_0 \times C_0 \rightarrow C_0$ which define, by pulling back and local exponential transpose, a morphism

$$d = {\pi_0}^*(u) \Rightarrow {\pi_1}^*(u) : C_1 \rightarrow C_0 \times C_0$$

of which both components $\mathrm{Cod} = \pi_2 \circ d$ and $\mathrm{Dom} = \pi_1 \circ d$ define the codomain and domain morphisms respectively of an internal category in $\mathbf{C}$. The "identity" morphism $C_0 \rightarrow C_1$ is obtained by internalizing the identity using exponential, and composition is defined in the same way, without any problems (see for instance [8]). This internal category will be noted $\mathrm{Full}_{\mathbf{C}}(u)$ and called the internal full subcategory generated by u.

We know furthermore that to any internal category C in the locally cartesian closed category $\mathbf{C}$, it is possible to associate a split fibration

$$p_C : [C] \rightarrow \mathbf{C}$$

(see for instance [7]). Let us recall this construction, which internalizes the "category of families of a (small) category indexed by sets". The category $[C]$ has objects the morphisms $x : X \to C_0$ ("families of objects of C indexed by X"). A morphisme $(x : X \to C_0) \to (y : Y \to C_0)$ in $[C]$ is given by a morphism $f : X \to Y$ ("function between sets of indexes") and a $\varphi : X \to C_1$ ("family of morphisms indexed by X"), morphisms in $\mathbf{C}$ such that both following diagrams be commutative

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & C_1 \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle d_0} \\ C_0 & = & C_0 \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\varphi} & C_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle d_1} \\ Y & \xrightarrow{y} & C_0 \end{array}$$

Composition is defined as follows. Let $(f,\varphi) : (x : X \to C_0) \to (y : Y \to C_0)$ and $(g,\psi) : (y : Y \to C_0) \to (z : Z \to C_0)$ be morphisms in $\mathbf{C}$. We set

$$(g,\psi) \circ (f,\varphi) = (g \circ f, c(\psi \circ f, \varphi))$$

where c is the internal composition of the internal category C. The split fibration $p_C : [C] \to \mathbf{C}$ is then the functor

$$\begin{array}{rcl} p_C : [C] & \to & \mathbf{C} \\ (x : X \to C_0) & \mapsto & X \\ (f,\varphi) & \mapsto & f \end{array}$$

When furthermore C is an internal full subcategory in $\mathbf{C}$ built on a morphism $u : U \to C_0$, (with the notation previously introduced, $C = \mathrm{Full}_{\mathbf{C}}(u)$), we have a cartesian full and faithful functor

$$\tau_{u,\mathbf{B}} : p_{\mathrm{Full}_{\mathbf{C}}(u)} \to \mathrm{Cod}$$

where Cod : $\mathbf{C}^2 \to \mathbf{C}$ is the codomain functor which is well known to be a fibration. This corresponds to the fact that C is a full subcategory of $\mathbf{C}$.

In a HLCCC $\mathcal{E}$, we shall consider this construction for the morphism T. The internal full subcategory $\mathrm{Full}_{\mathcal{E}}(T)$ will be called "the category of proposition". We shall note $\mathcal{P}$ the presheaf associated to the split fibration $p_{\mathrm{Full}_{\mathbf{C}}(u)}$. (Thus for any $X \in \mathcal{E}$, $\mathcal{P}(X)$ has as objects the morphisms $X \to \Omega$, and a morphism $\varphi \to \psi$, when $\varphi, \psi : X \to \Omega$, is simply a morphism $\widetilde{\varphi} \to \widetilde{\psi}$ in $\mathcal{E}/X$.)

1.3 A reflection principle

We know that in a topos there is a canonical factorization of any morphism into an epi followed by a mono (ie. the type associated with a predicate). We require a similar property in any dictos, and this will complete our axiomatization. This will allow to interpret "big sums" of propositions over types (which are not "strong sums", since this would be paradoxal, as it is well known), and the factorization we require here is deeply connected to the one described in [6], section 2.12. We use the same kind of reflection.

We give first the general definition of reflections:

Definition 1 *Let* $\mathbf{B}$ *and* $\mathbf{D}$ *be categories. We say that two functors* $I : \mathbf{B} \to \mathbf{D}$ *and* $R : \mathbf{D} \to \mathbf{B}$ *define a reflection if* I *is right adjoint to* R*, and if the counit of this adjunction is an iso.*

This notion must be understood as a kind of generalization of the "embedding-retractions" which are well-known in domain theory. I is a kind of embedding of $\mathbf{B}$ in $\mathbf{D}$ (actually, it is full and faithful since the co-unit is an iso). Sometimes, this functor will be called the "embedding part" of the reflection.

This notion admits an obvious generalization to the case where **B** and **D** are fibrations instead of simple categories. (Bénabou has shown that all what can be done with categories can as well be done with fibrations, see [1].)

Definition 2 *We say that a HLCCC $\mathcal{E}$ is reflexif (or that it is a dictos) if the cartesian functor $\tau_{u,\mathcal{E}}$ is the embedding part of a reflection between the fbrations $p_{\mathrm{Full}_{\mathcal{E}}(T)}$ and $\mathrm{Cod}_{\mathcal{E}}$.*

This means that for any object X in $\mathcal{E}$, the functor

$$\begin{array}{rcl} \mathbf{T}_X : \mathcal{P}(X) & \to & \mathcal{E}/X \\ \varphi & \mapsto & \tilde{\varphi} \\ (f:\varphi \to \psi) & \mapsto & f \end{array}$$

is the embedding part of a reflection. We shall note R_X the "retraction" part of this reflection. Furthermore, the requirement that the functor R be cartesian means that a kind of "Beck-Chevalley" condition be satisfied. This means that $f : X \to X'$ being a given morphism in $\mathcal{E}$, the following diagram must (pseudo-)commute:

$$\begin{array}{ccc} \mathcal{P}(X) & \xleftarrow{\mathcal{P}(f)} & \mathcal{P}(X') \\ {\scriptstyle \mathrm{R}_X}\uparrow & & {\scriptstyle \mathrm{R}_{X'}}\uparrow \\ \mathcal{E}/X & \xleftarrow{f^*} & \mathcal{E}/X' \end{array}$$

This reflection principle (assuming, to simplify a bit, that the counit is the identity, instead of being a more general iso) gives, for any $f : X \to Y$, morphism of $\mathcal{E}$, a unit morphism

$$\mu_f : X \to \widetilde{X}_{\mathrm{R}_X(f)}$$

such that

$$f = \widetilde{\mathrm{R}_X(f)} \circ \mu_f \quad \text{and} \quad \mathrm{R}_X(\mu_f) = \mathrm{Id}\ .$$

In that last equality, it is the morphism part of functor R_X which is applied. Furthermore, is $\varphi : X \to \Omega$ is a "predicate", we also have that $\mu_{\tilde{\varphi}} = \mathrm{Id}$.

We now explain why this reflection principle is a generalisation of the epi-mono factorization which holds in a elementary topos. For this, we need a categorical notion of orthogonality.

Definition 3 *Let $\mathcal{R}$ be a class of morphisms in a category* **C**. *Let $f : X \to Y$ be a morphism of* **C**. *We say that f if orthogonal to the class $\mathcal{R}$ if for any commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ U & \xrightarrow{r} & V \end{array}$$

where $r \in \mathcal{R}$, there exists a unique morphisms $g : Y \to U$ such that

$$g \circ f = a \quad \textit{and} \quad r \circ g = b$$

This definition implies readily that any decomposition of a morphism u of **C** in $r \circ f$ where $r \in \mathcal{R}$ and f orthogonal to $\mathcal{R}$ is unique (up to isomorphism). It is clear on the other hand that, in a topos, any epi is orthogonal to the class of monos (since monos are split). We show that the situation is similar in a dictos. First we make an obvious remark, which is nothing else but a rephrasement of the reflection condition:

Lemma 2 *For any $f : X \to Y$ in $\mathcal{E}$ and any decomposition of f:*

$$f = \widetilde{\psi} \circ h$$

there exists a unique morphism $v : \widetilde{Y}_{\mathrm{R}(f)} \to \widetilde{Y}_\psi$ such that

$$\widetilde{\psi} \circ v = \widetilde{\mathrm{R}(f)} \quad \textit{and} \quad v \circ \mu_f = h$$

Now we can state that

Proposition 1 *For any $f : X \to Y$ in $\mathcal{E}$, μ_f is orthogonal to the morphisms of the form $\widetilde{\varphi}$ where $\varphi : U \to \Omega$ is any morphism.*

Proof: Let $f : X \to Y$ and $\varphi : U \to \Omega$ be morphisms and assume given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_f} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \widetilde{U}_\varphi & \xrightarrow{\widetilde{\varphi}} & U \end{array}$$

where we have set $V = \widetilde{Y}_{\mathrm{R}(f)}$. We consider then the pullback built on $\widetilde{\varphi}$ and b:

$$\begin{array}{ccc} \widetilde{V}_\psi & \xrightarrow{\widetilde{\psi}} & V \\ {\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle b} \\ \widetilde{U}_\varphi & \xrightarrow{\widetilde{\varphi}} & U \end{array}$$

where we have set $\psi = \varphi \circ b$. Let $h : X \to \widetilde{V}_\psi$ be the unique morphism such that

$$b' \circ h = a \quad \text{et} \quad \widetilde{\psi} \circ h = \mu_f \ .$$

Since $\mathrm{R}(\mu_f) = \mathrm{Id}$ (because the unit of the adjunction is the identity), there exists (cf. previous lemma) a unique morphism $v : V \to \widetilde{V}_\psi$ such that

$$\widetilde{\psi} \circ v = \mathrm{Id} \quad \text{and} \quad v \circ \mu_f = h$$

Then let $u = b' \circ v : V \to \widetilde{U}_\varphi$. It is clear that this morphism renders both required triangles commutative. We conclude by verifying that it is alone satisfying these conditions. Actually, let u be such a morphism, and call v the unique morphism $V \to \widetilde{V}_\psi$ such that

$$\widetilde{\psi} \circ v = \mathrm{Id} \quad \text{and} \quad b' \circ v = u$$

Let $h' = v \circ \mu_f$. To conclude, it suffices to see that $h' = h$. But this results from the following equalities:

$$\widetilde{\psi} \circ h' = \mu_f \quad \text{and} \quad b' \circ h' = a$$

■

1.4 Logical functors between dictoses

Let us define a notion of morphism between dictoses.

Definition 4 *Let $\mathcal{E}$ and $\mathcal{F}$ be two dictoses. A logical functor between $\mathcal{E}$ and $\mathcal{F}$ is a functor $F : \mathcal{E} \to \mathcal{F}$ which is left exact, preserves the local exponentials, the generic morphisms T and the reflection.*

This is of course inspired by the homonym notion in topos theory.

Remark: It is not clear yet whether or not there exists a more relevant notion of morphisms between dictoses, like the geometric morphisms for toposes. This question should be clarified by the study of examples and by trying to generalize the construction of sheaves topose over a topological space.

We have in mind to give an important example of logical functor. In fact, we shall prove

Proposition 2 *Let $\mathcal{E}$ be a dictos. Let $X \in \mathcal{E}$ be an object in this dictos. Then the category $\mathcal{E}/X$ is a dictos. Furthermore, for any $u : X \to X'$, the functor $u^* : \mathcal{E}/X' \to \mathcal{E}/X$ is logical.*

Proof: First, let us prove that $\mathcal{E}/X$ is a dictos. We already know that it is locally cartesian closed by lemma 1. The object Ω_X of propositions in $\mathcal{E}/X$ is taken to be $\pi_2 : \Omega \times X \to X$, and in the same way $\Omega'_X = \pi_2 : \Omega \times X \to X$ defines the local object of proofs. As generic morphism $T_X : \Omega'_X \to \Omega_X$ we take of course $T \times X$ (where X stands for Id_X).

Now let $\varphi : f \to \Omega_X$ be a "predicate" in $\mathcal{E}/X$. Let $h : f \to g$ be a morphism, and let Y (resp. Z) be the codomain of h (resp. its domain). We define $\forall X\,(h)\,\varphi$ by

$$\forall X\,(h)\,\varphi = \langle \forall f\,(\pi_1 \varphi)\,,\, g\rangle : g \to \Omega_X$$

It is easy to check that it satisfies the required condition. For this just remark that, for any $\varphi : f \to \Omega_X$, we have

$$\widetilde{\varphi}^X = \widetilde{\pi_1 \varphi}$$

where $\hat{\varphi}^X$ is the type associated to φ in $\mathcal{E}/X$.

It remains to define a local version of the reflection which we shall note R^X. For any $h : f \to g$ morphism in $\mathcal{E}/X$ we set simply

$$\mathrm{R}^X_g(h) = \langle \mathrm{R}_Z(h)\,,\, g\rangle$$

where Z is the domain of g. And we let to the reader the straightforward verification that it defines the required reflection. Thus $\mathcal{E}/X$ is a dictos.

Let us proof the second part of the proposition. Let $u : X \to X'$ be a morphism. We already know by lemma 1 that the functor u^* is left exact and preserves local exponentials. The fact that it preserves $T : \Omega' \to \Omega$ results from the fact that, for any object $A \in \mathcal{E}$, the following diagram is a pullback:

$$\begin{array}{ccc} A \times X & \xrightarrow{\mathrm{Id}_A \times u} & A \times X' \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X & \xrightarrow{u} & X' \end{array}$$

It just remains to check that u^* preserves reflections. Let $h' : f' \to g'$ be an arrow in $\mathcal{E}/X'$ (where $f' : Y' \to X'$ and $g' : Z' \to X'$ are local objects). Then we have set

$$\mathrm{R}^{X'}_{g'}(h') = \langle \mathrm{R}_{Z'}(h')\,,\, g'\rangle$$

One easily checks that

$$u^*\left(\langle \mathrm{R}_{Z'}(h')\,,\, g'\rangle\right) = \langle \mathrm{R}_{Z'}(h') \circ u_{g'}\,,\, u^* g'\rangle\ .$$

On the other hand we get

$$\begin{aligned} \mathrm{R}^X_{g'}\left(u^*(h')\right) &= \langle \mathrm{R}_Z(u^*(h'))\,,\, u^* g'\rangle \\ &= \langle \mathrm{R}_Z(u_{g'}{}^* h')\,,\, u^* g'\rangle \\ &= \langle \mathrm{R}_{Z'}(h') \circ u_{g'}\,,\, u^* g'\rangle \end{aligned}$$

and this completes the proof. ∎

This proposition means that the notion of dictos is preserved by localization, and this has a clear logical significance: if we see a dictos as a logical theory, the theory we get by fixing a given hypothesis is again a theory of the same kind.

2 The internal category of propositions

This section is devoted to the announced study of the properties of the internal full subcategory of propositions $\mathrm{Full}_{\mathcal{E}}(T)$. We shall prove that it is an internal cartesian closed category which is internally small complete (more precisely, we shall only prove that it has all internal small products).

2.1 General results about reflections

Let $I : \mathbf{B} \to \mathbf{D}$ be the embedding part of a reflection between the categories $\mathbf{B}$ and $\mathbf{D}$, and let R be the retraction part of this reflection. Then

Lemma 3 *Let $F : \mathbf{J} \to \mathbf{C}$ be a functor such that $I \circ F$ admits a limit in $\mathbf{D}$. Then the functor F admits a limit in $\mathbf{B}$. More precisely we have, up to isomorphism*

$$\underleftarrow{\mathrm{Lim}}\, F = R(\underleftarrow{\mathrm{Lim}}\, IF) \ .$$

As a consequence, there exist in $\mathbf{B}$ all the limits that exist in $\mathbf{D}$.

We don't recall the proof of this well known result.

We shall also need the following

Lemma 4 *If $\mathbf{D}$ is cartesian closed (let $\Rightarrow_{\mathbf{D}}$ be its exponential), and if there exists a functor $\Rightarrow_{\mathbf{B}}: \mathbf{B}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{B}$ such that the following commutes (up to natural isomorphism)*

$$\begin{array}{ccc} \mathbf{B}^{\mathrm{op}} \times \mathbf{B} & \xrightarrow{\Rightarrow_{\mathbf{B}}} & \mathbf{B} \\ {\scriptstyle I^{\mathrm{op}} \times I}\downarrow & & \downarrow{\scriptstyle I} \\ \mathbf{D}^{\mathrm{op}} \times \mathbf{D} & \xrightarrow{\Rightarrow_{\mathbf{D}}} & \mathbf{D} \end{array}$$

then $\mathbf{B}$ is cartesian closed.

Proof: We already know that it is cartesian by the preceeding lemma. We just have to show that $\Rightarrow_{\mathbf{B}}$ defines an exponential on $\mathbf{B}$. Let X, Y, Z be three objects of $\mathbf{B}$. Then

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{B}}(X \times Y\ ,\ Z) & = & \mathrm{Hom}_{\mathbf{B}}(R(I(X) \times I(Y))\ ,\ Z) \\ & \cong & \mathrm{Hom}_{\mathbf{D}}(I(X) \times I(Y)\ ,\ I(Z)) \\ & \cong & \mathrm{Hom}_{\mathbf{D}}(I(X)\ ,\ I(Y) \Rightarrow I(Z)) \\ & \cong & \mathrm{Hom}_{\mathbf{D}}(I(X)\ ,\ I(Y \Rightarrow Z)) \\ & \cong & \mathrm{Hom}_{\mathbf{B}}(X\ ,\ Y \Rightarrow Z) \end{array}$$

and we conclude, since all these isomorphisms are natural. ∎

As a corollary we get

Lemma 5 *For any $X \in \mathcal{E}$, the category $\mathcal{P}(X)$ is cartesian closed.*

Proof: In view of the two preceeding lemmas, we just have to build a functor $\Rightarrow_{\mathcal{P}(X)}: \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$ such that the following diagram commute:

$$\begin{array}{ccc} \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) & \xrightarrow{\Rightarrow_{\mathcal{P}(X)}} & \mathcal{P}(X) \\ {\scriptstyle I^{\mathrm{op}} \times I}\downarrow & & \downarrow{\scriptstyle I} \\ \mathcal{E}/X^{\mathrm{op}} \times \mathcal{E}/X & \xrightarrow{\Rightarrow_{\mathcal{E}/X}} & \mathcal{E}/X \end{array} \qquad (1)$$

where $I : \mathcal{P}(X) \to \mathcal{E}/X$ is the obvious full embedding. We just have to define this functor $\Rightarrow_{\mathcal{P}(X)}$ on objects, since on arrow it is completely defined by the previous diagram. If $\varphi, \psi : X \to \Omega$ are objects of $\mathcal{P}(X)$, we take

$$\varphi \Rightarrow_{\mathcal{P}(X)} \psi = \forall\varphi\,(\psi\widetilde{\varphi})$$

and the axiom of impredicative product insures that the diagram 1 commutes. ∎

2.2 The internal structure of $\mathrm{Full}_{\mathcal{E}}(T)$

We prove now

Proposition 3 *The internal full subcategory* $\mathrm{Full}_{\mathcal{E}}(T)$ *is internally cartesian closed and has all internal small products.*

Proof: We use the caracterisation of internal full subcategories with internal finite and small products, and with internal exponential given in [12]. We first show that it has internally a terminal object. For this, we just have to find an arrow $[1] : 1 \to \Omega$ such there is a pullback of the form

$$\begin{array}{ccc} 1 & \longrightarrow & \Omega' \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle T} \\ 1 & \xrightarrow{[1]} & \Omega \end{array}$$

We take $[1] = \mathrm{R}_1(1)$. We just have to prove that the object $\widetilde{1}_1$ is terminal in $\mathcal{E}$. We know by lemma 3 that $\mathrm{R}_1(1)$ is terminal in $\mathcal{P}(1)$. But for any object $X \in \mathcal{E}$ we have

$$\mathrm{Hom}_{\mathcal{E}}\left(X\,,\,\widetilde{1}_1\right) \cong \mathrm{Hom}_{\mathcal{P}(1)}\left(\mathrm{R}_1(X)\,,\,\mathrm{R}_1(1)\right)$$

which is thus a set with a single element. This proves that $\widetilde{1}_1$ is terminal.

Let $\pi_1, \pi_2 : \Omega^2 \to \Omega$ be the two projections. To show that $\mathrm{Full}_{\mathcal{E}}(T)$ is cartesian we have to find an arrow $[\times] : \Omega^2 \to \Omega$ such that we have a pullback of the form

$$\begin{array}{ccc} U & \longrightarrow & \Omega' \\ {\scriptstyle \widetilde{\pi_1}\times\widetilde{\pi_2}}\downarrow & & \downarrow{\scriptstyle T} \\ \Omega^2 & \xrightarrow{[\times]} & \Omega \end{array}$$

We simply take $[\times] = \mathrm{R}_{\Omega^2}(\widetilde{\pi_1} \times \widetilde{\pi_2})$. We actually know by lemma 1 that the category $\mathcal{P}(\Omega^2)$ is cartesian. Hence, with respect to its product, we have just set

$$[\times] = \pi_1 \times \pi_2$$

Now it suffices to see that $\widetilde{\pi_1 \times \pi_2} \cong \widetilde{\pi_1} \times \widetilde{\pi_2}$. But this is obvious, since the functor $f \to \widetilde{f}$ preserves limits as a right adjoint.

To define internal exponential, we have to find an arrow $[\Rightarrow] : \Omega^2 \to \Omega$ such that we have a pullback of the form

$$\begin{array}{ccc} V & \longrightarrow & \Omega' \\ {\scriptstyle \widetilde{\pi_1}\Rightarrow\widetilde{\pi_2}}\downarrow & & \downarrow{\scriptstyle T} \\ \Omega^2 & \xrightarrow{[\Rightarrow]} & \Omega \end{array}$$

We take $[\Rightarrow] = \mathrm{R}_{\Omega^2}(\widetilde{\pi_1} \Rightarrow \widetilde{\pi_2})$. Again, we have to check that $\widetilde{[\Rightarrow]} \cong \widetilde{\pi_1} \Rightarrow \widetilde{\pi_2}$. But this results from the axiom of impredicative product. Actually, another equivalent definition for $[\Rightarrow]$ is

$$[\Rightarrow] = \forall\pi_1\,(\pi_2\widetilde{\pi_1})$$

Now we prove that $\mathrm{Full}_{\mathcal{E}}(T)$ has all internal small product (ie. products "indexed by objects of $\mathcal{E}$"). This amounts to find, for any object X of $\mathcal{E}$ an arrow $[\Pi_X] : \Omega^X \to \Omega$ such that we have a pullback of the form

$$\begin{array}{ccc} W & \longrightarrow & \Omega' \\ {\scriptstyle \Pi\pi_1(\widetilde{ev})}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ \Omega^X & \xrightarrow{[\Pi_X]} & \Omega \end{array}$$

where $\mathrm{ev} : \Omega^X \times X \to \Omega$ is the evaluation map and $\pi_1 : \Omega^X \times X \to \Omega^X$ is the first projection map. We simply take $[\Pi_X] = \forall\pi_1(\mathrm{ev})$ and the axiom of impredicative product gives the required pullback. ∎

As a corollary, we have another caracterisation of dictoses, coming from the work of M. Hyland :

Corollary 1 *A category is a dictos iff it is locally cartesian closed and has an internal full subcategory which is internally small complete.*

However we shall keep our caracterisation of dictoses which is more elementary, and closer to the idea of generalising topos theory. Furthermore, this notion may be easily weakened (cf. HLCCC's) and strengthened (for instance, it could seem natural to require all monos to be classified by T, since this phenomenon appears in any topos, and in $\omega-$**Set** as well, which is a non trivial dictos, as we shall see).

3 Reflection in $\omega-$Set

We won't recall how the category $\omega-$**Set** is a model of the theory of constructions (an HLCCC), since this is now a well known fact. We just want to express in this category the reflection principle, since it is very simple and nice. Remember just that an ω-set X is a pair $(|X|, \vdash_X)$ where $|X|$ is a set, called support of X and $\vdash_X$ is a relation over $\omega \times X$ such that for any $x \in |X|$ there exists a $n \in \omega$ such that $n \vdash_X x$. This means that n justifies x as an element of X. An ω-set is said to be modest if any integer justifies at most one point. For more details, see for instance [13], [2] or [3].

Let X be an ω-set. We associate to it a modest ω-set $\overline{X}$. Let $\sim$ be the transitive closure of the $\smile$ relation defined on $|X|$ by

$$x \smile x' \Leftrightarrow \exists n \in \omega \quad n \vdash_X x \text{ and } n \vdash_X x'$$

We set $|\overline{X}| = |X|/{\sim}$, and we define on this set of classes the following justification relation: if $\xi \in \overline{X}$ and $n \in \omega$, then

$$n \vdash_{\overline{X}} \xi \Leftrightarrow \exists x \in \xi \quad n \vdash_X x$$

With this justification relation, it is clear that $\overline{X}$ is a modest ω-set. Let P be any modest ω-set. Up to a trivial isomorphism, it is clear that $\overline{P} = P$. We check easily that

$$\mathrm{Hom}\,(X\ ,\ P) \cong \mathrm{Hom}\left(\overline{X}\ ,\ P\right)$$

actually, it suffices to remark that if $f : X \to P$ is a morphism of $\omega-$**Set** (i.e. a justified function), then

$$\forall x, x' \in X \quad x \smile x' \Rightarrow f(x) = f(x')$$

since P is modest. From this construction, we deduce the reflection principle: let $f : X \to Y$ be a morphism of $\omega-$**Set**. We define $\mathrm{R}_Y(f) : Y \to \Omega$ by setting

$$\forall y \in |Y| \quad \mathrm{R}_Y(f)(y) = \overline{f^{-1}(y)}\ .$$

Conclusion

We have just presented a few facts about dictoses. We hope that some well known results of topos theory may be extended cleanly to this framework. For instance we may expect a connection between dictoses and PL-categories (see [16]) similar to the one which exists between toposes and triposes (see [11]). We hope also to show that, for any internal category in a dictos, the category of internal presheaves over this internal category in the dictos is a dictos.

This study may be seen as a semantical approach to powerful type theories like the calculus of constructions, and we may hope that such an approach could have some applications in the study of these type theories which take more and more importance in synthesis of proved programs.

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